# $n$-ary algebras. Intervals arithmetic

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# Introduction

Ce mémoire comporte deux parties distinctes.

• La première partie concerne une étude d'algèbres $n$-aires. Une algèbre $n$-aire est un espace vectoriel sur lequel est définie une multiplication sur $n$ arguments. Classiquement les multiplications sont binaires, mais depuis l'utilisation en physique thorique de multiplications ternaires, comme les produits de Nambu, de nombreux travaux mathématiques se sont focalisés sur ce type d'algèbres. Deux classes d'algèbres $n$-aires sont essentielles: les algèbres $n$-aires associatives et les algèbres $n$-aires de Lie. Nous nous intéressons aux deux classes. Concernant les algèbres $n$-aires associatives, on s'intéresse surtout aux algèbres 3-aires partiellement associatives, c'est-à-dire dont la multiplication vérifie l'identité

$$((x_1 \cdot x_2 \cdot x_3) \cdot x_4 \cdot x_5) + (x_1 \cdot (x_2 \cdot x_3 \cdot x_4) \cdot x_5) + (x_1 \cdot x_2 \cdot (x_3 \cdot x_4 \cdot x_5)) = 0.$$

Ce cas est intéressant car les travaux connus concernant ce type d'algèbres ne distinguent pas les cas $n$ pair et $n$-impair. On montre dans cette thèse que le cas $n = 3$ ne peut pas être traité comme si $n$ était pair. On étudie en détail l'algèbre libre 3-aire partiellement associative sur un espace vectoriel de dimension finie. Cette algèbre est graduée : $L(V) = \oplus_{p \geq 0} L^p(V)$. On calcule précisément les dimensions des composantes pour $p = 1, 2, 3, 4, 5, 6, 7$. On donne dans le cas général un système de générateurs ayant la propriété qu'une base est donnée par la sous famille des éléments non nuls. Les principales conséquences sont

1. L'algèbre libre 3-aire partiellement associative est résoluble.
2. L'algèbre libre commutative 3-aire partiellement associative est telle que tout produit concernant 9 éléments est nul. soit $L_c(V) = \oplus_{0 \leq p \leq 3} L^p(V)$.
3. L'opérade quadratique correspondant aux algèbres 3-aires partiellement associatives ne vérifient pas la propriété de Koszul.

On s'intéresse ensuite à l'étude des produits $n$-aires sur les tenseurs. L'exemple le plus simple est celui d'un produit interne sur des matrices non carrées. Nous pouvons définir le produit 3-aire donné par $A \cdot^t B \cdot C$. On montre qu'il est nécessaire de généraliser un peu la définition de partielle associativité. Nous introduisons donc les produits $\sigma$ partiellement associatifs où $\sigma$ ets une permutation du groupe $\Sigma_p$.

Concernant les algèbres de Lie $n$-aires, deux classes d'algèbres ont été définies: les algèbres de Fillipov (aussi appelées depuis peu les algèbres de Lie-Nambu) et les algèbres $n$-Lie. Cette dernière notion est très générale. La condition de Jacobi sécrit

$$\sum_{\sigma \in \Sigma_{2n-1}} (-1)^{\epsilon(\sigma)} \mu(\mu(x_{\sigma(1)}, \cdots, x_{\sigma(n)}), x_{\sigma(n+1)}, \cdots, x_{\sigma(2n-1)}) = 0.$$

Quant aux algèbres de Lie de Fillipov, la condition est

$$\begin{aligned}[][u_1, \cdots, u_n], v_1, \cdots, v_{n-1}] &= [[u_1, v_1, \cdots, v_{n-1}], u_2, \cdots, u_n] + [u_1, [u_2, v_1, \cdots, v_{n-1}], u_3, \cdots, u_n] \\ &\quad + \cdots + [u_1, \cdots, [u_n, v_1, \cdots, v_{n-1}]]\end{aligned}$$

Cette dernière notion, très important dans l'étude de la mécanique de Nambu-Poisson, est un cas particulier de la première. Mais pour définir une approche du type Maurer-Cartan, c'est-à-dire définir une cohomologie scalaire, nous considérons dans ce travail les algèbres de Fillipov comme des algèbres $n$-Lie et développons un tel calcul dans le cadre des algèbres $n$-Lie. On s'intéresse également à la classification des algèbres $n$-aires nilpotentes.

Le dernier chapitre de cette partie est un peu à part et reflète un travail poursuivant mon mémoire de Master. Il concerne les algèbres de Poisson sur l'algèbre des polynômes. On commence à présenter le crochet de Poisson sous forme duale en utilisant des équations de Pfaff. On utilise cette approche pour classer les structures de Poisson sur $\mathbb{C}[X_1, X_2, X_3]$ non homogène. Le lien avec les algèbres de Lie est clair. Du coup on étend notre étude aux algèbres de Poisson dont l'algèbre de Lie sous jacent est rigideet on applique les résultats aux algèbres enveloppantes des algèbres de Lie rigides.

• La partie 2 concerne l'arithmétique des intervalles. Cette étude a été faite suite à une rencontre avec une société d'ingéniérie travaillant sur des problèmes de contrôle de paramètres, de problème inverse (dans quels domaines doivent évoluer les paramètres d'un robot pour que le robot ait un comportement défini). Dans le cadre de l'arithmétique des intervalles, les éléments sont les variables et les opérations sur les intervalles sont définies en suivant la règle: le résultant de l'opération de deux intervalles doit être l'intervalle contenant les résultants de la même opération sur les éléments des deux intervalles. Ces opérations, l'addition, la multiplication, la soustraction, ne suivent pas les règles de l'arithmétique classique. Par exemple la multiplication n'est pas distributive par rapport à l'addition, ce qui pose quelques problèmes calculatoires. Dans ce travail, on définit un modèle algébrique, enplongeant l'ensemble des intervalles dans une algèbre associative de dimen,sion 4. Ceci permet de mener un calcul formel algébrique sur les intervalles. Par contre le résultat du produit est plus large que le résultat espérer (il faut controler à tout instant les opérations, sinon très rapidement les calculs sur les intervalles peuvent résulter sur un intervalle résultat pouvant ̂tre parexemple égal à $\mathbb{R}$. Nous montrons qu'il existe une suite d'algèbre associative de dimension arbitraire, dans chacune desquelles nous pouvons plonger l'ensemble des intervalles mais avec uneprécision sur les calculs de plus enplus grande. On applique ces résultats à des problèmes de diagonalisation de matrices d'intervalles. Une approche des intervalles infiniment petits est également abordée.

# Part I

# n-ary algebras

# Chapter 1

# $(2k+1)$-ary partially associative algebras and coalgebras


An $n$-ary algebra is a vector space provided with a multiplication given by a $n$-linear map. A 3-ary algebra is partially associative if the product satisfies

$$((x_1 \cdot x_2 \cdot x_3) \cdot x_4 \cdot x_5) + (x_1 \cdot (x_2 \cdot x_3 \cdot x_4) \cdot x_5) + (x_1 \cdot x_2 \cdot (x_3 \cdot x_4 \cdot x_5)) = 0.$$

If $L(V) = \oplus_{p \geq 0} L^{2p+1}(V)$ denotes the free partially associative 3-ary algebra on a vector space $V$, we compute the dimensions of the components for $p \leq 7$ and give a system of generators for all the components. We show that this system is a basis if and only if none of these generators is zero. It permits to show that the corresponding operad doesn't verify the Koszul property. We describe the operadic cohomology of the $n$-ary partially associative algebras. In the last part we introduce the notions of $n$-ary partially and totally coassociative coalgebras generalizing (binary) coassociative coalgebras and extend the common properties between associative algebras and coassociative coalgebras to the $n$-ary case.


## 1.1 Introduction

There is no need to explain the interest of the class of associative algebras when we study binary algebras. So when we explore $n$-ary algebras, that is, algebras with an $n$-linear operation, it is natural to generalize the associativity in the binary case to the $n$-ary case. There exists two usual ways to generalize it, called partial and total associativity (see [17]) and more recently $\sigma$-partial and $\sigma$-total associativity ([55]). The partially and totally associative algebras for even $n$ where considered by Gnedbaye ([55]). He studied the corresponding free algebras and the associated operads. The results of [17] can be understood as a natural generalization of the binary case $n = 2$. But the case of odd $n$ behaves in a completely different way. An explanation can be done in terms of operads. If we consider for example the operad for partially associative $n$-ary algebras with $n = 2k+1 \leq 7$, it is non-Koszul ([43]) so the natural homology of the free algebra is not trivial, although the operad for partially associative $n$-ary algebras with $n = 2k$ is Koszul. If $\mathcal{P}$ is a quadratic operad generated by an operation of arity $n$ and degree $d$, then the generating operation of $\mathcal{P}^!$ has the same arity but degree $-d+n-2$, i.e. for $n$ odd, the Koszul duality does not preserve the parity of the degree of the generating operation.

In this paper we study the free 3-ary partially associative algebra generated by a finite set. Since the coefficients of the generating series of an operad coincide with the dimensions of the homogeneous components of the corresponding free algebra computing of the first components leads to the result that the operad for 3-ary partially associative algebras cannot be Koszul. A main consequence of this property concerns the deformation theory. If we write a deformation of an $n$-ary partially associative algebra as a formal series $\mu = \sum t^i \varphi_i$, thus the linear part $\varphi_1$ is a 2-cocycle of the deformation cohomology (see [42] and [43]). If

$n = 2$, $\mu_0$ is associative and this cohomology is the Hochschild cohomology of $\mu_0$, that is the natural cohomology given by the corresponding operad. If $n = 3$ and $\mu_0$ partially associative, as the corresponding operad is not Koszul, these cohomologies differs. In this case, following [43], we have to consider 3-ary multiplication of degree $d$. We determine explicitly the operadic complex for any $n$-ary partially associative multiplication of degree 1. If $n$ is even, we find the same result as in [15]. To understand the importance of the degree of the multiplication, we determine the spaces of cochains for a $n$-ary partially associative algebra with a multiplication of degree 0 and coboundary operators whose actions on cochains are similar to the previous one. In this case, we obtain a complex whose space of cochains depends on the partially associative multiplication.

In the last section we introduce the notions of coalgebras for $n$-ary algebras, that is, $n$-ary coalgebras. We generalize the common properties relating associative algebras and coassociative coalgebras. So the dual space of a $n$-ary partially coassociative coalgebra can be provided with a structure of an $n$-ary partially associative algebra, the dual space of a finite dimensional associative algebra can be provided with a structure of $n$-ary partially coassociative coalgebra structure and also, if $(A, \mu)$ is an associative algebra and $(M, \Delta)$ an $n$-ary totally coassociative coalgebra, the space $Hom(M, A)$ can be provided with an $n$-ary partially associative algebra structure.

## 1.2 Associative $n$-ary algebras

Let $\mathbb{K}$ be a field of characteristic zero. An $n$-ary algebra $(V, \mu)$ is a $\mathbb{K}$-vector space $V$ with a linear map

$$\mu : V^{\otimes n} \to V.$$

In what follows, $I_0 \otimes \mu$ and $\mu \otimes I_0$ means $\mu$ and, for any positive integer $k$, $I_k$ is the identity map of $End(V^{\otimes k})$.

**Definition 1** *The $n$-ary algebra $(V, \mu)$ is*

- *partially associative if $\mu$ satisfies*

$$\sum_{p=0}^{n-1} (-1)^{p(n-1)} \mu \circ (I_p \otimes \mu \otimes I_{n-p-1}) = 0 \tag{1.1}$$

- *totally associative if $\mu$ satisfies*

$$\mu \circ (\mu \otimes I_{n-1}) = \mu \circ (I_p \otimes \mu \otimes I_{n-p-1}), \tag{1.2}$$

*for any $p = 0, \cdots, n-1$.*

**Example : Gerstenhaber products.** Let $\mathcal{A}$ be a (binary) associative algebra and $H^*(\mathcal{A}, \mathcal{A})$ its Hochschild cohomology. The space of $k$-cochains is $\mathcal{C}^k(\mathcal{A}) = Hom_{\mathbb{K}}(\mathcal{A}^{\otimes k}, \mathcal{A})$. Gerstenhaber ([15]) defined a graded pre-Lie algebra $\oplus_k \mathcal{C}^k(\mathcal{A})$ with the product

$$\bullet_{n,m} : \mathcal{C}^n(\mathcal{A}) \times \mathcal{C}^m(\mathcal{A}) \to \mathcal{C}^{n+m-1}(\mathcal{A})$$

given by

$$(f \bullet_{n,m} g)(X_1 \otimes \cdots \otimes X_{n+m-1}) = \sum_{i=1}^{m} (-1)^{(i-1)(m-1)} f(X_1 \otimes \cdots \otimes g(X_i \otimes \cdots X_{i+m-1}) \otimes \cdots \otimes X_{n+m-1}).$$

The $k$-cochain $\mu$ which satisfies the identity $\mu \bullet_{k,k} \mu = 0$ and provides $\mathcal{A}$ with a $k$-ary partially associative structure.

**Remark.** There exists a generalization of the notions of partial and total associativity leading to a natural extension of the classical product of matrices to hypercubic matrices. Let $\sigma$ be an element of the symmetric group $\Sigma_n$ of degree $n$ and consider the endomorphism of $V^{\otimes n}$ given by

$$\phi_\sigma^V(e_{i_1} \otimes \cdots \otimes e_{i_n}) = e_{i_{\sigma^{-1}(1)}} \otimes \cdots \otimes e_{i_{\sigma^{-1}(n)}}.$$

An $n$-ary algebra $(V,\mu)$ is

- $\sigma$-partially associative if $\mu$ satisfies

$$\sum_{p=0}^{n-1}(-1)^{p(n-1)}\varepsilon(\sigma^p)\mu\circ(I_p\otimes(\mu\circ\phi^V_{\sigma^p})\otimes I_{n-p-1})=0 \tag{1.3}$$

where $\varepsilon(\tau)$ is the signature of the permutation $\tau$.

- $\sigma$-totally associative if $\mu$ satisfies

$$\mu\circ(\mu\otimes I_{n-1})=\mu\circ(I_p\otimes(\mu\circ\phi^V_{\sigma^p})\otimes I_{n-p-1}), \tag{1.4}$$

for any $p=0,\cdots,n-1$.

In the particular case, where $\sigma=Id$ we get partial and total associativity. In [55] we have defined a $2k+1$-product $s_k$-totally associative on the vector space $T^p_q(E)$ of tensors of $q$ contravariant and $p$ covariant type on a vector space $E$ where $s_k$ is the permutation

$$s_k(1,\cdots,2k+1)=(2k+1,2k,\cdots,2,1).$$

## 1.3 Free 3-ary partially associative algebras

The free $n$-ary totally associative algebras (for arbitrary $n$) and the free $n$-ary partially associative algebras for even $n$ are described in [17] and [43]. The methods used for the study of the free $n$-ary partially associative algebras do not work for odd $n$. In this section, we present the case $n=3$.

Let $L(V,\cdot)$ be the free 3-ary partially associative algebra on the $\mathbb{K}$-vector space $V$, where $x\cdot y\cdot z$ denotes the 3-ary multiplication of $L(V)$. This algebra admits a natural grading

$$L(V,\cdot)=\oplus_{p\geq 1}L^{2p+1}(V)$$

with $L^1(V)=V,\ L^3(V)=V^{\otimes 3}$. We denote by $F(V,\cdot)$ the free 3-ary algebra on $V$ corresponding to a 3-ary product (with no relations). This algebra is also graded

$$F(V,\cdot)=\oplus_{p\geq 0}F^{2p+1}(V)$$

with $F^1(V)=V,\ F^3(V)=V^{\otimes 3}$ and

$$F^{2p+1}(V)=\oplus_{(a,b,c)\in D(2p+1,3)}F^a(V)\otimes F^b(V)\otimes F^c(V),$$

where $D(k,3)$ is the set of triples $(a,b,c)$ of odd positif integers such that $a+b+c=k$. Then, for $p>1$, $L^{2p+1}(V)$ is a quotient space of $F^{2p+1}(V)$.

**i) Coding a vector of $F^{2p+1}(V)$.** We denote by $(v_1\cdot v_2\cdot v_3)$ a vector of $L^3(V)$ or $F^3(V)$ which is a 3-product of 3 vectors of $V$. An element of $F^{2p+1}(V)$ is a linear combination of vectors which are written as a word $(w_1\cdot w_2\cdot w_3)$ with $w_1\in F^a(V), w_2\in F^b(V), w_3\in F^c(V)$ and $(a,b,c)\in D(2p+1,3)$ so an element of $F^{2p+1}(V)$ is a linear combination of vectors $v_1\cdot v_2\cdots\cdot v_{2p}\cdot v_{2p+1}$ of length $2p+1$ with $p$ inserted parenthesis, each parenthesis containing exactly 3 vectors. For example $(v_1\cdot(v_2\cdot(v_3\cdot v_4\cdot v_5)\cdot v_6)\cdot v_7)\in F^7(V)$. If we consider a basis of $V$, then $F^{2p+1}(V)$ is generated by all the words of lenght $2p+1$ constructed on these basis vectors. Each of these words contains exactly $p$ parentheses. Such words will be called primitive vectors of $L^{2p+1}(V)$. We say that a left parenthesis is at the position $k$ if it is between the $k-1$ and the $k$ vectors appearing in the word. Thus the position of the parentheses can be coded by the position of the left parentheses. To simplify, we forget the first left parenthesis, which is always before the first vector (and the corresponding right parenthesis which is after the last vector). Thus in $F^5(V)$ the left parenthesis are coded by $\{1\},\{2\},\{3\}$ and corresponds respectively to the word $((v_1\cdot v_2\cdot v_3)\cdot v_4\cdot v_5),(v_1\cdot(v_2\cdot v_3\cdot v_4)\cdot v_5),(v_1\cdot v_2\cdot(v_3\cdot v_4\cdot v_5))$. In $F^7(V)$, the left parenthesis are coded by $\{11\},\{12\},\{13\},\{14\},\{15\},\{22\},\{23\},\{24\},\{25\},\{33\},\{34\},\{35\}$. For example $(v_1\cdot(v_2\cdot(v_3\cdot v_4\cdot v_5)\cdot v_6)\cdot v_7)$ corresponds to $\{23\}$.

**Lemma 1** *A $(p-1)$-sequence $\{n_1 \cdots n_{p-1}\}$ of positive integers is a coding of a primitive vector of $F^{2p+1}(V)$ or $L^{2p+1}(V)$ if and only if $1 \leq n_1 \leq 3, n_1 \leq n_2 \leq 5, \cdots, n_{p-2} \leq n_{p-1} \leq 2p-1$. Such a sequence is called admissible $(p-1)$-sequence or coding vector of length $p-1$. Moreover considering a coding vector $\{n_1 n_2 \cdots n_{p-1}\}$ and a vector $v_1 \otimes \cdots \otimes v_{2p+1}$ of $V^{2p+1}$ we get only one primitive vector of $F^{2p+1}(V)$ by bracketing with the defined coding.*

The condition on a $(p-1)$-sequence to be a coding of a primitive element of $F^{2p+1}(V)$ or $L^{2p+1}(V)$ traduce that the multiplication is ternary. For example 16 is not an admissible 2-sequence. There is no 3-product of 7 vectors corresponding to this sequence.

We denote by $\widetilde{C}_{p-1}$ the linear space generated by the coding vectors of length $p-1$. We assume that in this space all the coding vectors of length $p-1$ are independent. Thus $F^{2p+1}(V) = \widetilde{C}_{p-1} \otimes V^{\otimes 2p+1}$. For example, if $p = 2$, then

$$\widetilde{C}_1 = span(\{1\}, \{2\}, \{3\})$$

and

$$\dim F^5(V) = 3 \cdot \dim V^{\otimes 5}.$$

**ii) Coding the relations of $L^{2p+1}(V)$.**

The vector space $L^{2p+1}(V)$ is a quotient space of $F^{2p+1}(V)$. Let $R^{2p+1}(V)$ be the linear subspace of $F^{2p+1}(V)$ generated by the relations defining $L^{2p+1}(V)$. We denote by $C_{p-1}$ the linear subspace of $\widetilde{C}_{p-1}$ corresponding to the coding relations of $R^{2p+1}(V)$ for $p \geq 2$. Thus we have

$$R^{2p+1}(V) = C_{p-1} \otimes V^{\otimes^{2p+1}}$$

and

$$L^{2p+1}(V) = (\widetilde{C}_{p-1}/C_{p-1}) \otimes V^{\otimes^{2p+1}}.$$

**Examples**

1. If $p = 2$, $\dim \widetilde{C}_1 = 3$, $C_1 = span(\{1\} + \{2\} + \{3\})$ and

$$\dim L^5(V) = 2 \dim V^{\otimes^5}.$$

2. If $p = 3$, $\dim \widetilde{C}_2 = 12$, and

$$\begin{aligned} C_2 = \ & span(\{11\} + \{14\} + \{15\}, \{12\} + \{22\} + \{25\}, \{13\} + \{23\} + \{33\}, \\ & \{14\} + \{24\} + \{34\}, \{15\} + \{25\} + \{35\}, \{11\} + \{12\} + \{13\}, \\ & \{22\} + \{23\} + \{24\}, \{33\} + \{34\} + \{35\}). \end{aligned}$$

Thus $\dim C_2 = 8$ and

$$\dim L^7(V) = 4 \dim V^{\otimes 7}.$$

**Proposition 1** *All relations in $C_{p-1}$ are obtained from the relations in $C_{p-2}$ by the two following rules:*

- *Consider an element of $C_{p-2}$. For any coding vector $\{i_1, \cdots, i_{p-2}\}$ appearing in the linear presentation of the element), we add the index $i$ in front of this coding where $i$ is successively equal to $1, 2, 3$ and we replace $i_l$ by $i_l + (i-1)$ for all the elements $\{i_1, \cdots, i_{p-2}\}$ of the element of $C_{p-2}$.*

- *Consider an element of $C_{p-2}$. For all $\{i_1, \cdots, i_{p-2}\}$ appearing in the linear presentation of the element, we add the index $i$ in front of this coding where $i$ is successively equal to $1, 2, \cdots, 2p-1$ and, if $i_1 \leq i$, we conserve $i_1$, otherwise we replace $i_1$ by $i_1 + 2$, and we apply the same rule for all indices that follow. Then we reorder subscripts to get an admissible sequence (Lemma 4 ).*

This proposition permits to obtain the description of $L^{2p+1}$ directly from the one of $L^{2p-1}$.

**Example**. We consider $\{11\}+\{12\}+\{13\} \in C_2$. We obtain the following elements of $C_3$ (and relations in $F^9(V)$):

$$\left\{\begin{array}{ll}
\{111\}+\{112\}+\{113\} & \text{we added 1,} \\
\{222\}+\{223\}+\{224\} & \text{we added 2 and changed } i_l \text{ by } i_l+1, \\
\{333\}+\{334\}+\{335\} & \text{we added 3 and changed } i_l \text{ by } i_l+2, \\
\{111\}+\{114\}+\{115\} & \text{we added 1 and changed } i_l \text{ by } i_l+2 \text{ if } i_l>1, \\
\{112\}+\{122\}+\{125\} & \text{we added 2 and changed } i_l \text{ by } i_l+2 \text{ if } i_l>2, \\
\{113\}+\{123\}+\{133\} & \text{we added 3 and changed } i_l \text{ by } i_l+2 \text{ if } i_l>3, \\
\{114\}+\{124\}+\{134\} & \text{we added 4 and reorganized the sequence,} \\
\{115\}+\{125\}+\{135\} & \text{we added 5 and reorganized the sequence,} \\
\{116\}+\{126\}+\{136\} & \text{we added 6 and reorganized the sequence,} \\
\{117\}+\{127\}+\{137\} & \text{we added 7 and reorganized the sequence.}
\end{array}\right.$$

Thus 8 elements of $C_2$ lead to 80 relations of $\widetilde{C_3}$ which determine the space $R^9(V)$.

**Remark: Symmetric elements, symmetric relations.** Let $\{i_1 i_2 \cdots i_p\}$ be a coding vector of $\widetilde{C}_p$. It defines a vector of $F^{2p+3}$ which is written with $p$ parentheses. For this vector we consider the position of the right parentheses counted from the right side. This sequence $\{j_1 j_2 \cdots j_p\}$ satisfies Lemma 4 and belongs to $\widetilde{C}_p$. Let us consider the linear map

$$s : \widetilde{C}_p \to \widetilde{C}_p$$

given by

$$s(\{i_1 i_2 \cdots i_p\}) = \{j_1 j_2 \cdots j_p\}.$$

It satisfies

$$s^2 = Id.$$

A coding vector $\{i_1 i_2 \cdots i_p\}$ of $\widetilde{C_p}$ is symmetric if

$$s(\{i_1 i_2 \cdots i_p\}) = \{i_1 i_2 \cdots i_p\}.$$

More generally, a vector $v$ of $\widetilde{C_p}$ is symmetric if

$$s(v) = v.$$

The elements of $\widetilde{C_p}/C_p$ determine the relations of definition of $L^{2p+3}$. We call such a relation symmetric if the corresponding vector of $C_p$ is symmetric. The generating relation of $C_1$ is given by the symmetric vector $1+2+3$ of $\widetilde{C_1}$ because $s(1)=3$ and $s(2)=2$. This implies that the symmetric of any vector of $C_p$ is in $C_p$. In other words if we have a relation of definition of $L^{2p+1}(V)$ we have also the symmetric relation amongst the relations of definition of $L^{2p+1}(V)$.

**iii) Determination of $\dim L^{2p+1}(V)$ for $1 \leq p \leq 7$.**

**Proposition 2** *If $m = \dim V$, then*

- $\dim L^3(V) = \dim V^{\otimes^3} = m^3$.
- $\dim L^5(V) = 2m^5$ *and* $\{\{1\},\{3\} = s(\{1\})\}$ *is a basis of* $\widetilde{C_1}/C_1$.
- $\dim L^7(V) = 4m^7$ *and* $\{\{11\},\{13\},\{35\} = s(\{11\}),\{33\} = s(\{13\})\}$ *is a basis of* $\widetilde{C_2}/C_2$.
- $\dim L^9(V) = 5m^9$ *and* $\{\{113\},\{133\},\{355\} = s(\{113\}),\{335\} = s(\{133\}),\{117\}\}$ *is a basis of* $\widetilde{C_3}/C_3$.
- $\dim L^{11}(V) = 6m^{11}$ *and a basis of* $\widetilde{C_4}/C_4$ *is*

$$\{\{1133\},\{1335\},\{3557\} = s(\{1133\}),\{3355\} = s(\{1335\}),\{1177\},\{1379\} = s(\{1177\})\}$$

- $\dim L^{13}(V) = 7m^{13}$ *and* $\{\{11335\},\{13355\},\{35577\},\{33557\},\{11779\},\{13399\},\{11399\}\}$ *is a basis of* $\widetilde{C_5}/C_5$.
- $\dim L^{15}(V) = 8m^{15}$ *and*

$$\{\{113355\},\{133557\},\{355779\},\{335577\},\{117799\},\{1335(11)(13)\},\{11399(11)\},\{1339(11)(11)\}\}$$

*is a basis of* $\widetilde{C_6}/C_6$.

*Proof.* The first two cases are clear.

- $\dim L^7(V) = 4m^7$. To simplify, we use $ij$ instead of $\{ij\}$.

$$\begin{cases} 1) & 25 = 11 \\ 2) & 15 = -11-35, \\ 3) & 14 = 35, \\ 4) & 22 = 13, \\ 5) & 12 = -11-13, \\ 6) & 34 = -33-35, \\ 7) & 23 = -13-33, \\ 8) & 24 = 33, \end{cases}$$

a basis of $\widetilde{C_2}/C_2$ is

$$\{11, 13, 35 = s(11), 33 = s(13)\}.$$

- $\dim L^9(V) = 5m^9$. To simplify, we use $ijk$ instead of $\{ijk\}$. At this step for the first time some trivial monoidal identities $ijk = 0$ occur. More precisely, we have

$$\begin{aligned} 0 &= 111 = 114 = 115 = 125 = 135 = 136 = 144 = 147 = 222 = 225 = 226 = 236 = 246 = 247 = 255, \\ &= 333 = 336 = 337 = 347 = 357. \end{aligned}$$

The other identities are reduced to:

$$\begin{cases} a) & 113 = 122 = -112 = -137 = -227 = 237, \\ b) & 355 = s(113) = 145 = -146 = -155 = 346 = -356, \\ c) & 133 = 124 = -134 = 223 = -224 = -233, \\ d) & 335 = s(133) = -235 = -244 = 245 = -334 = 344, \\ e) & 117 = -116 = 126 = -157 = -256 = 257, \\ f) & 123 = -133 + 113, \\ g) & 345 = 335 - 355, \\ h) & 127 = 113 + 117, \\ i) & 156 = 117 + 355, \\ j) & 234 = 133 + 335. \end{cases}$$

Remark that the second line is symmetric to the first line, the third and fourth line are symmetric to each other. We have also that $s(117) = 157$ and more generally the symmetric of the fifth line is itself (modulo the sign). In the same way, $s(f) = g$, $s(h) = i$ and $s(j) = j$. For symmetry reasons, we choose as a basis of $\widetilde{C_3}/C_3$,

$$\{113, 133, 355, 335, 117\}.$$

- $\dim L^{11}(V) = 6m^{11}$. Any coding vector that is deduced from a trivial one in $\widetilde{C_3}/C_3$ is a trivial coding vector in $\widetilde{C_4}/C_4$. Then we have the new trivial vectors:

$$\begin{aligned} 0 &= 1122 = 1125 = 1126 = 1127 = 1137 = 1169 = 1179 = 1227 = 1237 = 1248 = 1266 = 1269 = 1346 \\ &= 1348 = 1377 = 1455 = 1458 = 1459 = 1468 = 1469 = 1555 = 1558 = 1559 = 1569 = 2233 = 2237 \\ &= 2238 = 2248 = 2277 = 2347 = 2348 = 2358 = 2359 = 2377. \end{aligned}$$

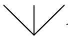
The non trivial identities are

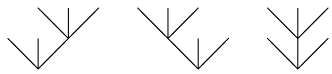
$$\begin{cases} 1133 & = -1233 = -1123 = 1124 = -1134 = 1223 = -1224 = -1249 = -1339 = 1349 = -2239 = 2249 \\ & = 2339 = -2349, \\ 1335 & = -1235 = -1244 = 1245 = -1334 = -1345 = -2234 = 2235 = 2244 = -2245 = 2334 = -2335 \\ & = -2344 = 1344, \\ 1177 & = -2578 = -1167 = 1168 = -1178 = 1267 = -1268 = -1277 = -2567 = 2568 = 2577 = -1568 \\ & = -1577 = 1578 \\ 1234 & = 1335 + 1133, \\ 1239 & = 1133 + s(1177), \\ 1278 & = s(1177) + 1177, \\ 2345 & = 1335 + s(1335). \end{cases}$$

and their symmetric identities. Thus, we can choose as a basis the following family

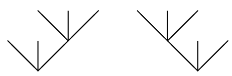
$$\{\{1133\}, \{1335\}, \{3557\} = s(1133), \{3355\} = s(1335), \{1177\}, \{1379\} = s(1177)\}.$$

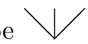
- $\dim L^{13}(V) = 7m^{13}$, $\dim L^{15}(V) = 8m^{15}$. In these cases the dimensions have been computed with Mathematica and the presented coding vectors form a basis of $\widetilde{C_p}/C_p$. For $n \geq 15$, the computations with Mathematica become impossible because of problems of memory. But in the next paragraph, we are going to describe generators of $\widetilde{C_p}/C_p$ in the general case.

**v) The general case.** In order to visualize and to better understand the relations that appear in any dimension, and especially to determine a minimal system of generators we represent each coding vector of $\widetilde{C}_p$ by a planar rooted tree with $(2p+3)$ leaves and only 3-branching nods (i.e. vertices have precisely 3 incoming edges), that is, three entries and one exit, since the multiplication is 3-ary (see [**?**, Section 4] or [45, II.1.5] for terminology). We denote by $\mathcal{T}_{2p+3}$ the set of all such trees. The leaves are the external edges that is the edges which have only one adjacent vertex. The root vertex of a tree is at the depth 0 and the level of a vertex $v$ is its distance to the root, that is, the number of vertices between $v$ and the root vertex. The level $k$ of a tree is then defined by the vertices of level $k$. The high of a tree is the maximal level of the vertices more one. We extend the definition to $\mathcal{T}_1$ by considering $|$, the one-point set consisting of the exceptional tree with one leg and no internal vertex. This tree will be considered of high 0. The set $\mathcal{T}_3$ contains only one tree  that represents the vector $\{\}$ of $\widetilde{C}_0$ and has the high 1. Likewise $\mathcal{T}_5$ contains the trees



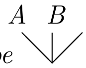
that represent respectively the vectors $\{3\}, \{1\} = s(\{3\}), \{2\}$ of $\widetilde{C}_1$, all of high 2. Note that since $\{2\} = \{1\} - \{3\}$, a basis of $\widetilde{C_1}/C_1$, is represented by the following trees:



A tree $T$ of $\mathcal{T}_{2p+3}$ is of type A B C  with $A \in \mathcal{T}_{2k_1+3}, B \in \mathcal{T}_{2k_2+3}$ and $C \in \mathcal{T}_{2k_3+3}$, $k_1 + k_2 + k_3 = p - 1$, $k_i \in [[-1, \cdots, p-1]] = \mathbb{Z} \bigcap [-1, \cdots, p-1]$ and its high $h(T)$ is $1 + sup(h(A), h(B), h(C))$.

As there is the one-to-one correspondence between the coding vectors and their corresponding trees we will identify the coding vector to its corresponding tree.

By subtree we mean a connected part of a tree obtained from this tree by cutting nodes.

**Definition 2** *We say that a tree $T$ of has no central branch if it has no subtree of type* $A$ $B$ $C$  *with $B$ different of the exceptional tree.*

For example, the trees corresponding to $\{1\}$ and $\{3\}$ are with no central branches whereas $\{2\}$ has one.

**Proposition 3** *A tree $T$ is with no central branches if and only it has not the tree corresponding to $\{2\}$ as a subtree.*

**Lemma 2** *There exists a system of generators of $\widetilde{C_p}/C_p$ which is represented by trees without central branches.*

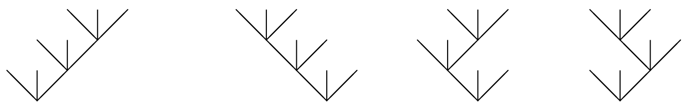
*Proof.* We use recursively the relation $\{2\} = \{1\} - \{3\}$ from the bottom to the top of the tree(s) to obtain a linear combination of trees without the tree $\{2\}$ as subtree. The procedure assure that a subtree $\{2\}$ can only appear higher in the trees obtained at the next steep of the reduction so applying this rule recursively we will obtain the announced result.

**Example.**

$$\{22\} = -\{12\} - \{25\} = \{11\} + \{13\} - \{25\}$$

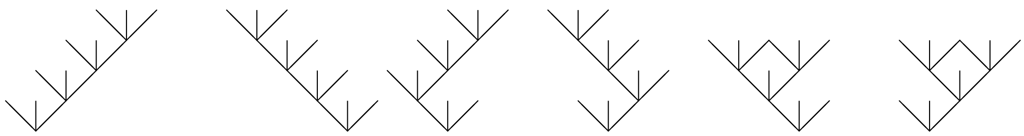
and the trees representing the last vector don't contain central branches.

**Remark.** The chosen basis of $\widetilde{C_2}/C_2$ is represented by the following trees



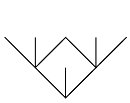
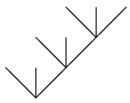
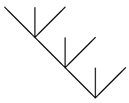
In $L^9(V)$ some identically null products appear for the first time, i.e. some coding vectors of $\widetilde{C_3}/C_3$ are zero. Those whose associated trees are without central branches are the following: $\{357\}$, $\{135\}$, $\{115\}$ and their symmetrical $\{111\}$, $\{333\}$, $\{337\}$. We deduce:

**Proposition 4** *Every tree that represents a coding vector and that contains one of the following trees as subtree*

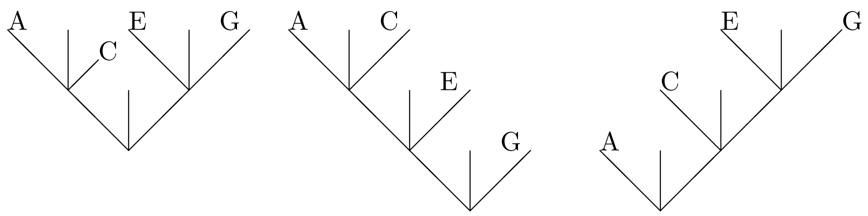


*represents the null vector.*

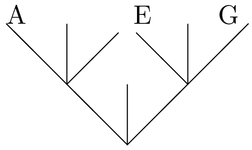
**Corollary 3** *Any tree with no central branches which has as subtree at level at least 1 the tree*

 *or*  *or*  *represents a null coding vector.*

We denote by $\mathcal{T}_1(A, C, E, G)$, $\mathcal{T}_2(A, C, E, G)$, $\mathcal{T}_3(A, C, E, G)$, the following trees where the letters $A, C, E, G$ represent grafted trees.



We will denote the exceptional tree by 0 that we will also call the trivial tree. For example, $\mathcal{T}_1(A, 0, E, G)$

corresponds to 


To simplify we shall say that two trees are equal if they represent two coding vectors that are equal; a tree is representing the null vector if its corresponding coding vector is null.

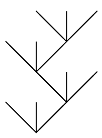
**Proposition 5** *Let us consider the tree* $\mathcal{T}_1(A, C, E, G)$.

1. *If* $G \neq 0$ *and* $A \neq 0$ *then* $\mathcal{T}_1(A, C, E, G) = 0$ *for any* $C$ *and* $E$.
2. *If* $G \neq 0$ *and* $A = 0$ *then* $\mathcal{T}_1(0, C, E, G) = -\mathcal{T}_2(0, C, E, G)$.
3. *If* $G = 0$ *and* $A \neq 0$ *then* $\mathcal{T}_1(A, C, E, 0) = -\mathcal{T}_3(A, C, E, 0)$.
4. *If* $G = 0$ *and* $A = 0$ *then* $\mathcal{T}_1(0, C, E, 0) = -\mathcal{T}_2(0, C, E, 0) - \mathcal{T}_3(0, C, E, 0)$.

*Proof.* From the identity $\{15\} = -\{11\} - \{35\}$, we obtain that $\mathcal{T}_1(A, C, E, G) = -\mathcal{T}_2(A, C, E, G) - \mathcal{T}_3(A, C, E, G)$. If $A \neq 0$, $\mathcal{T}_2(A, C, E, G) = 0$. If $G \neq 0$, $\mathcal{T}_3(A, C, E, G) = 0$. If $A = G = 0$, thus we obtain $\mathcal{T}_1(0, C, E, 0) = -\mathcal{T}_2(0, C, E, 0) - \mathcal{T}_3(0, C, E, 0)$.

**Corollary 4** *Consider a tree* $\mathcal{T}_1(A, C, E, G)$. *If it is non representing the null vector it can only be of the type* $\mathcal{T}_1(A, 0, E, 0)$ *or* $\mathcal{T}_1(0, C, 0, G)$. *Moreover, the tree* $\mathcal{T}_1(A, 0, E, 0)$ *is uniquely determined by the highs of the trees* $A$ *and* $E$ *and the tree* $\mathcal{T}_1(0, C, 0, G)$ *is uniquely determined by the highs of the trees* $C$ *and* $G$.

*Proof.* In all other cases the trees have subtrees representing the null vector so the trees themselves are representing the null vector. In fact consider $T = \mathcal{T}_1(A, C, E, G)$. If $G \neq 0, A \neq 0$ the tree represents the null vector for any $C$ and $E$. For $G \neq 0, A = 0$ the tree represents the null vector if $E$ is non trivial because it has the tree $\{2\}$ as subtree at level at least 1. The same happen if $G = 0$ and $A \neq 0$ meaning that if $C$ is non trivial, the tree $\{2\}$ is a subtree at level at least 1 of the initial tree which then represents the null vector. If $G = 0, A = 0$ then the tree is trivial if $C \neq 0$ and $E \neq 0$ because $\mathcal{T}_2(0, C, E, 0)$ (containing the tree $\{2\}$ at level 1) is representing the null vector. It then remains $\mathcal{T}_1(0, 0, E, 0)$, $\mathcal{T}_1(0, C, 0, 0)$ and $\mathcal{T}_1(0, 0, 0, 0)$. So all the trees $\mathcal{T}_1(A, C, E, G)$ which are not representing the null vector are of the type $\mathcal{T}_1(A, 0, E, 0)$ (with no condition on $A$ and $E$) or $\mathcal{T}_1(0, C, 0, G)$ (with no condition on $C$ and $G$). The second part of the corollary also uses that the trees $\mathcal{T}_1(A, 0, E, 0)$ and $\mathcal{T}_1(0, C, 0, G)$ must not have subtrees representing the null vector. For $\mathcal{T}_1(A, 0, E, 0)$, the tree $A$, if it is not the exceptional tree or the tree $\{\}$, can also just have one tree at the level one and this tree must be at the right side and all other levels are fixed in a same way. So $A$ is of type



We get $E$ by similar reasoning. The case $\mathcal{T}_1(0, C, 0, G)$ can be treated in the same way.

**Theorem 5** *Any element of* $\widetilde{C_{p-1}}/C_{p-1}$ *is a linear combination of the coding vectors:*

1. $p = 4k$.

$$\begin{cases} v_1 = \{113355 \cdots (p-3)(p-3)(p-1)\}, v_2 = \{13355 \cdots (p-1)(p-1)\}, \\ v_5 = \{117799 \cdots (p+1)(p+1)(p+3)\}, v_6 = \{11399 \cdots (p+3)(p+3)\}, \\ v_7 = \{1133(11)(11) \cdots (p+3)(p+3)(p+5)\}, \cdots, \\ v_{\frac{p}{2}+2} = \{1133 \cdots (\frac{p}{2}-3)(\frac{p}{2}-3)(\frac{p}{2}-1)(p+1)(p+1) \cdots (p+\frac{p}{2}-1)(p+\frac{p}{2}-1)\}, \\ \text{their symmetrics}, \\ v_{\frac{p}{2}+3} = \{1133 \cdots (\frac{p}{2}-1)(\frac{p}{2}-1)(p+3)(p+3)(p+5)(p+5) \cdots (p+\frac{p}{2}+1)\}. \end{cases}$$

$v_3 = s(v_1), v_4 = s(v_2), v_{\frac{p}{2}+4} = s(v_5), \cdots, v_{p+1} = s(v_{\frac{p}{2}+2})$

2. $p = 4k + 2$.

$$\begin{cases} v_1 = \{1133 \cdots (p-3)(p-3)(p-1)\}, v_2 = \{133 \cdots (p-1)(p-1)\}, \\ v_5 = \{117799 \cdots (p+1)(p+1)(p+3)\}, v_6 = \{11399 \cdots (p+3)(p+3)\}, \cdots, \\ v_{\frac{p}{2}+2} = \{1133 \cdots (\frac{p}{2}-2)(\frac{p}{2}-2)(p+1)(p+1) \cdots (p+\frac{p}{2}-2)(p+\frac{p}{2}-2), (p+\frac{p}{2})\}, \\ \text{their symmetrics}, \\ v_{\frac{p}{2}+3} = \{1133 \cdots (\frac{p}{2}-2)(\frac{p}{2}-2)(\frac{p}{2})(p+3)(p+3)(p+5)(p+5) \cdots (p+\frac{p}{2})(p+\frac{p}{2})\}. \end{cases}$$

$v_3 = s(v_1), v_4 = s(v_2), v_{\frac{p}{2}+4} = s(v_5), \cdots, v_{p+1} = s(v_{\frac{p}{2}+2})$

3. $p = 4k + 1$

$$\begin{cases} v_1 = \{1133 \cdots (p-2)(p-2)\}, v_2 = \{13355 \cdots (p-2)(p-2)p\}, \\ v_5 = \{1177 \cdots (p+2)(p+2)\}, v_6 = \{11399 \cdots (p+2)(p+2)(p+4)\}, \cdots, \\ v_{\frac{p+5}{2}} = \{1133 \cdots (\frac{p-7}{2})(\frac{p-7}{2})(\frac{p-3}{2})(\frac{p-3}{2})(p+2)(p+2)(p+4)(p+4) \cdots (p+\frac{p-1}{2})(p+\frac{p-1}{2})\}, \\ \text{their symmetrics}. \end{cases}$$

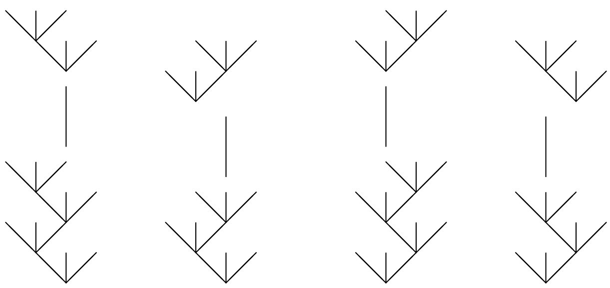
$v_3 = s(v_1), v_4 = s(v_2), v_{\frac{p+7}{2}} = s(v_5), \cdots, v_{p+1} = s(v_{\frac{p+5}{2}})$

4. $p = 4k + 3$

$$\begin{cases} v_1 = \{1133 \cdots (p-2)(p-2)\}, v_2 = \{13355 \cdots (p-2)(p-2)p\}, \\ v_5 = \{1177 \cdots (p+2)(p+2)\}, v_6 = \{11399 \cdots (p+2)(p+2)(p+4)\}, \\ v_{\frac{p+5}{2}} = \{1133 \cdots (\frac{p-5}{2})(\frac{p-5}{2})\frac{p-1}{2}(p+2)(p+2)(p+4)(p+4) \cdots (p+\frac{p-3}{2})(p+\frac{p-3}{2})(p+\frac{p+1}{2})\}, \\ \text{their symmetrics}. \end{cases}$$

$v_3 = s(v_1), v_4 = s(v_2), v_{\frac{p+7}{2}} = s(v_5), \cdots, v_{p+1} = s(v_{\frac{p+5}{2}})$

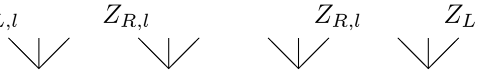
Consider $Z_{L,l}$ the left zigzag tree of high $l$ and $Z_{R,l}$ the right zigzag tree of high $l$ i.e



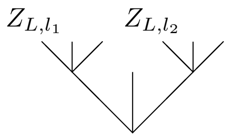
$Z_{L,2s}$ $Z_{L,2s+1}$ $Z_{R,2s}$ $Z_{L,2s+1}$

Then the trees corresponding to the vectors of the theorem are

$Z_{L,l}$ $Z_{R,l}$ $Z_{R,l}$ $Z_{L,l}$

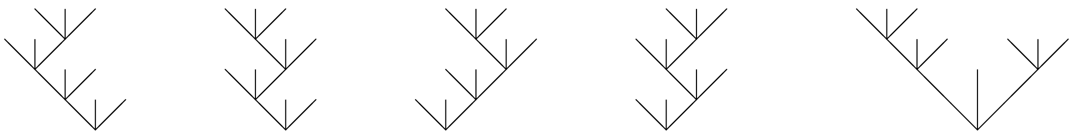


with $l = p - 1$ and

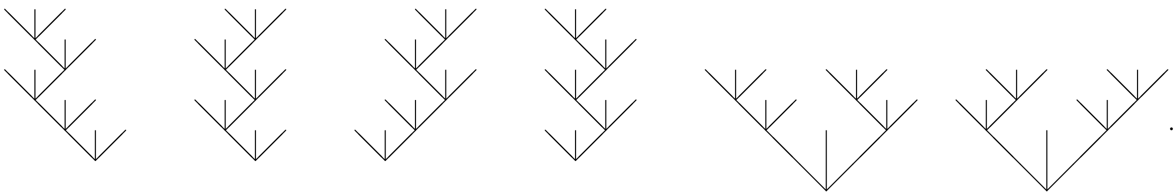



with $l_1 + l_2 + 3 = p$, $1 \leq l_1 \leq \lceil \frac{p-3}{2} \rceil$ (where $\lceil x \rceil$ denotes the ceiling function) and the symmetric of these trees (except for the last one when $p$ is even)

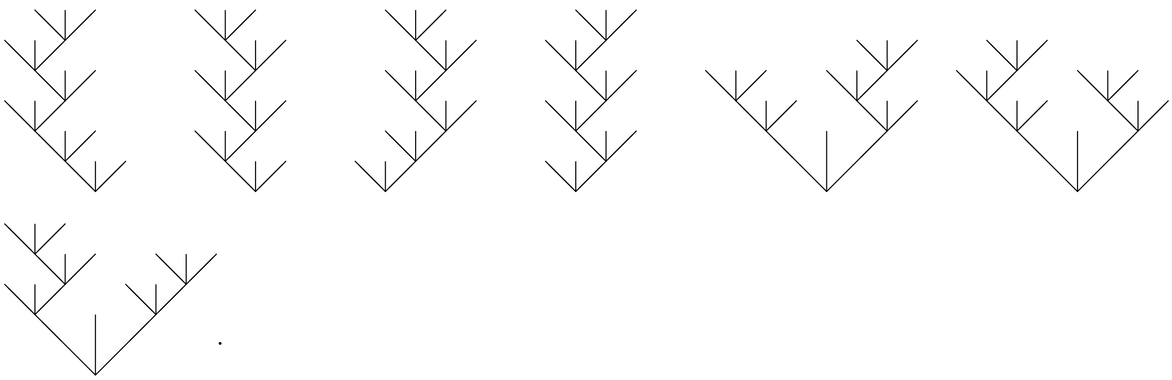
**Examples.** If $2p+1 = 9$, then $p = 4$ and the generating coding vectors are $\{113\}, \{133\}$, $s(\{113\}), s(\{133\})$ and $\{117\}$ corresponding respectively to the trees



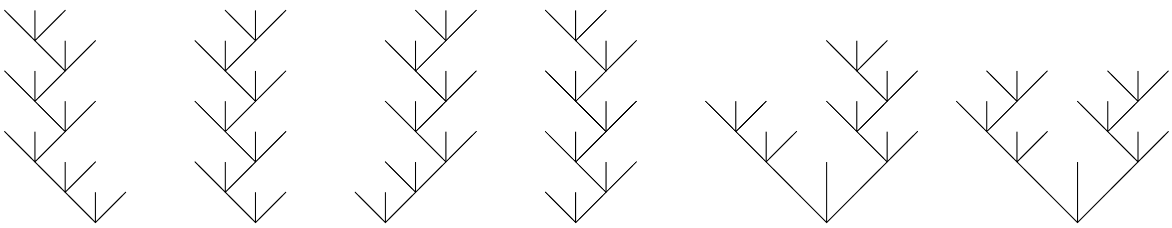
If $2p+1 = 11$, then $p = 5$ and the generating coding vectors are $\{1133\}, \{1335\}, s(\{1133\}), s(\{1335\}), \{1177\}$ and $s(\{1177\})$. corresponding respectively to the trees

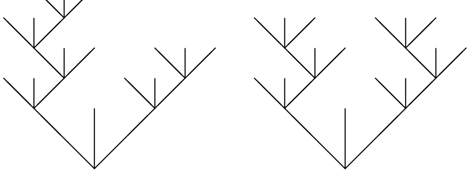


If $2p+1 = 13$, then $p = 6$ and the generating coding vectors are $\{11335\}, \{13355\}$, $s(\{11335\})$, $s(\{13355\})$, $\{11779\}$, $\{11399\}$ and $s(\{11779\})$ corresponding respectively to the trees



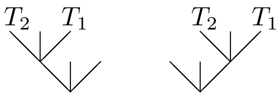
Finally, if $2p + 1 = 15$, $p = 7$, the generating coding vectors are $\{113355\}$, $\{133557\}$, $s(\{113355\})$, $s(\{133557\})$, $\{117799\}$, $\{11399(11)\}$, $s(\{117799\})$ and $s(\{11399(11)\})$ corresponding respectively to the trees





Let's notice that these families of the generators are the bases defined in Proposition 6.

*Proof.* By Lemma 2, any tree in $\mathcal{T}_{2p+1}$ is a linear combination of trees which does not have the tree $\{2\}$ as a subtree, that is without central branches. So, we are going to find a system of generators with trees grafted to the level 2 on the trees $\{1\}$, $\{3\}$ and $\{15\}$ by following the rules of Lemma 2, Proposition 4, its consequence and Proposition 5. This means that we consider, because of Lemma 2, trees of type




and $\mathcal{T}_1(A, C, E, G)$. Let us consider a tree with $\{1\}$ at the bottom. To obtain a tree in $\mathcal{T}_{2p+1}$ we will graft a tree at a level at least 2. There are only two ways to do it, which correspond to the two first trees of the theorem. In fact, we can not graft two non trivial trees at the level 2 because we would get a tree with $\{115\}$ as a subtree, so representing the null vector. So we can graft only one tree on the left or on the right at level 2 (i.e. $T_1$ or $T_2$ is the trivial tree) and in both cases there is only one possible tree to obtain a tree in $\mathcal{T}_{2p+1}$ which is not representing the null vector ($T_2 = Z_{L,p-2}$ and $T_1 = 0$ or $T_1 = Z_{R,p-2}$ and $T_2 = 0$). The case of a tree with $\{3\}$ at the bottom can be treated in a similar way. Thus it only remains to examine the trees with $\{15\}$ at the bottom. By Proposition 5 and its corollary, these trees are the trees $\mathcal{T}_1(A, 0, E, 0)$ with high $h(A)$ of $A$, less than $p-3$ and $\mathcal{T}_1(0, C, 0, G)$ with high $h(C)$, of $C$ less than $p-3$. But if $h(A) = 0$, then the tree $\mathcal{T}_1(A, 0, E, 0) = \mathcal{T}_1(0, 0, E, 0)$ is, up to the sign, a tree with $\{3\}$ at the bottom and if $h(C) = 0$, the tree $\mathcal{T}_1(0, C, 0, G) = \mathcal{T}_1(0, 0, 0, G)$ is, up to the sign, a tree with $\{1\}$ at the bottom. Moreover $h(A) \geq 0$, $\mathcal{T}_1(A, 0, E, 0) = -\mathcal{T}_1(0, C', 0, G') = -s(\mathcal{T}_1(A', 0, E', 0))$ with $h(C') \geq 0$ and $h(A') = p - 2 - h(A) + 1$. Then instead of considering the trees $\mathcal{T}_1(A, 0, E, 0)$ with non trivial $A$, we can take the trees presented in the theorem which where chosen for symmetries reasons.

**Corollary 6** *For any $p \geq 3$, $\dim L^{2p+1}(V) \leq (p+1)(\dim V)^{2p+1}$. If $3 \leq p \leq 7$ then $\dim L^{2p+1}(V) = (p+1)(\dim V)^{2p+1}$.*

We denote by $\mathcal{G}_{2p+1}$ the generating family of vectors of $\widetilde{C_{p-1}}/C_{p-1}$ defined in Theorem 5.

**Theorem 7** *Assume that, for every $p$, every vector $v$ of $\mathcal{G}_{2p+1}$ is non zero. Then*

$$\dim L^{2p+1}(V) = (p+1)(\dim V)^{2p+1}.$$

*Proof.* To each coding vector $v \in \mathcal{G}_{2p+1}$ we assign the tree $t_v \in \mathcal{T}_{2p+1}$. We count the leaves of $t_v$ from left to right. If $t$ and $t'$ are two trees of $\mathcal{T}_{2k_1+1}$ and $\mathcal{T}_{2k_2+1}$ we denote $t \circ_i t'$ the tree of $\mathcal{T}_{2(k_1+k_2)+1}$ obtained by grafting the tree $t'$ at the leaf $i$ of $t$. Let $\sum_{i=1}^{p+1} a_i t_{v_i} = 0$ be a linear combination of the trees $t_{v_i}$, $v_i \in \mathcal{G}_{2p+1}$. Let $u$ be the tree of $\mathcal{T}_3$. By grafting $u$ twice in adapted places, we will prove successively that all the $a_i$ are zero. Consider the trees $t_{v_i} \circ_1 u$. From Corollary 3, all these trees are representing the null vector except of $t_{v_3} \circ_1 u$ and $t_{v_4} \circ_1 u$. Thus the equality $\sum_{i=1}^{p+1} a_i t_{v_i} = 0$ implies $a_3 t_{v_3} \circ_1 u + a_4 t_{v_4} \circ_1 u = 0$. If we then graft $u$ at the highest left leaf of $t_{v_3} \circ_1 u$ and $t_{v_4} \circ_1 u$, exactly one of the two obtained trees of $\mathcal{T}_{2(p+2)+1}$ is zero, the other one is by hypothesis non zero so its corresponding coefficient is zero. Thus coming back to $a_3 t_{v_3} \circ_1 u + a_4 t_{v_4} \circ_1 u = 0$ we get that $a_3 = a_4 = 0$. For symmetric reasons the coefficients corresponding to the symmetric of these trees, that is, $t_{v_1}$ and $t_{v_2}$ are zero. Then $\sum_{i=5}^{p+1} a_i t_{v_i} = 0$ concerns only trees of type $\mathcal{T}_1(A, 0, E, 0)$ with non trivial $A$ and of type $\mathcal{T}_1(0, C, 0, G)$ with non trivial $G$. By grafting $u$ at the highest level the equality $\sum_{i=5}^{p+1} a_i t_{v_i} = 0$ is reduced to a linear combination of two trees associated to vectors of $\mathcal{G}_{2p+3}$. We repeat this process to $\sum_{i=5}^{p+1} a_i t_{v_i} = 0$. Grafting $u$ at the correct place, $\sum_{i=5}^{p+1} a_i t_{v_i} = 0$ is reduced to only to two terms. With a new grafting the corresponding linear combination is reduced to one term and its coefficient is zero. This implies that the second coefficient of the linear combination containing two terms are zero. Repeating this process we prove that all the coefficients are zero. So all the $a_i$ are zero and the vectors of $\mathcal{G}_{2p+1}$ are independent and $\mathcal{G}_{2p+1}$ defines a basis of $\widetilde{C_{p-1}}/C_{p-1}$.

**Corollary 8** *If there exists $p$ such that $v_2 \in \mathcal{G}_{2p+1}$ is zero, then $L^{2k+1} = 0$ for some $k \geq p$.*

*Proof.* In fact $v_2$ is a subtree of any tree of $L^{2k+1}$ for some $k$ greater than $p$.

**Remark.** Recall that for any vector space $V$, the associated tensor algebra $T(V)$ is the unique solution, up to isomorphism, of the universal problem which determine from a linear application $f : V \to A$ in an associative algebra $A$, a morphism of associative algebra $T(V) \to A$. The construction of this algebra comes from the isomorphisms

$$\Phi_{n,m} : T^{\otimes n}(V) \otimes T^{\otimes m}(V) \to T^{\otimes (n+m)}(V)$$

defined by

$$\Phi_{n,m}((x_1 \otimes x_2 \otimes \cdots \otimes x_n) \otimes (y_1 \otimes y_2 \cdots \otimes y_m)) = x_1 \otimes x_2 \otimes \cdots x_n \otimes y_1 \otimes y_2 \otimes \cdots \otimes y_m.$$

In fact the multiplication $\mu$ of $T(V)$ is given by

$$\mu((x_1 \otimes x_2 \cdots \otimes x_n) \otimes (y_1 \otimes y_2 \cdots \otimes y_m)) = \Phi_{n,m}((x_1 \otimes x_2 \cdots \otimes x_n) \otimes (y_1 \otimes y_2 \cdots \otimes y_m))$$

and the associativity of the multiplication follows from

$$\Phi_{n+m,p} \bullet (\Phi_{n,m} \otimes Id_p) = \Phi_{n+m,p} \bullet (Id_n \otimes \Phi_{m,p}).$$

We can define another isomorphism no longer adapted to the associative structure but adapted to the $n$-ary structure. For this we consider the family of vectorial isomorphisms

$$\Psi_{n,m,p} : T^{\otimes n}(V) \otimes T^{\otimes m}(V) \otimes T^{\otimes p}(V) \to T^{\otimes n+m+p}(V)$$

satisfying

$$\begin{cases} \Psi_{n,m+p+q,r} \bullet (Id_n \otimes \Psi_{m,p,q} \otimes Id_r) & = -2\Psi_{n,m+p+q,r} \bullet (Id_{n+m} \otimes \Psi_{p,q,r}) \\ & = -2\Psi_{n,m+p+q,r} \bullet (\Psi_{n,m,p} \otimes Id_{q+r}). \end{cases}$$

## 1.4 Consequences

### 1.4.1 On solvability of ternary partially associative algebras

**Definition 9** *Let $(A, \mu)$ be a ternary partially associative algebra. We denote by $\mathcal{D}^0(A) = A, \mathcal{D}^1(A) = \mu(A, A, A)$ and more generally $\mathcal{D}^{p+1}(A) = \mu(\mathcal{D}^p(A), \mathcal{D}^p(A), \mathcal{D}^p(A))$. We say that $(A, \mu)$ is solvable if there is an integer $p$ such that $\mathcal{D}^p(A) = \{0\}$.*

**Theorem 10** *The free partially associative algebra $L(V, \cdot)$ is solvable.*

*Proof.* Studying $L^7(V)$, we have seen that $\{147\} = 0$. This means that for any vector $v_1, \cdots, v_9$ in $V$, $(v_1 \cdot v_2 \cdot v_3) \cdot (v_4 \cdot v_5 \cdot v_6) \cdot (v_7 \cdot v_8 \cdot v_9) = 0$. Thus $\mathcal{D}^2(L(V, \cdot)) = 0$ and $L(V, \cdot)$ is solvable.

**Corollary 11** *Any ternary partially associative algebra is solvable.*

### 1.4.2 Commutative ternary partially associative algebras

A ternary partially associative algebras is commutative if $\mu(x_1, x_2, x_3) = \mu(x_{\sigma(1)}, X_{\sigma(2)}, X_{\sigma(3)})$ for any permutation $\sigma$.

**Theorem 12** *Let $\mathcal{A}(V)$ be the free commutative ternary partially associative algebra on the vector space $V$. Thus $\mathcal{A}^9(V) = 0$.*

*Proof.* We have $\mathcal{A}(V) = \oplus_{k\in\mathbb{N}}\mathcal{A}^{2k+1}(V)$. We have $\mathcal{A}^1(V) = V$ and $\mathcal{A}^3(V)$ is of dimension $C_3^{n+2} = \frac{(n+2)(n+1)n}{6}$ since the commutativity implies $x_1 \cdot x_2 \cdot x_3 = x_{\sigma(1)} \cdot x_{\sigma(2)} \cdot x_{\sigma(3)}$ for any $\sigma \in \Sigma_3$ so we get a basis of $\mathcal{A}^3(V)$ considering $e_{i_1} \otimes e_{i_2} \otimes e_{i_3}$ with $i_1 \leq i_2 \leq i_3,\ i_1, i_2, i_3 \in [[1, \cdots, n]]$. Clearly $\mathcal{A}^{2k+1}(V) \subset L^{2k+1}(V)$. The generators of $L^9(V)$ follows from the coding vectors $113, 133, 117, 355 = s(113), 335 = s(133)$. Consider the vector 113. It corresponds to a product $((x_1 \cdot x_2 \cdot (x_3 \cdot x_4 \cdot x_5)) \cdot x_6 \cdot x_7) \cdot x_8 \cdot x_9$. The commutativity implies that this product is equal to $(((x_3 \cdot x_4 \cdot x_5) \cdot x_1 \cdot x_2) \cdot x_6 \cdot x_7) \cdot x_8 \cdot x_9$ which corresponds to the coding vector 111. But $111 = 0$. Thus $113 = 0$. Likewise we have $133 = 111 = 0$. By symmetry $355 = 335 = 0$. We have also $117 = 144 = 0$. Thus any commutative product on 9 elements is trivial.

**Corollary.** Let $A$ be a commutative ternary partially associative algebra. Consider the following sequence $\mathcal{C}^0(A) = A, \mathcal{C}^p(A) = \oplus_{i+j+k=p-1,\ i\geq j\geq k}\mu(\mathcal{C}^i(A), \mathcal{C}^j(A), \mathcal{C}^k(A))$. Then $\mathcal{C}^k(A) = 0$ for any $k \geq 4$.

### 1.4.3 The operad $\partial$ of 3-ary partially associative algebras

¿From Proposition 2, we can deduce the following result (see also [43] and [51])

**Theorem 13** *The operad $\partial$ of 3-ary partially associative algebras is not Koszul.*

*Proof.* Recall that the quadratic operad $\partial$ is a sequence $(\partial(2k+1))_{k\in\mathbb{N}}$ of vector spaces where any $\partial(2k+1)$ is also provided with a structure of $\Sigma_{2k+1}$-module, where $\Sigma_n$ is the symmetric group of degree $n$. Moreover, the $\Sigma_3$-module $\partial(3)$ is isomorphic with the group algebra $\mathbb{K}[\Sigma_3]$, and the $\Sigma_5$-module $\partial(5)$ is the quotient space of tree copies of $\mathbb{K}[\Sigma_5]$ by the operadic ideal defined by the relation of partial associativity. The previous calculus show that

$$\dim(\partial(3)) = \dim \mathbb{K}[\Sigma_3] = 6,\ \dim(\partial(5)) = 2 \times \dim \mathbb{K}[\Sigma_5] = 240$$

and more generally

$$\dim(\partial(2k+1)) = (k+1)\ \dim \mathbb{K}[\Sigma_{2k+1}]$$

for $k = 3, 4, 5, 6, 7$. By definition, the generating function of an operad $\mathcal{P}$ is

$$g_{\mathcal{P}}(x) := \sum_{a\geq 1} \frac{1}{a!}\chi(\mathcal{P}(a))x^a,$$

where

$$\chi(\mathcal{P}(a)) := \sum_i (-1)^i \dim(\mathcal{P}_i(a)).$$

(see [45]). If a quadratic operad $\mathcal{P}$ is Koszul, then its generating function and the generating function of its dual $\mathcal{P}^!$ are tied by the functional equation ([45])

$$g_{\mathcal{P}}(-g_{\mathcal{P}^!}(-x)) = x. \tag{1.5}$$

The generating function of $\partial$ is then written

$$g_{\partial}(x) = x + x^3 + 2x^5 + 4x^7 + 5x^9 + 6x^{11} + 7x^{13} + 8x^{15} + \cdots$$

Let $s$ be a formal power series satisfying

$$g_{\partial}(-s(-x)) = x.$$

We find

$$s(x) = x - x^3 + x^5 - 19x^{11} + O[x]^{12}.$$

But the dual operad of $\partial$ is the operad $t\mathcal{A}ss_1^3$, of totally associative algebras with generating ternary operation of degree 1 (see [43]) and its generating series is

$$g_{t\mathcal{A}ss_1^3}(x) = x - x^3 + x^5.$$

So $\partial$ cannot be a Koszul operad.

## 1.5 Cohomology and deformations of $(2k+1)$-ary partially associative algebras

### 1.5.1 Cohomology of deformations

Let $(V, \mu_0)$ be an $n$-ary partially associative algebra and $\mu(t) = \mu_0 + \sum_{i\geq 1} t^i \phi_i$ a formal deformation of $\mu_0$ (see [19] and [13] for the terminology). As $\mu(t)$ is a $n$-ary partially associative algebra product, the $n$-linear map $\phi_1$ satisfies the following linear identity:

$$\mu_0 \circ (\sum_{p=0}^{n} I_p \otimes \phi_1 \otimes I_{n-p-1}) + \phi_1 \circ (\sum_{p=0}^{n} I_p \otimes \mu_0 \otimes I_{n-p-1}) = 0, \tag{1.6}$$

where $I_k$ is the identity map on $(V)^{\otimes k}$ and $I_0 \otimes f = f \otimes I_0 = f$ for any $n$-linear map $f$. By [42] there exists a complex $(\mathcal{C}^*, \partial^*)$ such that (1.6) corresponds to $\partial^2(\phi_1) = 0$. This cohomology which governs deformations is called the cohomology of deformations. When the corresponding quadratic operad is Koszul, this cohomology coincides with the natural cohomology defined by the operad (operadic cohomology, for definition see [45]). But if $\mu_0$ is the product of a 3-ary partially associative algebra, Theorem 13 implies that the corresponding operad is not Koszul. It follows that the two cohomologies are different.

**Remarks**

1. If $n$ is even, the operad $p\mathcal{A}ss_0^n$ for $n$-ary partially associative algebras is Koszul and the cohomology of deformations coincides with the operadic cohomology. This cohomology is described in [17]. But in [17], the author doesn't distinguish the odd and even cases and the results concerning the odd case are not correct (see also [51]).
2. In [43], we develop a general approach to deformations of algebras over non Koszul operads. In particular we define the notion of dual operads for $n$-ary algebras. Earlier, this notion was given in [17] but it was wrong. Our definition is given in terms of multiplication with non trivial degree.

### 1.5.2 $n$-ary-product of degree $d$

Let $\mathcal{C}^k(V)$ be the linear space of $k$-linear maps of $V$. For any $f \in \mathcal{C}^n(V)$ and $g \in \mathcal{C}^m(V)$ the Gerstenhaber product $f \bullet_{n,m} g$ is

$$f \bullet_{n,m} g = \sum_{i=1}^{n} (-1)^{(i-1)(m-1)} f \circ_i g$$

with

$$(f \circ_i g)(X_1 \otimes \cdots \otimes X_{n+m-1}) = f(X_1 \otimes \cdots \otimes g(X_i \otimes \cdots X_{i+m-1}) \otimes \cdots \otimes X_{n+m-1}).$$

The product $\circ_i$ is the comp-i operations of Gerstenhaber.

**Definition 14** *An $n$-ary product $\mu$ is of degree $d$ if we have for $1 \leq j \leq n$,*

$$(\mu \circ_j \mu) \circ_i \mu = \begin{cases} (-1)^{d^2} (\mu \circ_i \mu) \circ_{j+n-1} \mu & \text{if } 1 \leq i \leq j-1, \\ \mu_j \circ (\mu \circ_{i-j+1} \mu) & \text{if } j \leq i \leq n+j-1, \\ (-1)^{d^2} (\mu \circ_{i-n+1} \mu) \circ_j \mu & \text{if } i \geq j+n. \end{cases}$$

**Examples.**

1. Degree 0 partially associative 2-ary algebras are classical associative algebras.
2. If $n = 2$, $d = 1$ and $\mu$ associative, then $\mu$ satisfies

$$\begin{cases} \mu \circ_1 \mu = \mu \circ_2 \mu, \\ (\mu \circ_1 \mu) \circ_1 \mu = \mu \circ_1 (\mu \circ_1 \mu), \\ (\mu \circ_1 \mu) \circ_2 \mu = \mu \circ_1 (\mu \circ_2 \mu), \\ (\mu \circ_1 \mu) \circ_3 \mu = -(\mu \circ_2 \mu) \circ_1 \mu, \\ (\mu \circ_2 \mu) \circ_2 \mu = \mu \circ_2 (\mu \circ_1 \mu), \\ (\mu \circ_2 \mu) \circ_3 \mu = \mu \circ_2 (\mu \circ_2 \mu). \end{cases}$$

3. If $n = 3, d = 1$ and if $\mu$ is totally associative, we have

$$\mu \circ_1 \mu = \mu \circ_2 \mu = \mu \circ_3 \mu,$$

and

$$\begin{cases} (\mu \circ_1 \mu) \circ_4 \mu = -(\mu \circ_2 \mu) \circ_1 \mu, \\ (\mu \circ_1 \mu) \circ_5 \mu = -(\mu \circ_3 \mu) \circ_1 \mu, \\ (\mu \circ_2 \mu) \circ_5 \mu = -(\mu \circ_3 \mu) \circ_2 \mu. \end{cases}$$

### 1.5.3 A cohomology for $2k+1$-ary partially associative algebras with operation in degree 1

In what follows, we denote by $\bullet$ any Gerstenhaber product $\bullet_{p,q}$.

**Lemma 3** *(Graded pre-Lie identity)*
*Let $\varphi_1 \in C^n(V)$, $\varphi_2 \in C^m(V)$ and $\varphi_3 \in C^p(V)$, and let $|\varphi_1|, |\varphi_2|, \varphi_3|$ be respective degrees of $\varphi_1, \varphi_2, \varphi_3$. Then*

$$(\varphi_1 \bullet \varphi_2) \bullet \varphi_3 - \varphi_1 \bullet (\varphi_2 \bullet \varphi_3) = (-1)^{(m-1)(p-1)}(-1)^{|\varphi_2||\varphi_3|}((\varphi_1 \bullet \varphi_3) \bullet \varphi_2 - \varphi_1 \bullet (\varphi_3 \bullet \varphi_2)).$$

¿From this lemma we directly deduce, considering $\varphi_2 = \varphi_3 = \mu$ and using that $\mu \bullet \mu = 0$ :

**Proposition 6** *Let $(V, \mu)$ be a $(2k+1)$-ary partially associative algebra with a multiplication $\mu$ of degree 1. Then, for any $\varphi \in \mathcal{C}^k(V)$ we have $(\varphi \bullet \mu) \bullet \mu = 0$.*

Let

$$\delta : C^n(V) \longrightarrow C^{n+2k}(V)$$

be the 1 degree operation defined by

$$\delta\varphi = \mu \bullet \varphi - (-1)^{|\varphi|}\varphi \bullet \mu \tag{1.7}$$

where $|\varphi|$ is the degree of $\varphi$. The graded pre-Lie identity gives

$$(\mu \bullet \mu) \bullet \varphi - \mu \bullet (\mu \bullet \varphi) = (-1)^{|\varphi|}((\mu \bullet \varphi) \bullet \mu - \mu \bullet (\varphi \bullet \mu)).$$

This implies

$$\delta(\delta\varphi) = 0$$

The complex

$$(C^{2k+1}(V), \delta)_{k \geq 1}$$

gives the operadic cohomology. In fact, it was proved in [43], that the quadratic operad associated with $n$-ary partially associative multiplication of degree 1 is Koszul. When $n$ is even and the multiplication of degree 0, this complex also corresponds to the operadic cohomology. But this is false if $n$ is odd and the multiplication of degree 0. In the following section we construct a complex whose coboundary operators satisfy identities similar to the identity (1.7).

### 1.5.4 A complex associated to a product of $(2k+1)$-ary partially associative algebra in degree 0

In this section, we assume that $\mu$ is of degree 0.

**Lemma 4** *Let $\mu$ be an $n$-ary partially associative product on a vector space $V$ and $\varphi \in \mathcal{C}^k(V)$.*
*1) If $n$ is even, then $(\varphi \bullet \mu) \bullet \mu = 0$.*
*2) If $n$ is odd, then*

$$(\varphi \bullet \mu) \bullet \mu = \sum_{1 \leq p \leq q-n \leq k-1} (\varphi \circ_p \mu) \circ_q \mu. \tag{1.8}$$

*where $\circ_i$ is the comp-$i$ operation.*

*Proof.* We have the pre-Lie identity ([15]):

$$(\varphi \bullet_{k,n} \mu) \bullet_{k+n-1,n} \mu - \varphi \bullet_{k,2n-1} (\mu \bullet_{n,n} \mu) = (-1)^{(n-1)^2}[(\varphi \bullet_{k,n} \mu) \bullet_{k+n-1,n} \mu - \varphi \bullet_{k,2n-1} (\mu \bullet_{n,n} \mu)].$$

As $\mu \bullet_{n,n} \mu = 0$, the previous equation reduces to

$$(\varphi \bullet_{k,n} \mu) \bullet_{k+n-1,n} \mu = (-1)^{(n-1)^2} (\varphi \bullet_{k,n} \mu) \bullet_{k+n-1,n} \mu.$$

If $n$ is even, it implies that $(\varphi \bullet_{k,n} \mu) \bullet_{k+n-1,n} \mu = 0$ and this case is similar to the above section. But if $n$ is odd, the previous identity is trivial. Computing directly $(\varphi \bullet_{k,n} \mu) \bullet_{k+n-1,n} \mu$, we obtain identity (1.8) of Lemma 4.

Assume that $n$ is odd. Lemma 4 shows that $\delta_i^{k+1} \circ \delta_i^k \neq 0$. To define a cohomology in this case, we restrict the spaces of cochains. Let $\chi^k(V)$ be the subspace of $\mathcal{C}^k(V)$ given by

$$\chi^k(V) = \left\{\varphi \in \mathcal{C}^k(V), (\varphi \bullet \mu) \bullet \mu = (\mu \bullet \varphi) \bullet \mu = \mu \bullet (\varphi \bullet \mu) = 0\right\}.$$

Pre-Lie identity applied to the triple $(\mu, \varphi, \mu)$ implies

$$(\mu \bullet \varphi) \bullet \mu = \mu \bullet (\varphi \bullet \mu) - \mu \bullet (\mu \bullet \varphi),$$

and if $\varphi \in \chi^k(V)$ we have also $\mu \bullet (\mu \bullet \varphi) = 0$.

**Proposition 7** *Let $\partial^k : \chi^k(V) \rightarrow \varphi^{k+n-1}(V)$ be the linear map defined by*

$$\partial^k \varphi = (-1)^{k-1} \mu \bullet \varphi - \varphi \bullet \mu.$$

*Then $Im(\partial^k) \subset \chi^{k+n-1}(V)$ and*

$$\partial^{k+n-1} \circ \partial^k = 0.$$

*For any $i$,*

$$(\chi^{k(n-1)+i}(V), \delta_i^k)_{k \geq 0}$$

*is a complex, where $\delta_i^j = \partial^{i+j(n-1)}$.*

*Proof.* Let $\varphi$ be in $\chi^k(V)$. Let us prove that $\partial^k \varphi \in \chi^{k+n-1}(V)$. We have

$$(\partial^k \varphi \bullet \mu) \bullet \mu = (-1)^{k-1}((\mu \bullet \varphi) \bullet \mu) \bullet \mu - ((\varphi \bullet \mu) \bullet \mu) \bullet \mu = 0,$$

$$(\mu \bullet \partial^k \varphi) \bullet \mu = (-1)^{k-1}(\mu \bullet (\mu \bullet \varphi)) \bullet \mu - (\mu \bullet (\varphi \bullet \mu)) \bullet \mu = 0,$$

and

$$\mu \bullet (\partial^k \varphi \bullet \mu) = (-1)^{k-1} \mu \bullet ((\mu \bullet \varphi) \bullet \mu) - \mu \bullet ((\varphi \bullet \mu) \bullet \mu) = 0.$$

Thus $\partial^k \varphi \in \chi^{k+n-1}(V)$. But

$$\begin{array}{ll}(\partial^{k+n-1} \circ \partial^k)\varphi & = \partial^{k+n-1}((-1)^{k-1}\mu \bullet \varphi - \varphi \bullet \mu) \\ & = \mu \bullet (\mu \bullet \varphi) + (-1)^{k-1}\mu \bullet (\varphi \bullet \mu) + (-1)^k (\mu \bullet \varphi) \bullet \mu + (\varphi \bullet \mu) \bullet \mu = 0,\end{array}$$

so

$$\partial^{k+n-1} \circ \partial^k = 0.$$

**Remark: Graded $n$-ary algebras and $n$-ary super-algebras.** We have just studied $n$-ary algebras with multiplications with a non trivial degree. It is also easy to define a notion of graded $n$-ary algebra. Let $\Gamma$ be an abelian group. If $V = \oplus_{\gamma \in \Gamma} V_\gamma$ is a $\Gamma$-graded vector space and if $\mu$ is a $n$-ary multiplication on $V$, the $n$-ary algebra $(V, \mu)$ is a graded $n$-ary algebra if

$$\mu(V_{\gamma_1}, \cdots, V_{\gamma_n}) \subset V_{\gamma_1 + \cdots + \gamma_n}$$

for any $\gamma_1, \cdots, \gamma_n \in \Gamma$. If $\Gamma = \mathbb{Z}_3$, such a graded algebra will be called super algebra.

## 1.6 Extension of the notion of coassociative algebras for $n$-ary algebras

If $n = 2$, then $n$-ary partially associative algebras are just associative algebras and we can define coassociative coalgebras with the well-known relations between these two structures. In fact, the dual space of a coassociative algebra can be provided with a structure of associative algebra, the dual space of a finite dimensional associative algebra can be provided with a structure of coassociative coalgebra structure and also. In addition, if $(A, \mu)$ is an associative algebra and $(M, \Delta)$ a coassociative coalgebra, the space $Hom(M, A)$ can be provided with an associative algebra structure. All these notions can be extended to $n$-ary algebras. An $n$-ary partially associative algebra has a product $\mu$ satisfying:

$$\sum_{p=0}^{n-1} (-1)^{p(n-1)} \mu \circ (I_p \otimes \mu \otimes I_{n-1-p}) = 0.$$

We can give the following definition of partially coassociative $n$-ary coalgebra.

**Definition 15** *An $n$-ary comultiplication on a $\mathbb{K}$-vector space $M$ is a map*

$$\Delta : M \to M^{\otimes n}.$$

*An $n$-ary partially coassociative coalgebra is a $\mathbb{K}$-vector space $M$ provided with an $n$-ary comultiplication $\Delta$ satisfying*

$$\sum_{p=0}^{n-1} (-1)^{p(n-1)} (I_p \otimes \Delta \otimes I_{n-1-p}) \circ \Delta = 0.$$

*An $n$-ary totally coassociative coalgebra is a $\mathbb{K}$-vector space $M$ provided with an $n$-ary comultiplication $\Delta$ satisfying*

$$(I_p \otimes \Delta \otimes I_{n-1-p}) \circ \Delta = (\Delta \otimes I_{n-1}) \circ \Delta,$$

*for any $p \in \{0, \cdots, n-1\}$.*

If $(\mathcal{A}, \mu)$ is an $n$-ary algebra and $(M, \Delta)$ an $n$-ary coalgebra we set

$$A(\mu) = \sum_{p=0}^{n-1} (-1)^{p(n-1)} \mu \circ (I_p \otimes \mu \otimes I_{n-1-p}),$$

$$\tilde{A}(\Delta) = \sum_{p=0}^{n-1} (-1)^{p(n-1)} (I_p \otimes \Delta \otimes I_{n-1-p}) \circ \Delta.$$

Then an $n$-ary algebra $(\mathcal{A},\mu)$ is partially associative if and only if $A(\mu)=0$, and an $n$-ary coalgebra $(M,\Delta)$ is partially coassociative if and only if $\tilde{A}(\Delta)=0$.
For any natural number $k$ and any $\mathbb{K}$-vector spaces $E$ and $F$, we denote by

$$\lambda_k : Hom(E,F)^{\otimes k} \longrightarrow Hom(E^{\otimes k},F^{\otimes k})$$

the natural embedding

$$\lambda_k(f_1\otimes ...\otimes f_k)(x_1\otimes ...\otimes x_k)=f_1(x_1)\otimes ...\otimes f_k(x_k).$$

**Proposition 8** *The dual space of an $n$-ary partially coassociative coalgebra is provided with a structure of $n$-ary partially associative algebra.*

*Proof.* Let $(M,\Delta)$ be an $n$-ary partially coassociative coalgebra. We consider the multiplication on the dual vector space $M^*$ of $M$ defined by

$$\mu=\Delta^*\circ\lambda_n.$$

It provides $M^*$ with an $n$-ary partially associative algebra structure. In fact we have

$$\mu(f_1\otimes f_2\otimes\cdots\otimes f_n)=\mu_{\mathbb{K}}\circ\lambda_n(f_1\otimes f_2\otimes\cdots\otimes f_n)\circ\Delta \tag{1.9}$$

for all $f_1,\cdots,f_n\in M^*$, where $\mu_{\mathbb{K}}$ is the multiplication in $\mathbb{K}$. Equation (1.9) becomes

$$\begin{aligned}\mu\circ(I_p\otimes\mu\otimes I_{n-1-p})&(f_1\otimes f_2\otimes\cdots\otimes f_{2n-1})\\ &=\mu_{\mathbb{K}}\circ(\lambda_n(f_1\otimes\cdots\otimes f_p\otimes\mu(f_{p+1}\otimes\cdots\otimes f_{p+n})\otimes f_{p+n+1}\otimes\cdots\otimes f_{2n-1}))\circ\Delta\\ &=\mu_{\mathbb{K}}\circ\lambda_n(f_1\otimes\cdots\otimes f_p\otimes(\mu_{\mathbb{K}}\circ\lambda_n(f_{p+1}\otimes\cdots\otimes f_{p+n})\circ\Delta)\otimes f_{p+n+1}\otimes\cdots\otimes f_{2n-1})\circ\Delta\\ &=\mu_{\mathbb{K}}\circ(I_p\otimes\mu_{\mathbb{K}}\otimes I_{n-1-p})\circ\lambda_{2n-1}(f_1\otimes\cdots\otimes f_{2n-1})\circ(I_p\otimes\Delta\otimes I_{n-1-p})\circ\Delta.\end{aligned}$$

Using associativity and commutativity of the multiplication in $\mathbb{K}$, we obtain

$$\forall p\in\{0,\cdots,n-1\},\quad \mu_{\mathbb{K}}\circ(I_p\otimes\mu_{\mathbb{K}}\otimes I_{n-1-p})=\mu_{\mathbb{K}}\circ(\mu_{\mathbb{K}}\otimes I_{n-1}),$$

so

$$\begin{aligned}&\sum_{p=0}^{n-1}(-1)^{p(n-1)}\mu\circ(I_p\otimes\mu\otimes I_{n-1-p})\\ &\quad=\mu_{\mathbb{K}}\circ(\mu_{\mathbb{K}}\otimes I_{n-1})\circ\lambda_{2n-1}(f_1\otimes\cdots\otimes f_{2n-1})\circ\sum_{p=0}^{n-1}(-1)^{p(n-1)}(I_p\otimes\Delta\otimes I_{n-1-p})\circ\Delta=0\end{aligned}$$

and $(M^*,\mu)$ is an $n$-ary partially associative algebra. □

**Proposition 9** *The dual vector space of a finite dimensional $n$-ary partially associative algebra has an $n$-ary partially coassociative coalgebra structure.*

*Proof.* Let $\mathcal{A}$ be a finite dimensional $n$-ary partially associative algebra and let $\{e_i\}_{i=1,\dots,n}$ be a basis of $\mathcal{A}$. If $\{f_i\}$ is the dual basis, then $\{f_{i_1}\otimes\cdots\otimes f_{i_n}\}$ is a basis of $(\mathcal{A}^*)^{\otimes n}$. We define a coproduct $\Delta$ on $\mathcal{A}^*$ by

$$\Delta(f)=\sum_{i_1,\cdots,i_n} f(\mu(e_{i_1}\otimes\cdots\otimes e_{i_n}))f_{i_1}\otimes\cdots\otimes f_{i_n}.$$

In particular

$$\Delta(f_k)=\sum_{i_1,\cdots,i_n} C^k_{i_1\cdots,i_n}f_{i_1}\otimes\cdots\otimes f_{i_n},$$

where $C^k_{i_1\cdots,i_n}$ are the structure constants of $\mu$ related to the basis $\{e_i\}$. Then $\Delta$ is a comultiplication of an $n$-ary partially coassociative coalgebra. □

Now we study the convolution product. Let us recall that if $(\mathcal{A},\mu)$ is an associative $\mathbb{K}$-algebra and $(M,\Delta)$ a coassociative $\mathbb{K}$-coalgebra then the convolution product

$$f \star g = \mu \circ \lambda_2(f \otimes g) \circ \Delta$$

provides $Hom(M,\mathcal{A})$ with an associative algebra structure. This result can be extended to the $n$-ary partially associative algebras and partially coassociative coalgebras.

**Proposition 10** *Let $(\mathcal{A},\mu)$ be an $n$-ary partially associative algebra and $(M,\Delta)$ an $n$-ary totally coalgebra. Then the algebra $(Hom(M,\mathcal{A}),\star)$ is an $n$-ary partially associative algebra, where $\star$ is the convolution product:*

$$f_1 \star f_2 \star \cdots \star f_n = \mu \circ \lambda_n(f_1 \otimes f_2 \otimes \cdots \otimes f_n) \circ \Delta.$$

*Proof.* Let us compute the convolution product of functions of $Hom(M,\mathcal{A})$. For any $i = 1,\cdots,n$ we have

$$\begin{aligned}
&f_1 \star \cdots \star f_{i-1} \star (f_i \star f_{i+1} \star \cdots \star f_{i+n-1}) \star f_{i+n} \star \cdots \star f_{2n-1} \\
&= \mu \ \circ \lambda_n(f_1 \otimes f_2 \otimes \cdots \otimes f_{i-1} \otimes (f_i \star \cdots \star f_{i+n-1}) \otimes f_{i+n} \otimes \cdots \otimes f_n) \ \circ \ \Delta \\
&= \mu \ \circ \ \lambda_n(f_1 \otimes \cdots f_{i-1} \otimes (\mu \circ \lambda_n(f_i \otimes \cdots \otimes f_{i+n-1}) \circ \Delta) \otimes f_{i+n} \otimes f_{2n-1}) \ \circ \ \Delta \\
&= \mu \ \circ \ (I_{i-1} \otimes \mu \otimes I_{n-i}) \circ \lambda_{2n-1}(f_1 \otimes f_2 \otimes \cdots \otimes f_{2n-1}) \circ (I_{i-1} \otimes \Delta \otimes I_{n-i}) \ \circ \ \Delta.
\end{aligned}$$

Since $\Delta$ is an $n$-ary totally associative product, we have

$$\begin{aligned}
&A(\star)(f_1 \otimes \cdots \otimes f_{2n-1}) \\
&= \sum_{p=0}^{n-1} (-1)^{p(n-1)} \mu \circ (I_p \otimes \mu \otimes I_{n-1-p}) \circ \lambda_{2n-1}(f_1 \otimes \cdots \otimes f_{2n-1}) \circ (I_p \otimes \Delta \otimes I_{n-1-p}) \circ \Delta \\
&= \left( \sum_{p=0}^{n-1} (-1)^{p(n-1)} \mu \circ (I_p \otimes \mu \otimes I_{n-1-p}) \right) \circ \lambda_{2n-1}(f_1 \otimes \cdots \otimes f_{2n-1}) \circ (\Delta \otimes I_{n-1}) \circ \Delta = 0,
\end{aligned}$$

which proves the result.

# Chapter 2

# The $n$-ary algebra of tensors and of hypercubic matrices

We define a ternary product and more generally a $(2k+1)$-ary product on the vector space $T_q^p(E)$ of tensors of type $(p,q)$ that is contravariant of order $p$ and covariant of order $q$ and total order $(p+q)$. This product is totally associative up to a permutation $s_k$ of order $k$ (we call this property a $s_k$-totally associativity). When $p=2$ and $q=1$, we obtain a $(2k+1)$-ary product on the space of bilinear maps on $E$ with values on $E$, which is identified to the cubic matrices. If we call a $l$-matrix a square tableau with $l \times \cdots \times l$ entrances (if $l=3$ we have the cubic matrices and we speak about hypercubic matrices as soon as $l>3$), then the $(2k+1)$-ary product on $T_q^p(E)$ gives a $(2k+1)$-product on the space of $(p+q)$-matrices. We describe also all these products which are $s_k$-totally associative. We compute the corresponding quadratic operads and their dual.

## 2.1 On $n$-ary associative algebras

### 2.1.1 Recall: $n$-ary partially and totally associative algebras

A $n$-ary algebra is a pair $(V,\mu)$ where $V$ is a vector space on a commutative field $\mathbb{K}$ of characteristic 0 and $\mu$ a linear map

$$\mu : V^{\otimes n} \to V$$

where $V^{\otimes n}$ denotes the $n$-tensor product $V \otimes \cdots \otimes V$ ($n$ times).

A $n$-ary algebra is *partially associative* if $\mu$ satisfies

$$\sum_{p=0}^{n-1} (-1)^{p(n-1)} \mu \circ (I_p \otimes \mu \otimes I_{n-p-1}) = 0, \tag{2.1}$$

for any $p = 0, \cdots, n-1$, where $I_j : V^{\otimes j} \to V^{\otimes j}$ is the identity map and $I_0 \otimes \mu = \mu \otimes I_0 = \mu$. For example, if $n=2$ we have the classical notion of binary associative algebra.

A $n$-ary algebra is *totally associative* if $\mu$ satisfies

$$\mu \circ (\mu \otimes I_{n-1}) = \mu \circ (I_p \otimes \mu \otimes I_{n-p-1}), \tag{2.2}$$

for all $p = 0, \cdots, n-1$. For $n=2$, the two notions of partially and totally associativity coincide with the classical notion of associativity. A totally associative $(2p)$-ary algebra is partially associative. A totally associative $(2p+1)$-ary algebra is partially associative if and only if $\mu$ is 2-step nilpotent (i.e. $\mu \circ_i \mu = 0$ for any $i = 1, \cdots, n$ with $\mu \circ_i \mu = \mu \circ (I_{i-1} \otimes \mu \otimes I_{n-i})$) .

### 2.1.2 Definition of $n$-ary $\sigma$-partially and $\sigma$-totally associative algebras

We can generalize Identities (2.1) and (2.2) using actions of the symmetric group on $n$ elements $\Sigma_n$. This generalization is in the spirite of the binary $\mathbb{K}[\Sigma_3]$-associative algebras introduced and developped in [23] and [53].

**Definition 16** *For a permutation $\sigma$ in $\Sigma_n$ define a linear map*

$$\Phi_\sigma^V : V^{\otimes^n} \rightarrow V^{\otimes^n}$$

*by*

$$\Phi_\sigma^V(e_{i_1} \otimes \cdots \otimes e_{i_n}) = e_{i_{\sigma^{-1}(1)}} \otimes \cdots \otimes e_{i_{\sigma^{-1}(n)}}.$$

*A $n$-ary algebra $(V, \mu)$ is* $\sigma$**-partially associative** *if*

$$\sum_{p=0}^{n-1} (-1)^{p(n-1)} (-1)^{p\varepsilon(\sigma)} \mu \circ (I_p \otimes (\mu \circ \Phi_{\sigma^p}^V) \otimes I_{n-p-1}) = 0, \tag{2.3}$$

*for all $p = 0, \cdots, n-1$,*
*and* $\sigma$**-totally associative** *if*

$$\mu \circ (\mu \otimes I_{n-1}) = \mu \circ (I_p \otimes (\mu \circ \Phi_{\sigma^p}^V) \otimes I_{n-p-1}), \tag{2.4}$$

*for all $p = 0, \cdots, n-1$.*

**Example** If $n = 3$ and $\sigma = \tau_{12}$ is the transposition exchanging 1 and 2 then a $\tau_{12}$-totally associative algebra satisfies

$$\mu(\mu(e_1, e_2, e_3), e_4, e_5) = \mu(e_1, \mu(e_3, e_2, e_4), e_5) = \mu(e_1, e_2(\mu(e_3, e_4, e_5)),$$

and a $\tau_{12}$-partially associative algebra satisfies

$$\mu(\mu(e_1, e_2, e_3), e_4, e_5) - \mu(e_1, \mu(e_3, e_2, e_4), e_5) + \mu(e_1, e_2(\mu(e_3, e_4, e_5)) = 0.$$

## 2.2 A $(2p + 1)$-ary product on the vector space of tensors $T_1^2(E)$

### 2.2.1 The tensor space $T_1^2(E)$

Let $E$ be a finite dimensional vector space over a field $\mathbb{K}$ of characteristic 0. We denote by $T_1^2(E) = E \otimes E \otimes E^*$ the space of tensors covariant of degree 1 and contravaviant of degree 2. The space $T_1^2(E)$ is identified to the space of linear maps

$$\mathcal{L}(E \otimes E, E) = \{\varphi : E \otimes E \rightarrow E \text{ linear}\}.$$

Let $\{e_1, \cdots, e_n\}$ be a fixed basis of $E$. The structure constants $\{C_{ij}^k\}$ of $\varphi \in T_1^2(E)$ are defined by

$$\varphi(e_i \otimes e_j) = \sum_{k=1}^{n} C_{ij}^k e_k.$$

**Definition 17** *The dual map of $\varphi \in T_1^2(E)$ is the tensor $\widetilde{\varphi} \in T_2^1(E) \simeq \mathcal{L}(E, E \otimes E)$ defined by*

$$\begin{array}{cccc} \widetilde{\varphi}: & E & \rightarrow & E \otimes E \\ & e_k & \mapsto & \displaystyle\sum_{1 \leq i,j \leq n} C_{ij}^k e_i \otimes e_j. \end{array}$$

If $\varphi$ is considered as a multiplication on $E$, then $\widetilde{\varphi}$ is a coproduct. For example, if $\varphi$ is an associative product then $\widetilde{\varphi}$ is the corresponding coassociative coproduct (often denoted by $\Delta$).

### 2.2.2 Definition of a 3-ary product on $T_1^2(E)$

Let $\varphi_1, \varphi_2, \varphi_3$ be in $T_1^2(E)$. We define a 3-ary product $\mu$ by

$$\mu(\varphi_1, \varphi_2, \varphi_3) = \varphi_1 \circ \widetilde{\varphi_2} \circ \varphi_3. \tag{2.5}$$

As $\widetilde{\varphi_2} : E \to E \otimes E$, then $\varphi_1 \circ \widetilde{\varphi_2} \circ \varphi_3 \in T_1^2(E)$ and $\mu$ is well defined. Let us compute its stucture constants. We denote by $C_{ij}^k(l)$ the structure constants of $\varphi_l$ $(l = 1, 2, 3)$.

$$\begin{aligned} \mu(\varphi_1, \varphi_2, \varphi_3)(e_i \otimes e_j) = \ & \varphi_1 \circ \widetilde{\varphi_2} \circ \varphi_3(e_i \otimes e_j) \\ &= \sum_{k=1}^{n} C_{ij}^k(3) \varphi_1 \circ \widetilde{\varphi_2}(e_k) \\ &= \sum_{k=1}^{n} \sum_{1 \leq l,m \leq n} C_{ij}^k(3) C_{lm}^k(2) \varphi_1(e_l \otimes e_m) \\ &= \sum_{t=1}^{n} \sum_{k=1}^{n} \sum_{1 \leq l,m \leq n} C_{ij}^k(3) C_{lm}^k(2) C_{lm}^t(1) e_t. \end{aligned}$$

Thus if $\mu(\varphi_1, \varphi_2, \varphi_3)(e_i \otimes e_j) = \sum_{t=1}^{n} A_{ij}^t(1,2,3) e_t$ we get

$$A_{ij}^t(1,2,3) = \sum_{1 \leq k,l,m \leq n} C_{ij}^k(3) C_{lm}^k(2) C_{lm}^t(1).$$

**Proposition 11** *The 3-ary product in $T_1^2(E)$ given by*

$$\mu(\varphi_1, \varphi_2, \varphi_3) = \varphi_1 \circ \widetilde{\varphi_2} \circ \varphi_3$$

*satisfies*

$$\begin{aligned} \mu(\mu(\varphi_1, \varphi_2, \varphi_3), \varphi_4, \varphi_5) &= \mu(\varphi_1, \mu(\varphi_4, \varphi_3, \varphi_2), \varphi_5) \\ &= \mu(\varphi_1, \varphi_2, \mu(\varphi_3, \varphi_4, \varphi_5)), \end{aligned}$$

*that is this product is $\tau_{13}$-totally associative.*

*Proof.* We have

$$\begin{aligned} \mu(\mu(\varphi_1, \varphi_2, \varphi_3), \varphi_4, \varphi_5)(e_i \otimes e_j) &= (\varphi_1 \circ \widetilde{\varphi_2} \circ \varphi_3) \circ \widetilde{\varphi_4} \circ \varphi_5(e_i \otimes e_j) \\ &= \sum_t \left[ \sum_{k,l,m} C_{ij}^k(5) C_{lm}^k(4) A_{lm}^t(1,2,3) \right] e_t \\ &= \sum_t \left[ \sum_{k,l,m} \sum_{u,r,s} \left( C_{ij}^k(5) C_{lm}^k(4) C_{lm}^u(3) C_{rs}^u(2) C_{rs}^t(1) \right) \right] e_t. \end{aligned}$$

Thus the structure constant $A_{ij}^t((1,2,3),4,5)$ of this tensor is

$$A_{ij}^t((1,2,3),4,5) = \sum_{1 \leq \substack{k,l,m \\ u,r,s} \leq n} C_{ij}^k(5) C_{lm}^k(4) C_{lm}^u(3) C_{rs}^u(2) C_{rs}^t(1).$$

Similary

$$\begin{aligned} \mu(\varphi_1, \varphi_2, \mu(\varphi_3, \varphi_4, \varphi_5))(e_i \otimes e_j) &= \varphi_1 \circ \widetilde{\varphi_2} \circ (\varphi_3 \circ \widetilde{\varphi_4} \circ \varphi_5)(e_i \otimes e_j) \\ &= \sum_t \left[ \sum_{u,r,s} A_{ij}^u(3,4,5) C_{rs}^u(2) C_{rs}^t(1) \right] e_t \\ &= \sum_t \left[ \sum_{u,r,s} \left( \sum_{k,l,m} C_{ij}^k(5) C_{lm}^k(4) C_{lm}^u(3) \right) C_{rs}^u(2) C_{rs}^t(1) \right] e_t. \end{aligned}$$

Thus

$$A_{ij}^t(1,2,(3,4,5)) = \sum_{\substack{k,l,m\\u,r,s}} C_{ij}^k(5)C_{lm}^k(4)C_{lm}^u(3)C_{rs}^u(2)C_{rs}^t(1),$$

and

$$A_{ij}^t(1,2,(3,4,5)) = A_{ij}^t((1,2,3),4,5).$$

We also have

$$\begin{aligned}\mu(\varphi_1,\mu(\varphi_2,\varphi_3,\varphi_4),\varphi_5)(e_i\otimes e_j) &= \varphi_1\circ(\widetilde{\varphi_2\circ\widetilde{\varphi_3}\circ\varphi_4})\circ\varphi_5(e_i\otimes e_j)\\ &= \sum_t\left[\sum_{k,l,m} C_{ij}^k(5)A_{lm}^k(2,3,4)C_{lm}^t(1)\right]e_t\\ &= \sum_t\left[\sum_{k,l,m}\sum_{u,r,s} C_{ij}^k(5)C_{lm}^u(4)C_{rs}^u(3)C_{rs}^k(2)C_{lm}^t(1)\right]e_t,\end{aligned}$$

and

$$A_{ij}^t(1,(2,3,4),5) = \sum_{\substack{k,l,m\\u,r,s}} C_{ij}^k(5)C_{lm}^u(4)C_{rs}^u(3)C_{rs}^k(2)C_{lm}^t(1).$$

This shows that

$$A_{ij}^t((1,2,3),4,5) = A_{ij}^t(1,(4,3,2),5).$$

**Remarks.**

1. We can define in this way other non equivalent products by:

$$\begin{cases}\mu_2(\varphi_1,\varphi_2,\varphi_3)=\varphi_3\circ\widetilde{\varphi_2}\circ\varphi_1,\\ \mu_3(\varphi_1,\varphi_2,\varphi_3)=\varphi_1\circ\widetilde{\varphi_2}\circ{}^t\varphi_3,\\ \mu_4(\varphi_1,\varphi_2,\varphi_3)=\varphi_3\circ\widetilde{\varphi_2}\circ{}^t\varphi_1,\end{cases}$$

where ${}^t\varphi(e_i\otimes e_j)=\varphi(e_j\otimes e_i)$.

2. If we identify a tensor $\varphi$ to its structure constants $\{C_{ij}^k\}$ and if we consider the family $\{C_{ij}^k\}$ as a cubic matrix $\{C_{ijk}\}$ with 3-entries, the product $\mu$ on $T_1^2(E)$ gives a 3-ary product on the cubic matrices. This last product has been studied in [1].

### 2.2.3 A $(2k+1)$-ary product on $T_1^2(E)$

Let $\varphi_1,\cdots,\varphi_{2k+1}$ be in $T_1^2(E)$. We define a $(2k+1)$-ary product $\mu_{2k+1}$ on $T_1^2(E)$ putting

$$\mu_{2k+1}(\varphi_1,\cdots,\varphi_{2k+1})=\varphi_1\circ\widetilde{\varphi_2}\circ\cdots\circ\varphi_{2k-1}\circ\widetilde{\varphi_{2k}}\circ\varphi_{2k+1}.$$

Let $s_k$ be the permutation of $\Sigma_{2k+1}$ defined by

$$s_k(1,2,\cdots,2k+1)=(2k+1,2k,\cdots,2,1),$$

that is $s_k=\tau_{1\,2k+1}\circ\tau_{2\,2k}\circ\cdots\circ\tau_{k-1\,k+1}=\Pi_{i=i}^k\tau_{i\,2k+1-i}$. It satisfies $(s_k)^{2p}=Id$ and $(s_k)^{2p+1}=s_k$ for any $p$ (it is a symmetry).

Recall that the $(2k+1)$-ary product $\mu_{2k+1}$ is a $s_k$-totally associative product if

$$\mu_{2k+1}\circ(\mu_{2k+1}\otimes I_{2k})=\mu_{2k+1}\circ(I_p\otimes(\mu_{2k+1}\circ\Phi_{s_k^p})\otimes I_{2k-p}),$$

for $p=1,\cdots,2k$. In particular, we have

$$\mu_{2k+1}\circ(\mu_{2k+1}\otimes I_{2k})=\mu_{2k+1}\circ(I_{2q}\otimes\mu_{2k+1}\otimes I_{2k-2q}),$$

for any $q=1,\cdots,k$.

**Proposition 12** *The product $\mu_{2k+1}$ is $s_k$-totally associative.*

*Proof.* In fact if we put

$$\mu_{2k+1}(\varphi_1,\cdots,\varphi_{2k+1})(e_i\otimes e_j)=\sum_t A_{ij}^t(1,2,\cdots,2k+1)e_t,$$

then $A_{ij}^t(1,2,\cdots,2k+1)=$

$$\sum_{\substack{a_1,\cdots,a_{k+1}\\ k_1,\cdots,k_k}} C_{ij}^{k_1}(2k+1)C_{a_1a_2}^{k_1}(2k)C_{a_1a_2}^{k_2}(2k-1)\cdots C_{a_{2k-1}a_{2k}}^{k_k}(2)C_{a_{2k-1}a_{2k}}^{t}(1).$$

More precisely the line of superscripts is

$$(k_1,k_1,k_2,k_2,\cdots,k_k,k_k,t),$$

and the line of subscripts

$$((i,j),(a_1,a_2),(a_1,a_2),(a_3,a_4),(a_3,a_4),\cdots,(a_{2k-1},a_{2k}),(a_{2k-1},a_{2k})).$$

Let us consider

$$\mu_{2k+1}\circ(I_l\otimes(\mu_{2k+1}\circ\Phi_{s_k^l})\otimes I_{2k-l})(\varphi_1,\cdots,\varphi_{4k+1})(e_i\otimes e_j)=\sum B_{ij}^t e_t.$$

Thus for $l=2r$, we get

$$\begin{array}{rl} B_{ij}^t= & \sum C_{ij}^{k_1}(4k+1)C_{a_1a_2}^{k_1}(4k)C_{a_1a_2}^{k_2}(4k-1)\cdots C_{a_{2k-2r-1}a_{2k-2r}}^{k_{k-r}}(2k+l+2)\\ & A_{a_{2k-2r-1}a_{2k-2r}}^{k_{k-r+1}}(l+1,\cdots,2k+l+1)C_{a_{2k-2r+1}a_{2k-2r+2}}^{k_{k-r+1}}(l)\cdots C_{a_{4k-1}a_{4k}}^{t}(1),\end{array}$$

such that the line of superscripts is

$$(k_1,k_1,k_2,k_2,\cdots,k_{k-r},h_1,h_1,\cdots,h_k,h_k,k_{k-r+1},k_{k-r+1},\cdots,k_k,k_k,t),$$

where the terms $h_1,\cdots,h_k,k_{k-r+1}$ correspond to the factor $A_{a_{2k-2r-1}a_{2k-2r}}^{k_{k-r+1}}(l+1,\cdots,2k+l+1)$. Such a line is the same as the line of superscripts of

$$\mu_{2k+1}\circ(\mu_{2k+1}\otimes I_{2k})(\varphi_1,\cdots,\varphi_{4k+1})(e_i\otimes e_j).$$

The line of subscripts is

$$((i,j),(a_1,a_2),(a_1,a_2),\cdots,(a_{2k-2r-1},a_{2k-2r}),(a_{2k-2r-1},a_{2k-2r}),(\beta_1\beta_2),\cdots,(\beta_{2k-1},\beta_{2k}),$$
$$(a_{2k-2r-1},a_{2k-2r}),\cdots,(a_{4k-1},a_{4k})).$$

So

$$\mu_{2k+1}\circ((\mu_{2k+1}\otimes I_{2k})=\mu_{2k+1}\circ(I_l\otimes(\mu_{2k+1}\circ\Phi_{s_k^l})\otimes I_{2k-l}),$$

for $l=2r$. Assume now that $l=2r+1$. In this case $B_{ij}^t$ is of the form

$$\sum\cdots C_{a_{2k-2r-1}a_{2k-2r}}^{k_{k-r+1}}(2k+l+2)A_{a_{2k-2r+1}a_{2k-2r+2}}^{k_{k-r+1}}(2k+l+1,\cdots,l+1)C_{a_{2k-2r+1}a_{2k-2r+2}}^{k_{k-r+1}}(l)\cdots.$$

We find the same list of exponents and of indices that for $\mu_{2k+1}\circ(\mu_{2k+1}\otimes I_{2k})$. This finishes the proof.

**Consequences.**

1. The product $\mu_{2k+1}$ on $T_1^2(E)$ induces directly a $(2k+1)$-ary products on cubic matrices.
2. All the other products which are $s_k$-totally associative corresponds to

$$\left\{\begin{array}{rcl} \mu_{2k+1}^2(\varphi_1,\cdots,\varphi_{2k+1}) & = & \varphi_{2k+1}\circ\widetilde{\varphi_{2k}}\circ\cdots\widetilde{\varphi_2}\circ\varphi_1,\\ \mu_{2k+1}^3(\varphi_1,\cdots,\varphi_{2k+1}) & = & \mu_{2k+1}({}^t\varphi_1,\varphi_2,\cdots,\varphi_{2k+1}),\\ \mu_{2k+1}^4(\varphi_1,\cdots,\varphi_{2k+1}) & = & \mu_{2k+1}^2(\varphi_1,\cdots,\varphi_{2k},{}^t\varphi_{2k+1}).\end{array}\right.$$

and more generally

$$\mu_{2k+1}({}^t\varphi_1,\varphi_2,{}^t\varphi_3,\cdots,\varphi_{2k+1})$$

or

$$\mu_{2k+1}({}^t\varphi_1,\varphi_2,{}^t\varphi_3,\cdots,\varphi_{2k+1}).$$

## 2.3 Generalisation: a $(2k+1)$-ary product on $T_q^p(E)$

### 2.3.1 The vector space $T_q^p(E)$

Let $E$ be a finite $m$-dimensional $\mathbb{K}$-vector space. The vector space $T_q^p(E)$ is the space of tensors which are contravariant of degree $p$ and covariant of degree $q$. In $\{e_1, \cdots, e_m\}$ is a fixed basis of $E$, a tensor $t$ of $T_q^p(E)$ is written

$$t = \sum_{\substack{1 \leq i_k, j_l \leq n \\ 1 \leq k \leq p \\ 1 \leq l \leq q}} t_{i_1, \cdots, i_p}^{j_1, \cdots, j_q} e_{i_1} \otimes \cdots \otimes e_{i_p} \otimes e^{j_1} \otimes \cdots \otimes e^{j_q}$$

where $(e^1, \cdots, e^m)$ is the dual basis of $(e_1, \cdots, e_m)$. As

$$T_q^p(E) = T_0^p(E) \otimes T_q^0(E),$$

then the tensor space

$$T(E) = \sum_{p,q=0}^{\infty} T_q^p(E)$$

is an associative algebra with product

$$\begin{array}{ccc} T_q^p(E) \times T_m^l(E) & \rightarrow & T_{q+m}^{p+l}(E) \\ (K, L) & \mapsto & K \otimes L \end{array}.$$

But this product is not internal on each component $T_q^p(E)$. In this section we will define internal $(2p-1)$-ary-product on the components.

The vector space $T_q^p(E)$ is isomorphic to the space $\mathcal{L}(E^{\otimes p}, E^{\otimes q})$ of linear maps

$$t : E^{\otimes^p} \rightarrow E^{\otimes^q}.$$

We define the structure constants by

$$t(e_{i_1} \otimes \cdots \otimes e_{i_p}) = \sum C_{i_1 \cdots i_p}^{j_1 \cdots j_q} e_{j_1} \otimes \cdots \otimes e_{j_q}.$$

For such a map we define $\widetilde{t}$ by

$$\begin{array}{cccc} \widetilde{t} : & E^{\otimes^q} & \rightarrow & E^{\otimes^p} \\ & (e_{j_1} \otimes \cdots \otimes e_{j_q}) & \mapsto & \sum C_{i_1 \cdots i_p}^{j_1 \cdots j_q} e_{i_1} \otimes \cdots \otimes e_{i_p}. \end{array}$$

### 2.3.2 A $(2k+1)$-ary product on $T_q^p(E)$

**Definition 18** *The map $\mu$ defined by:*

$$\mu(\varphi_1, \cdots, \varphi_{2k+1}) = \varphi_{2k+1} \circ \widetilde{\varphi_{2k}} \circ \varphi_{2k-1} \circ \cdots \circ \widetilde{\varphi_2} \circ \varphi_1, \tag{2.6}$$

*for any $\varphi_1, \cdots, \varphi_{2k+1} \in T_r^s(E)$ is a $(2k+1)$-ary product on $T_r^s(E)$.*

We take an odd number of map $\varphi_i$ so we get compostions of $\widetilde{\varphi_{j+1}} \circ \varphi_j : E^{\otimes^p} \rightarrow E^{\otimes^p}$ for $j = 1, \cdots, 2k-1$ and finally compose with $\varphi_{2k-1} : E^{\otimes^p} \rightarrow E^{\otimes^q}$ so $\mu$ is well defined.

**Proposition 13** *The $(2k+1)$-ary product $\mu$ on $T_q^p(E)$ defined by (2.6) is $s_k$-totally associative.*

*Proof.* The proof is similar to the proof of Proposition 12 concerning an $(2k+1)$-ary product on $T_1^2(E)$. In fact we have
$$\mu(\varphi_1, \cdots \varphi_{2p+1})(e_{i_1} \otimes \cdots \otimes e_{i_p}) = \sum A_{i_1 \cdots i_p}^{r_1 \cdots r_q} e_{r_1} \otimes \cdots \otimes e_{r_q},$$
and
$$A_{i_1 \cdots i_p}^{r_1 \cdots r_q} = C_{i_1 \cdots i_p}^{j_1 \cdots j_q}(2k+1) C_{l_1 \cdots l_p}^{j_1 \cdots j_q}(2k) C_{l_1 \cdots l_p}^{m_1 \cdots m_q}(2k-1) \cdots C_{s_1 \cdots s_p}^{r_1 \cdots r_q}(1),$$
that is the line of superscripts is
$$(j_1 \cdots j_q)(j_1 \cdots j_q)(m_1 \cdots m_q)(m_1 \cdots m_q) \cdots (n_1 \cdots n_q)(n_1 \cdots n_q)(r_1 \cdots r_q),$$
and the line of subscripts is
$$(i_1 \cdots i_p)(l_1 \cdots l_p)(l_1 \cdots l_p) \cdots (s_1 \cdots s_p)(s_1 \cdots s_p).$$
Using the same arguments that in Proposition 12, changing pairs by $p$-uples and $q$-uples, we obtain the announced result.

**Remark.** We can also use the same trick that in Consequences 2. to find others $s_k$-totally associative products on $T_q^p(E)$.

**Applications.** This product can be translated as a product of "hypercubic matrices" that is square tableau of length $p+q$. This generalizes in a natural way the classical associative product of matrices.

## 2.4 Current $(2k+1)$-ary $s_k$-totally associative algebras

The name refers to current Lie algebras which are Lie algebras of the form $L \otimes A$ where $L$ is a Lie algebra and $A$ is a associative commutative algebra, equipped with bracket
$$[x \otimes a, y \otimes b]_{L \otimes A} = [x, y]_L \otimes ab.$$
We want to generalize this notion to $(2k+1)$-ary $s_k$-totally associative algebras. The problem is to find a category of $(2k+1)$-ary algebras such that its tensor product with a $(2k+1)$-ary $s_k$-totally associative algebra gives a $(2k+1)$-ary $s_k$-totally associative algebra with obvious operation on the tensor product. Such a tensor product will be called *current* $(2k+1)$*-ary* $s_k$*-totally associative algebra*. We first focus on the ternary case and $s_1 = \tau_{13}$.

Let $(V, \mu)$ be a 3-ary algebra where $\mu$ is a $\tau_{13}$-totally associative product on $V$ (for example $V = T_1^2(E)$ and $\mu$ is defined by (2.5) ) so $\mu$ satisfies Equations (2.4) for $\sigma = \tau_{13}$, that is,
$$\mu(\mu(e_1, e_2, e_3), e_4, e_5) = \mu(e_1, \mu(e_4, e_3, e_2), e_5) = \mu(e_1, e_2(\mu(e_3, e_4, e_5)),$$
for any $e_1, e_2, e_3$ in $V$. Let $(W, \tilde{\mu})$ be a 3-ary algebra. Then the tensor algebra $(V \otimes W, \mu \otimes \tilde{\mu})$ is a 3-ary $\tau_{13}$-totally associative algebra if and only if
$$(\mu \otimes \tilde{\mu})(v_1 \otimes w_1 \otimes v_2 \otimes w_2 \otimes v_3 \otimes w_3) = \mu(v_1, v_2, v_3) \otimes \tilde{\mu}(w_1, w_2, w_3)$$
satisfies the $\tau_{13}$-totally associativity relation. But
$$\begin{cases} (\mu \otimes \tilde{\mu}) \circ (\mu \otimes \tilde{\mu} \otimes I_4) = \mu \circ (\mu \otimes I_2) \otimes \tilde{\mu} \circ (\tilde{\mu} \otimes I_2), \\ (\mu \otimes \tilde{\mu}) \circ (I_2 \otimes (\mu \otimes \tilde{\mu}) \circ \Phi_{\tau_{13}}^{V \otimes W} \otimes I_2) = \mu \circ (I \otimes \mu \circ \Phi_{\tau_{13}}^{V} \otimes I) \otimes \tilde{\mu} \circ (I \otimes \tilde{\mu} \circ \Phi_{\tau_{13}}^{W} \otimes I), \\ (\mu \otimes \tilde{\mu}) \circ (I_4 \otimes \mu \otimes \tilde{\mu}) = \mu \circ (I_2 \otimes \mu) \otimes \tilde{\mu} \circ (I_2 \otimes \tilde{\mu}), \end{cases}$$
then $(\mu \otimes \tilde{\mu}) \circ (\mu \otimes \tilde{\mu} \otimes I_4) - (\mu \otimes \tilde{\mu}) \circ (I_4 \otimes \mu \otimes \tilde{\mu}) = 0$ is equivalent to
$$\mu \circ (\mu \otimes I_2) \otimes \tilde{\mu} \circ (\tilde{\mu} \otimes I_2) - \mu \circ (I_2 \otimes \mu) \otimes \tilde{\mu} \circ (I_2 \otimes \tilde{\mu}) = 0. \tag{2.7}$$

But $\mu \circ (\mu \otimes I_2) = \mu \circ (I_2 \otimes \mu)$. Thus Equation (2.7) is equivalent to

$$\mu \circ (\mu \otimes I_2) \otimes [\tilde{\mu} \circ (\tilde{\mu} \otimes I_2) - \tilde{\mu} \circ (I_2 \otimes \tilde{\mu})] = 0,$$

and

$$\tilde{\mu} \circ (\tilde{\mu} \otimes I_2) = \tilde{\mu} \circ (I_2 \otimes \tilde{\mu}).$$

Similary

$$\begin{array}{l} (\mu \otimes \tilde{\mu}) \circ (\mu \otimes \tilde{\mu} \otimes I_4) - (\mu \otimes \tilde{\mu}) \circ (I_4 \otimes \mu \otimes \tilde{\mu}) \\ \mu \circ (\mu \otimes I_2) \otimes \left[\tilde{\mu} \circ (\tilde{\mu} \otimes I_2) - \tilde{\mu} \circ (I \otimes \tilde{\mu} \circ \Phi^W_{\tau_{13}} \otimes I)\right] = 0, \end{array}$$

which leads to

$$\tilde{\mu} \circ (\tilde{\mu} \otimes I_2) = \tilde{\mu} \circ (I \otimes \tilde{\mu} \circ \Phi^W_{\tau_{13}} \otimes I).$$

So $\mu \otimes \tilde{\mu}$ is $\tau_{13}$-totally associative if and only if $\tilde{\mu}$ is $\tau_{13}$-totally associative.

**Proposition 14** *Let $(V, \mu)$ be a 3-ary $\tau_{13}$-totally associative algebra and $(W, \tilde{\mu})$ be a 3-ary algebra. Then $(V \otimes W, \mu \otimes \tilde{\mu})$ is a 3-ary $\tau_{13}$-totally associative algebra if and only if $(W, \tilde{\mu})$ is also of this type.*

This result can be extended for $(2k+1)$-ary $s_k$-totally associative algebras.

**Proposition 15** *Let $(V, \mu)$ be a $(2k+1)$-ary $s_k$-totally associative algebra and $(W, \widetilde{\mu})$ be a $(2k+1)$-ary algebra. Then $(V \otimes W, \mu \otimes \tilde{\mu})$ is a $(2k+1)$-ary $s_k$-totally associative algebra if and only if $(W, \tilde{\mu})$ is also of this type.*

*Proof.* The product $\mu$ is a $(2k+1)$-ary $s_k$-totally associative product so satisfies

$$\begin{array}{rcl} \mu \circ (\mu \otimes I_{2k}) & = & \mu \circ (I_{2q} \otimes \mu \otimes I_{2k-2q}) \\ & = & \mu \circ (I_{2q+1} \otimes \mu \circ \Phi^V_{s_k^q} \otimes I_{2k-2q-1}), \end{array}$$

for any $q = 0, \cdots, k$. The system

$$\begin{array}{l} (\mu \otimes \widetilde{\mu}) \circ ((\mu \otimes \widetilde{\mu}) \otimes I_{4k}) - (\mu \otimes \widetilde{\mu}) \circ (I_{4q} \otimes (\mu \otimes \widetilde{\mu}) \circ \Phi^{V \otimes W}_{s_k^q} \otimes I_{4k-2q}) = \\ \mu \circ (\mu \otimes I_{2k}) \otimes \widetilde{\mu} \circ (\widetilde{\mu} \otimes I_{2k}) - \mu \circ (I_q \otimes \mu \circ \Phi^V_{s_k^q} \otimes I_{2k-q}) \otimes \tilde{\mu} \circ (I_q \otimes \widetilde{\mu} \circ \Phi^W_{s_k^q} \otimes I_{2k-q}) = 0, \end{array}$$

for any $q = 0, \cdots, k$ is equivalent to

$$\mu \circ (\mu \otimes I_{2k}) \otimes \left[\widetilde{\mu} \circ (\widetilde{\mu} \otimes I_{2k}) - \tilde{\mu} \circ (I_q \otimes \widetilde{\mu} \circ \Phi^W_{s_k^q} \otimes I_{2k-q})\right] = 0,$$

for any $q = 0, \cdots, k$. Then $\mu \otimes \tilde{\mu}$ is $(2k+1)$-ary $s_k$-totally associative if and only if

$$\widetilde{\mu} \circ (\widetilde{\mu} \otimes I_{2k}) - \tilde{\mu} \circ (I_q \otimes \widetilde{\mu} \circ \Phi^W_{s_k^q} \otimes I_{2k-q}) = 0$$

for any $q = 0, \cdots, k$ that is $\tilde{\mu}$ is a $(2k+1)$-ary $s_k$-totally associative product.

## 2.5 The operads $\partial$, $3\text{-}tot_{\tau_{13}}\mathcal{A}ss$

### 2.5.1 On the operad $\partial$

We denote by $\partial$ the quadratic operad of 3-ary -i.e. ternary- partially associative algebras (with operation in degree 0). In [29] we compute the free 3-ary partially associative algebra based on a finite dimensional vector space $V$. In [52] we notice that the dual operad is in general defined in the graded framework, compute it, as the knowledge of the dual is fundamental to study if the operad is Kozsul or not. We prove in [52] that $\partial$ is not Koszul. Note that this result contradicts some affirmations of the Koszulity of the operad $\partial$. This confusion can be explained by observing the general case of the operad $n$-$pa\mathcal{A}ss$ for $n$-ary partially associative algebras with operation of degree 0. If $n$ is even ([17]), $n$-$pa\mathcal{A}ss$ is Koszul and the dual operad $n$-$pa\mathcal{A}ss^!$ is the operad $n$-$tot\mathcal{A}ss$ for $n$-ary totally associative algebras with operation of degree 0 (which is also Koszul). But if $n = 2k+1$, the operad $n$-$pa\mathcal{A}ss^!$ is not $n$-$tot\mathcal{A}ss$ but $n$-$tot^1\mathcal{A}ss$ for totally associative algebras with *operation of degree* 1 and this operad is not Koszul (see[52]). As a consequence we deduce that for $n$ odd, the operadic cohomology (which always exits) is not the cohomology which governs deformations (which also always exits contrary to what is written in [3]). Remark that in [29] we have also defined a cohomology of Hochschild type for 3-ary partially associative algebras with some extra conditions.

### 2.5.2 The operad 3-$tot_{\tau_{13}}\mathcal{A}ss$

We denote by 3-$tot_{\tau_{13}}\mathcal{A}ss$ the quadratic operad for 3-ary $\tau_{13}$-totally associative algebras that is satisfying Relation (2.3) for $\sigma = \tau_{13}$. Let $\mu$ be a 3-ary multiplication, and

$$E_{3\text{-}tot_{\tau_{13}}\mathcal{A}ss}(m) = \begin{cases} <\mu>\simeq \mathbb{K}[\Sigma_3], & \text{if } m=3 \text{ and} \\ 0, & \text{if } m \neq 3. \end{cases}$$

We simply say that $E_{3\text{-}tot_{\tau_{13}}\mathcal{A}ss} = E_{3\text{-}tot_{\tau_{13}}\mathcal{A}ss}(3)$. The ideal of relation is generated by the $\mathbb{K}[\Sigma_5]$-closure $R_{3\text{-}tot_{\tau_{13}}\mathcal{A}ss}$ of the $\tau_{13}$-associativity

$$\begin{cases} r_1 = \mu(\mu\otimes I_2) - \mu(I\otimes\mu\cdot\tau_{13}\otimes I), \\ r_2 = \mu(\mu\otimes I_2) - \mu(I_2\otimes\mu), \end{cases}$$

where $\mu\cdot\sigma = \mu\circ\Phi_\sigma$ for $\sigma\in\Sigma_3$.

If $\Gamma(E_{3\text{-}tot_{\tau_{13}}\mathcal{A}ss})$ denotes the free operad generated by $E_{3\text{-}tot_{\tau_{13}}\mathcal{A}ss}$, we get that $R_{3\text{-}tot_{\tau_{13}}\mathcal{A}ss} \subset \Gamma(E_{3\text{-}tot_{\tau_{13}}\mathcal{A}ss})(5)$. The operad for 3-ary $\tau_{13}$-totally associative algebras is then the quadratic 3-ary operad

$$3\text{-}tot_{\tau_{13}}\mathcal{A}ss = \Gamma(E_{3\text{-}tot_{\tau_{13}}\mathcal{A}ss})/(R_{3\text{-}tot_{\tau_{13}}\mathcal{A}ss}),$$

that is 3-$tot_{\tau_{13}}\mathcal{A}ss(m) = \Gamma(E_{3\text{-}tot_{\tau_{13}}\mathcal{A}ss})(m)/(R_{3\text{-}tot_{\tau_{13}}\mathcal{A}ss})(m)$.

### 2.5.3 The current operad $\widetilde{3\text{-}tot_{\tau_{13}}\mathcal{A}ss}$

In [54] we have defined, for a quadratic operad $\mathcal{P}$, the current operad $\widetilde{\mathcal{P}}$ that is, the maximal operad $\widetilde{\mathcal{P}}$ such that the tensor product of a $\mathcal{P}$-algebra $A$ and a $\widetilde{\mathcal{P}}$-algebra $B$ is a $\mathcal{P}$-algebra with the usual product on $A\otimes B$. Let us compute $\widetilde{3\text{-}tot_{\tau_{13}}\mathcal{A}ss}$.

**Proposition 16** *The current operad of the operad* 3-$tot_{\tau_{13}}\mathcal{A}ss$ *is* 3-$tot_{\tau_{13}}\mathcal{A}ss$ *itself that is*

$$\widetilde{3\text{-}tot_{\tau_{13}}\mathcal{A}ss} = 3\text{-}tot_{\tau_{13}}\mathcal{A}ss.$$

This result follows from the Proposition 14.

### 2.5.4 The dual operad 3-$tot_{\tau_{13}}\mathcal{A}ss^!$

For $n$-ary quadratic operad $\mathcal{P} = \Gamma(E)/(R)$ with $E = E(n)$, the dual (quadratic $n$-ary) operad is defined as follows

$$\mathcal{P}^! = \Gamma(\overline{E})/(R^\perp),$$

where $\overline{E} =\uparrow^{n-2} E^\# \otimes sgn_n$, $\uparrow^{n-2}$ denotes the suspension iterated $(n-2)$ times, $\#$ the linear dual and $R^\perp \subset \Gamma(\overline{E})(2n-1)$ is the annihilator of $R\subset\Gamma(E)(2n-1)$ with respect to the pairing between $\Gamma(\overline{E})(2n-1)$ and $\Gamma(E)(2n-1)$.

**Proposition 17** *The dual operad of* 3-$tot_{\tau_{13}}\mathcal{A}ss$ *is*

$$3\text{-}tot_{\tau_{13}}\mathcal{A}ss^! = 3\text{-}pa^1_{\tau_{13}}\mathcal{A}ss,$$

*that is the operad for $\tau_{13}$-partially associative algebras with operation in degree* 1.

*Proof.* The operad $\mathcal{P} = 3\text{-}tot_{\tau_{13}}\mathcal{A}ss$ is the quadratic operad defined by

$$\mathcal{P} = \Gamma(E)/(R),$$

where $\mu$ a ternary operation of degree 0, $\Gamma(E)$ the free operad generated by $E =< \mu >$ and $R \subset \Gamma(E)$ is the generated as $\mathbb{K}[\Sigma_5]$-module by the relations

$$\begin{cases} \mu(\mu \otimes I_2) - \mu(I \otimes \mu \cdot \tau_{13} \otimes I), \\ \mu(\mu \otimes I_2) - \mu(I_2 \otimes \mu). \end{cases}$$

We consider

$$\mu \circ_s \mu = \mu(I_{s-1} \otimes \mu \otimes I_{3-s}),$$

which "plugs" $\mu$ into the $s$-st input of $\mu$ and

$$(f \cdot \sigma)(i_1, i_2, \cdots, i_m) = f(i_{\sigma^{-1}(1)}, i_{\sigma^{-1}(2)}, \cdots, i_{\sigma^{-1}(m)}),$$

if $f \in \Gamma(\mu)(m), \sigma \in \Sigma_m$.

We get $\overline{E}(3) =< \alpha >$ where $\alpha$ is a ternary operation of degree 1 satisfying $< \mu, \alpha >= 1$. The pairing between $\Gamma(E)(5)$ and $\Gamma(\overline{E})(5)$ is given by

$$\begin{array}{l} < (\mu \circ_j \mu)(i_1, i_2, i_3, i_4, i_5), (\alpha \circ_j \alpha > (i_1, i_2, i_3, i_4, i_5)) \\ \qquad =< \mu, \alpha > sgn_5 \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ i_1 & i_2 & i_3 & i_4 & i_5 \end{pmatrix} = sgn_5 \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ i_1 & i_2 & i_3 & i_4 & i_5 \end{pmatrix}, \end{array}$$

for $j = 1, 2, 3$. So

$$\begin{array}{l} < (\mu \circ_1 \mu - \mu \circ_2 \mu \cdot \tau_{13})(1,2,3,4,5), (\alpha \circ_1 \alpha - \alpha \circ_2 \alpha \cdot \tau_{13} + \alpha \circ_3 \alpha)(1,2,3,4,5) > \\ \quad =< \mu \circ_1 \mu, \alpha \circ_1 \alpha > + < \mu \circ_2 \mu \cdot \tau_{13}, \alpha \circ_2 \alpha \cdot \tau_{13} > \\ \quad = 1- < \mu \circ_2 \mu, \alpha \circ_2 \alpha >= 1 - 1 = 0, \end{array}$$

$$\begin{array}{l} < (\mu \circ_1 \mu - \mu \circ_3 \mu)(1,2,3,4,5), (\alpha \circ_1 \alpha - \alpha \circ_2 \alpha \cdot \tau_{13} + \alpha \circ_3 \alpha)(1,2,3,4,5) > \\ \quad =< \mu \circ_1 \mu, \alpha \circ_1 \alpha > - < \mu \circ_3 \mu, \alpha \circ_3 \alpha >= 1 - 1 = 0. \end{array}$$

The dual operad is then the quadratic operad

$$\mathcal{P}^! = \Gamma(\alpha)/(R^\perp),$$

with $\alpha$ ternary operation of degree 1 and $R^\perp$ generated by

$$\alpha(\alpha \otimes I_2) - \alpha(I \otimes \alpha \cdot \tau_{13} \otimes I) + \alpha(I_2 \otimes \alpha).$$

So this operad is the operad of ternary $\tau_{13}$-partially associative algebras with operations of degree 1.

**Remark.** A direct computation similar to [52] shows that

$$dim\mathcal{P}(3) = 6,\ dim\mathcal{P}(5) = 5!\,,\ dim\mathcal{P}(7) = 7! \quad .$$

The generating function of $\mathcal{P}$ is similar to the generating function of 3-$tot\mathcal{A}ss$. Likewise the generating function of 3-$pa^1_{\tau_{13}}\mathcal{A}ss$ is the generating function of 3-$pa^1\mathcal{A}ss$. ¿From [33] the operads 3-$tot\mathcal{A}ss$ and 3-$pa^1\mathcal{A}ss$ are Koszul. We conclude that 3-$tot_{\tau_{13}}\mathcal{A}ss$ is Koszul.

# Chapter 3

# n-ary Lie algebras

The notion of $n$-ary algebras, that is vector spaces with a multiplication concerning $n$-arguments, $n \geq 3$, became fundamental since the works of Nambu. Here we first present general notions concerning $n$-ary algebras and associative $n$-ary algebras. Then we will be interested in the notion of $n$-Lie algebras, initiated by Filippov, and which is attached to the Nambu algebras. We study the particular case of nilpotent or filiform $n$-Lie algebras to obtain a beginning of classification. This notion of $n$-Lie algebra admits a natural generalization in Strong Homotopy $n$-Lie algebras in which the Maurer Cartan calculus is well adapted.

## 3.1 $n$-ary algebras

### 3.1.1 Basic definitions

Let $\mathbb{K}$ be a commutative field of characteristic zero and $V$ a $\mathbb{K}$-vector space. Let $n$ be in $\mathbb{N}$, $n \geq 2$. A $n$-ary algebra structure on $V$ is given by a linear map

$$\mu : V^{\otimes n} \to V.$$

We denote by $(V, \mu)$ such an algebra. Classical algebras (associative algebras, Lie algebras, Leibniz algebras for example) are binary that is given by a 2-ary product. In this paper, we are interested in the study of $n$-ary algebras for $n \geq 3$. A subalgebra of the $n$-ary algebra $(V, \mu)$ is a vector subspace $W$ of $V$ such that the restriction of $\mu$ to $W^{\otimes n}$ satisfies $\mu(W^{\otimes n}) \subset W$. In this case $(W, \mu)$ is also a $n$-ary algebra.

**Definition 19** *Let $(V, \mu)$ be a $n$-ary algebra. An ideal of $(V, \mu)$ is a subalgebra $(I, \mu)$ satisfying*

$$\mu(V^{\otimes p} \otimes I \otimes V^{\otimes n-p-1}) \subset I,$$

*for all $p = 0, \cdots, n-1$ and where $V^{\otimes 0} \otimes I = I \otimes V^{\otimes 0} = I$.*

**Definition 20** *Let $(V_1, \mu_1)$ and $(V_2, \mu_2)$ be $n$-ary algebras. A morphism of $n$-ary algebras is a linear map $\varphi : V_1 \to V_2$ satisfying*

$$\mu_2 \circ \varphi^{\otimes n} = \varphi \circ \mu_1.$$

In this case, the linear kernel $Ker\varphi$ of the morphism $\varphi$ is an ideal of $(V_1, \mu_1)$. In fact, if $v \in Ker\varphi$, then

$$\varphi(\mu_1(v_1 \otimes \cdots \otimes v \otimes \cdots \otimes v_{n-1})) = \mu_2(\varphi(v_1) \otimes \cdots \otimes \varphi(v) \otimes \cdots \otimes \varphi(v_{n-1})) = 0.$$

To simplify notations, we identify the linear map $\mu$ on $V^{\otimes n}$ with the corresponding $n$-linear map on $V^n$. Then we write $\mu(v_1 \otimes \cdots \otimes v_n)$ as well as $\mu(v_1 \cdot v_2 \cdots v_n)$.

### 3.1.2 Anticommutative $n$-ary algebras

Let $(V,\mu)$ be a $n$-ary algebra. It is called anticommutative if $\mu(v_1\otimes\cdots\otimes v_n)=0$ whenever $v_i=v_j$ for $i\neq j$. Since $\mathbb{K}$ is of characteristic 0, this is equivalent to

$$\mu(v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(n)})=(-1)^{\varepsilon(\sigma)}\mu(v_1\otimes\cdots\otimes v_n),$$

for any $\sigma$ in the symmetric group $\Sigma_n$ where $(-1)^{\varepsilon(\sigma)}$ is the signum of the permutation $\sigma$. If $\mu$ is an antisymmetric $n$-ary multiplication, we write

$$[v_1,\cdots,v_n]$$

instead of $\mu(v_1\otimes\cdots\otimes v_n)$.

### 3.1.3 Symmetric and commutative $n$-ary algebras

A $n$-ary algebra $(V,\mu)$ is called symmetric if it satisfies

$$\mu(v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(n)})=\mu(v_1\otimes\cdots\otimes v_n),$$

for all $v_1,\cdots,v_n\in V$ and for all $\sigma\in\Sigma_n$. It is called commutative if

$$\sum_{\sigma\in\Sigma_n}(-1)^{\varepsilon(\sigma)}\mu(v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(n)})=0,$$

for all $v_1,\cdots,v_n\in V$. Of course, any symmetric $n$-ary algebra is commutative.

### 3.1.4 Derivations

Let $(V,\mu)$ a $n$-algebra.

**Definition 21** *A derivation of the $n$-algebra $(V,\mu)$ is a linear map*

$$D:V\to V$$

*satisfying*

$$D(\mu(v_1,\cdots,v_n))=\sum_{i=1}^{n}\mu(v_1,\cdots,D(v_i),\cdots,v_n),$$

*for any $v_1,\cdots,v_n\in V$.*

All derivations of $(V,\mu)$ generate a subalgebra of Lie algebra $gl(V)$. It is called the algebra of derivations of $V$ and denoted by $Der(V)$.

**Remark.** For any $v_1,\cdots,v_{n-1}$ in $V$, let $ad(v_1,\cdots,v_{n-1})$ be the linear map given by

$$ad(v_1,\cdots,v_{n-1})(v)=\mu(v_1,\cdots,v_{n-1},v).$$

Then this linear map is a (inner) derivation if and only if the product $\mu$ satisfies

$$\mu(v_1,\cdots,v_{n-1},\mu(u_1,\cdots,u_n))=\sum_{i=1}^{n}\mu(u_1,\cdots,\mu(v_1,\cdots,v_{n-1},u_i),\cdots,u_n). \tag{3.1}$$

We will study such a product for $n$-Lie algebras. If $n=2$, this shows that the maps $ad(X)$ are derivations of $(V,\mu)$ if and only if the binary product satisfies

$$\mu(v_1,\mu(u_1,u_2))=\mu(\mu(v_1,u_1),u_2)+\mu(u_1,\mu(v_1,v_2))$$

and $(V,\mu)$ is a Leibniz algebra ([11]). Thus, for any $n$, a $n$-algebra $(V,\mu)$ satisfying Equation (3.1) is called $n$-Leibniz algebra.

### 3.1.5 Simple, nilpotent $n$-ary algebras

**Definition 22** *A $n$-ary algebra $(V,\mu)$ is called simple if*

- *$\mu$ is not abelian (i.e $\mu(V\cdots V)\neq 0$).*
- *Any ideal is isomorphic to $V$ or is equal to 0.*

We define the derived series by

$$\begin{cases} V^{(1)}=V,\\ V^{(k)}=\mu(V^{(k-1)},V^{(k-1)},V,\cdots,V),\end{cases}$$

and the lower central series by

$$\begin{cases} V^{1}=V,\\ V^{k}=\mu(V^{k-1},V,V,\cdots,V).\end{cases}$$

**Definition 23** *The $n$-ary algebra $(V,\mu)$ is called*

- Solvable *if there is an integer $k$ such that $V^{(k)}=0$.*
- Nilpotent *if there is an integer $k$ such that $V^{k}=0$.*

The definitions presented here are the definitions given in [36].

## 3.2 $n$-Lie algebras

Many notions of $n$-Lie algebras have been presented to generalize Lie algebras for $n$-ary algebras. The first one is probably due to Filippov ([14]). These algebras have been studied from an algebraic point of view (classification, simplicity, nilpotency, representations) and because of their relations with the Nambu mechanic. The second one is the notion introduced with the strong homotopy algebra point of view. In this paper we are concerned by the two approaches. To distinguish these different definitions we will call $n$-Lie algebras the first one and Lie $n$-algebras or sh-$n$-Lie algebras in the second one. In this section, we study Filippov algebras.

**Definition 24** *An anticommutative $n$-ary algebra is a $n$-ary Lie algebra or simpler $n$-Lie algebra if the following Jacobi identity holds:*

$$[[u_1,\cdots,u_n],v_1,\cdots,v_{n-1}]=\sum_{i=1}^{n}[u_1,\cdots,u_{i-1},[u_i,v_1,\cdots,v_{n-1}],u_{i+1},\cdots,u_n],$$

*for any $u_1,\cdots,u_n,v_1,\cdots,v_{n-1}\in V$.*

This last condition is called Jacobi identity for $n$-Lie algebras.

### 3.2.1 Fundamental examples

1. This example was given by Fillipov. Let $A$ be a $n$-dimensional vector space on $\mathbb{K}$. Let $\{v_1,\cdots,v_{n+1}\}$ be a basis of $V$. The following product

$$[v_1,v_2,\cdots,\hat{v_i},\cdots,v_{n+1}]=(-1)^{n+1+i}v_i,$$

for $i=1,\cdots,n+1$ provides $A$ with a $n$-Lie algebra structure. We denote this algebra $A_{n+1}$.

**Theorem 25** *If $\mathbb{K}=\mathbb{C}$, every simple $n$-Lie algebra is of dimension $n+1$ and it is isomorphic to $A_{n+1}$.*

2. Let $A = \mathbb{K}[X_1, \cdots, X_n]$ be the associative algebras of $n$ indeterminates polynomials. We consider the product

$$[P_1, \cdots, P_n] = Jac(P_1, \cdots, P_n),$$

where $Jac$ denotes the Jacobian, that is the determinant of the Jacobian matrix of partial derivatives of $P_1, \cdots, P_n$. Provided with this product, $A$ is an infinite dimensional $n$-Lie algebra.

3. The Nambu brackets. It generalizes directly the previous example. Let $A = \mathcal{C}^\infty(\mathbb{R}^3)$ be the algebra of differential functions on $\mathbb{R}^3$. This algebra is considered as classical observables on the three dimensional space $\mathbb{R}^3$ with coordinates $x, y, z$. We consider on $A$ the 3-product

$$\{f_1, f_2, f_3\} = Jac(f_1, f_2, f_3).$$

This product is a 3-Lie algebra product which generalizes the usual Poisson bracket from binary to ternary operations.

4. ([59]). Let $A = \mathbb{K}[X_1, \cdots, X_n]$ be the associative algebra of $n$ indeterminates polynomials. Let $I_r$ be the linear subspace of $A$ linearly generated by the monomials of $A$ of degree greater than or equal to $r$. Clearly $I_r$ is a subspace of $I_3$ as soon as $r \geq 3$. We define $J_r = I_3/I_r$ for $r > 3$. For any $Q_1, \cdots, Q_n \in J_r$ we put

$$[Q_1, \cdots, Q_n] = Jac(Q_1, \cdots, Q_n).$$

This product is a $n$-Lie algebra bracket and $Q$ is a finite dimensional nilpotent $n$-Lie algebra.

5. Every $n$-Lie algebra of dimension $n$ is abelian.

### 3.2.2 Nilpotent $n$-Lie algebras

In the first section we have defined nilpotency for general $n$-ary algebras. Since any $n$-Lie algebra is a $n$-Leibniz algebra, any adjoint operator $ad(v_1, \cdots, v_{n-1})$ is a derivation.

**Theorem 26** *([36]) For any finite dimensional nilpotent Lie algebras, the adjoint operators are nilpotent. Conversely, if the adjoint operators of the $n$-Lie algebra $V$ are nilpotent, then $V$ is nilpotent.*

Assume that $V$ is a finite dimensional complex nilpotent $n$-Lie algebra. We will generalize the notion of characteristic sequence of Lie algebras to $n$-Lie algebras. We consider the set of generators of $V$ which is isomorphic to $V/V^2$.

**Lemma 5**

$$\dim V/V^2 \geq n.$$

Let us consider a free family $\{v_1, \cdots, v_{n-1}\}$ of $n-1$ vectors of $V - V^2$. The operator $ad(v_1, v_2, \cdots, v_{n-1})$ is a linear nilpotent operator of $V$ admitting $v_1, \cdots, v_{n-1}$ as eigenvectors associated to the eigenvalue 0. We consider now the ordered sequence of the similitude invariants (the dimensions of Jordan blocks ) of this operator. It is of type $(c_1, \cdots, c_k, 1, \cdots, 1)$ with at least $n-1$ invariant equal to 1, corresponding to the dimension of the eigenspace generated by the eigenvectors $v_i$. We assume that $c_1 \geq \cdots \geq c_k \geq 0$. We denote this sequence $c(v_1, \cdots, v_{n-1})$.

**Definition 27** *The characteristic sequence of the nilpotent $n$-Lie algebra is the sequence*

$$c(V) = max\{c(v_1, \cdots, v_{n-1})\},$$

*where $(v_1, \cdots, v_{n-1})$ are $n-1$ independent vectors of $V - V^2$, the order relation being the lexicographic order.*

Assume that $\dim V = p$. The possible extremal values of $c(V)$ are

- $(1, \cdots, 1)$ and $V$ is an abelian $n$-Lie algebra,

- $(p-n+1, 1, \cdots, 1)$. This sequence corresponds to a nilpotent operator $ad(v_1, v_2, \cdots, v_{n-1})$ with a biggest nilindex.

**Definition 28** *A p-dimensional complex (or real) nilpotent n-Lie algebra is called filiform is its characteristic sequence is equal to* $(p-n+1, \underbrace{1, \cdots, 1}_{n-1})$.

**Examples**

- We consider $n = 3$ and $p = 4$. The characteristic sequence is equal to $(2, 1, 1)$. Let $\{v_1, v_2, v_3, v_4\}$ be a basis of $V$ such that the characteristic sequence of $ad(v_1, v_2)$ is $(2, 1, 1)$. If $\{v_3 v_4\}$ is the Jordan basis of this operator then we have
$$[v_1, v_2, v_3] = v_4.$$
¿From the classification of [8], we deduce that we have obtained the only filiform 3-Lie algebra of dimension 4.

- We generalize easily this example. Let $V$ be the $p$-dimensional 3-Lie algebra given by
$$\begin{cases} [X_1, X_2, X_3] = X_4, \\ [X_1, X_2, X_4] = X_5, \\ \cdots \\ [X_1, X_2, X_{p-1}] = X_p. \end{cases}$$
It is also a filiform 3-Lie algebra. It is a model ([25]) of the filiform 3-Lie algebras of dimension $p$, that is any $p$-dimensional filiform 3-Lie algebras can be contracted on this algebra.

- Every filiform 5-dimensional 3-Lie algebra is isomorphic to
$$\begin{cases} [X_1, X_2, X_3] = X_4, \\ [X_1, X_2, X_4] = X_5, \\ [X_1, X_3, X_4] = aX_5, \\ [X_2, X_3, X_4] = bX_5. \end{cases}$$

### 3.2.3 Graded filiform $n$-Lie algebras

Let $f$ be a derivation of a complex filiform $n$-Lie algebra $V$. We assume that $f$ is diagonalizable. The decomposition of eigenspaces of $V$ gives a graduation of this $n$-Lie algebra. We consider the maximal abelian subalgebra of $Der(V)$ given by the diagonalizable derivations of $V$ which commute with $f$. We denote this algebra $T(f)$.

**Definition 29** *The rank of* $V$ *is the biggest dimension amongst the dimensions of* $T(f)$ *for any diagonalizable derivation* $f$.

**Proposition 18** *The rank of any filiform n-Lie algebra is equal to or smaller than n.*

*Proof.* We consider the model given by
$$[X_1, X_2, \cdots, X_{n-1}, X_i] = X_{i+1},$$
for $i = n+1, \cdots, p-1$, with $p = \dim V$. We can assume that $X_1, X_2, \cdots, X_n$ are eigenvectors. If we put
$$f(X_t) = \lambda_t X_t,$$
for $t = 1, \cdots, n$, then other eigenvalues are
$$\lambda_i = \lambda_1 + \cdots + \lambda_{n-1} + \lambda_{i-1}$$

and this implies

$$\lambda_i = (n-i)(\lambda_1 + \cdots + \lambda_{n-1}) + \lambda_n.$$

Thus $\lambda_1, \cdots, \lambda_n$ are the independent roots of this algebra which is then of rank $n$. Let $V_1$ be any filiform $n$-Lie algebra of dimension $p$. There exists $(X_1, \cdots, X_{n-1})$ independent vectors in $V_1 - V_1^2$ such that the characteristic sequence of $V_1$ is given by the nilpotent operator $ad(X_1, \cdots, X_{n-1})$. We consider the corresponding Jordan basis of $V_1$. It satisfies

$$[X_1, X_2, \cdots, X_{n-1}, X_i] = X_{i+1}$$

and other brackets are linear combinations of $X_{n+1}, \cdots, X_p$. Let $f_t$ be the endomorphism given by $f_t(X_l) = X_l$ if $1 \leq l \leq n$ and $f_t(X_l) = tX_l$ for $n+1 \leq l \leq p$. This endomorphism generates a contraction of $V_1$ in the model $V$. We deduce that the rank of $V_1$ is smaller than the rank of $V$.

- Let us consider the filiform 3-algebra

$$\begin{cases} [X_1, X_2, X_3] = X_4, \\ [X_1, X_2, X_4] = X_5, \\ [X_1, X_3, X_4] = aX_5, \\ [X_2, X_3, X_4] = bX_5. \end{cases}$$

Its rank is equal to 2. In fact, in the basis $\{X_1, X_2, X_3 - aX_2, X_4, X_5\}$, the algebra writes

$$\begin{cases} [X_1, X_2, X_3] = X_4, \\ [X_1, X_2, X_4] = X_5, \\ [X_1, X_3, X_4] = 0, \\ [X_2, X_3, X_4] = bX_5. \end{cases}$$

Any diagonalizable derivation which admits this basis as eigenvectors basis, satisfies

$$f(X_i) = \lambda_i X_i$$

with

$$\lambda_3 = \lambda_1, \lambda_4 = 2\lambda_1 + \lambda_2, \lambda_5 = 3\lambda_1 + 2\lambda_2.$$

Then the rank is 2.

- For $n = 2$, we have the following important result: any Lie algebra which admits a nonsingular derivation is nilpotent. This is false as soon as $n \geq 3$. We have the interesting example ([59]): consider the $n$-Lie algebra given by

$$[X_1, X_2, \cdots, X_n] = X_2.$$

This algebra admits a non singular derivation but it is not nilpotent.

- In a forthcoming paper we develop the classification of filiform 3-Lie algebras whose rank is not 0.

## 3.3 sh-$n$-Lie algebras or Lie $n$-algebras

### 3.3.1 Definition

**Definition 30** *Let $\mu$ be a $n$-ary skewsymmetric product on a vector space $A$. We say that $(A, \mu)$ is a sh-$n$-Lie algebra (or a Lie $n$-algebra) if $\mu$ satisfies the (sh)-Jacobi's identity:*

$$\sum_{\sigma \in Sh(n,n-1)} (-1)^{\epsilon(\sigma)} \mu(\mu(x_{\sigma(1)}, \cdots, x_{\sigma(n)}), x_{\sigma(n+1)}, \cdots, x_{\sigma(2n-1)}) = 0,$$

*for any $x_i \in A$, where $Sh(n, n-1)$ is the subset of $\Sigma_{2n-1}$ defined by:*

$$Sh(n, n-1) = \{\sigma \in \Sigma_{2n-1}, \sigma(1) < \cdots < \sigma(n), \sigma(n+1) < \cdots < \sigma(2n-1)\}.$$

Moreover, we assume that $\mu$ is of degree $n-2$.

For example, if $n = 3$, we have the following (sh)-Jacobi's identity, writing (123)45 in place of $\mu(\mu(x_1, x_2, x_3), x_4, x_5)$:

$$(123)45 - (124)35 + (125)34 + (134)25 - (135)24 + (145)23 - (234)15 + (235)14 \\ -(245)13 + (345)12 = 0.$$

### 3.3.2 $n$-Lie algebras and sh-$n$-Lie algebras

**Proposition 19** *Any n-Lie algebra is a sh-n-Lie algebra.*

*Proof.* The Jacobi condition for $n$-Lie algebras writes

$$\mu \circ (\mu \otimes I_{n-1}) \circ \Phi_v = 0,$$

where $v \in \mathbb{K}[\Sigma_{2n-1}]$, the algebra group of the symmetric group $\Sigma_{2n-1}$ on $2n-1$ elements, given by

$$v = \quad Id + \sum_{i=1}^{n} (-1)^i (i, n+1, \cdots, 2n-1, 1, 2, \cdots, i-1, \widehat{i}, i+1, \cdots, n),$$

where $(i, n+1, \cdots, 2n-1, 1, 2, \cdots, i-1, \widehat{i}, i+1, \cdots, n)$ is the permutation

$$\begin{pmatrix} 1 & 2 & \cdots & n & n+1 & n+2 & \cdots & \cdots & \cdots & \cdots & 2n-1 \\ i & n+1 & \cdots & 2n-1 & 1 & 2 & \cdots & i-1 & i+1 & \cdots & n \end{pmatrix}.$$

Let

$$w = \sum_{\sigma \in \Sigma_{2n-1}} (-1)^{\epsilon(\sigma)} \sigma.$$

We have in $\mathbb{K}[\Sigma_{2n-1}]$, $w \circ v = \alpha(n) w$ with $\alpha(n) = 1 - n$ if $n$ is odd and $\alpha(n) = 1 + n$ if $n$ is even. This shows that the vector $w$ is in the invariant subspace of $\mathbb{K}[\Sigma_{2n-1}]$ generated by the vector $v$. This means that the identity

$$\mu \circ (\mu \otimes I_{n-1}) \circ \Phi_v = 0$$

implies

$$\mu \circ (\mu \otimes I_{n-1}) \circ \Phi_w = 0$$

which is equivalent to the Jacobi identity for sh-$n$-Lie algebras.

**Proposition 20** *A sh-n-Lie algebra is a n-Lie algebra if and only if any adjoint operator is a derivation.*

*Proof.* We have seen that a $n$-Lie algebra is a $n$-Leibniz algebras and these last are characterized by the fact that any adjoint operator is a derivation.

**Remark. Colored Lie algebras, colored $n$-Lie algebras.** Let us consider a binary algebra with a skew symmetric product satisfying a colored Jacobi identity:

$$\alpha[[X_i, X_j], X_k] + \beta[[X_j, X_k], X_l + \gamma[[X_k, X_i], X_j] = 0,$$

for any $i < j < k$, the constants $\alpha, \beta, \gamma$ being in $\mathbb{K}$. This identity is related to the vector $v = \alpha Id + \beta c + \gamma c^2$ of $\mathbb{K}[\Sigma_3]$. Let $w = Id - \tau_{12} - \tau_{13} - \tau_{23} + c + c^2$ the vector of $\mathbb{K}[\Sigma_3]$. Since $\mathbb{K}$ is of characteristic 0, the Jacobi identity, is equivalent to

$$\mu \circ (\mu \otimes Id) \circ \Phi_w = 0.$$

But in $\mathbb{K}[\Sigma_3]$ we have

$$w \circ v = (\alpha + \beta + \gamma) w.$$

Then, if $\alpha + \beta + \gamma \neq 0$, the colored Lie algebra satisfies the (non colored) Jacobi condition. This minimizes the interest of the notion of colored Lie algebras. It is the same for colored $n$-Lie algebras.

### 3.3.3 3-Lie admissible algebras

To simplify notations, we consider the case $n = 3$. In this case the product is of degree 1. A 3-ary algebra $(A, \cdot)$ is called 3-Lie admissible if the antisymmetric product

$$[v_1, v_2, v_3] = \sum_{\sigma \in \Sigma_3} (-1)^{\varepsilon(\sigma)} v_{\sigma(1)} \cdot v_{\sigma(2)} \cdot v_{\sigma(3)}$$

is a sh-3-Lie product.

**Proposition 21** *A $n$-ary algebra $(A, \cdot)$ is 3-Lie admissible if and only if we have*

$$\sum_{\sigma \in \Sigma_5} (-1)^{\varepsilon(\sigma)} ((v_{\sigma(1)} \cdot v_{\sigma(2)} \cdot v_{\sigma(3)}) \cdot v_{\sigma(4)} \cdot v_{\sigma(5)} + v_{\sigma(1)} \cdot (v_{\sigma(2)} \cdot v_{\sigma(3)} \cdot v_{\sigma(4)}) \cdot v_{\sigma(5)}$$
$$+ (v_{\sigma(1)} \cdot v_{\sigma(2)} \cdot (v_{\sigma(3)} \cdot v_{\sigma(4)} \cdot v_{\sigma(5)})) = 0,$$

*for any $v_1, v_2, v_3, v_4, v_5 \in A$.*

**Examples.**

- Any 3-ary partially associative algebra is 3-Lie admissible.
- In [55], a notion of $\sigma$-associative algebra have been introduced in the space of tensors $(2, 1)$ based on a vector space. In case of symmetric tensor, this product can be simplified. A symmetric tensor is defined by its structure constants $T_{ijk}$ which satisfy

  $$T_{ijk} = T_{jki} = T_{kij}.$$

  The 3-product $T \cdot U \cdot V$ whose structure constants are

  $$(T \cdot U \cdot V)_{ijk} = \sum_l T_{lij} U_{lki} V_{ljk}$$

  is 3-Lie admissible. Moreover the associated sh-3-Lie algebra is a 3-Lie algebra.

### 3.3.4 Maurer-Cartan equations

We assume in this section that any $n$-Lie algebras is of finite dimension. To simplify the presentation, we assume also that $n = 3$. Let $V$ be a finite dimensional 3-Lie algebras. Let $\{v_1, \cdots, v_p\}$ be a basis of $V$. The structure constants of $V$ related to this basis are given by

$$\{v_i, v_j, v_k\} = \sum_{l=1}^{l} C^l_{i,j,k} v_l$$

and satisfy

$$C^l_{ijk} = (-1)^{\varepsilon(\sigma)} C^l_{\sigma(i)\sigma(j)\sigma(k)},$$

for any $\sigma \in \Sigma_3$. The Jacobi condition writes

$$C^t_{ijk} C^s_{tlm} - C^t_{ijl} C^s_{tkm} + C^t_{ijm} C^s_{tjk} + C^t_{ikl} C^s_{tjm} - C^t_{ikm} C^s_{tjl} + C^t_{ilm} C^s_{tjl}$$
$$- C^t_{jkl} C^s_{tim} + C^t_{jkm} C^s_{til} - C^t_{jlm} C^s_{tik} + C^t_{klm} C^s_{tij} = 0,$$

for any $i < j < k, l < m$ and $s, t = 1, \cdots, p$. Let $\{\omega_1, \cdots, \omega_p\}$ be the dual basis of $\{v_1, \cdots, v_p\}$. We consider the graded exterior algebra $\Lambda(V) = \oplus \Lambda^k$ of $V$ and the linear operator

$$d : \Lambda^1(V) = V^* \rightarrow \Lambda^3(V)$$

given by

$$d\omega_l = \sum_{i<j<k} C^l_{ijk}\omega_i \wedge \omega_j \wedge \omega_k.$$

If we denote also by $d$ the linear operator

$$d : \Lambda^3(V) = V^* \to \Lambda^5(V)$$

defined by

$$d(\omega_i \wedge \omega_j \wedge \omega_k) = d\omega_i \wedge \omega_j \wedge \omega_k + \omega_i \wedge d\omega_j \wedge \omega_k + \omega_i \wedge \omega_j \wedge d\omega_k,$$

we obtain

$$\begin{aligned} d(d\omega_l) &= \sum_{i<j<k} C^l_{ijk}(d\omega_i \wedge \omega_j \wedge \omega_k + \omega_i \wedge d\omega_j \wedge \omega_k + \omega_i \wedge \omega_j \wedge d\omega_k \\ &= \sum_{i<j<k} C^l_{ijk}(C^i_{lst}\omega_l \wedge \omega_s \wedge \omega_t \wedge \omega_j \wedge \omega_k + C^j_{lst}\omega_i \wedge \omega_l \wedge \omega_s \wedge \omega_t \wedge \omega_k \\ &\quad +C^k_{lst}\omega_i \wedge \omega_j \wedge \omega_l \wedge \omega_s \wedge \omega_t). \end{aligned}$$

In this summand, all the products containing two equal factors are zero (this justifies to use the exterior algebra). In the same way, the Jacobi condition related to five vectors is trivial as soon as two vectors are equal. In fact, if we compute the Jacobi condition for the vectors $(X_1, X_2, X_3, X_1, X_1)$ we find $0 = 0$ and for the vector $(X_1, X_2, X_3, X_1, X_5)$ we find

$$\begin{aligned} &[[X_1, X_2, X_3], X_1, X_5] + [[X_1, X_2, X_5], X_3, X_1] - [[X_1, X_3, X_5], X_2, X_1] \\ &-[[X_2, X_3, X_1], X_1, X_5] - [[X_2, X_1, X_5], X_1, X_3] - [[X_3, X_1, X_5], X_1, X_2] \quad = \quad 0, \end{aligned}$$

that is, $0 = 0$. Thus the Jacobi condition concerns a family of 5 independent vectors. Let us return to the computation of $d(d\omega)$. The coefficient of $d(d\omega_l)$ related for example to $\omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \omega_4 \wedge \omega_5$ corresponds to the coefficient of $X_l$ in the Jacobi condition related to $(X_1, X_2, X_3, X_4, X_5)$. Thus

$$d(d\omega_l) = 0.$$

These relations can be called the Maurer-Cartan equations.

**Remark.** We cannot use the same calculus to obtain Maurer-Cartan equations adapted to the structure of $n$-Lie algebras. This means that the Maurer-Cartan equations of a $n$-Lie algebra are the Maurer-Cartan equations of this algebra considered as a sh-$n$-Lie algebra. In the classical case of Lie algebras, we have also such a situation. For example, when we consider the 2-step nilpotent Lie algebras, defined by the 2-step Jacobi condition

$$[[X_i, X_j], X_k] = 0,$$

there is no exterior calculus adapted to this special Jacobi condition. The Maurer-Cartan equations of a 2-step nilpotent algebra are the Maurer-Cartan equations of this algebra considered as a Lie algebra.

# Part II

# Arithmetic of Intervals

# Chapter 4

# An algebraic approach to the set of intervals.

In this chapter we present the set of intervals as a normed vector space. We define also a four-dimensional associative algebra whose product gives the product of intervals in any cases. With this approach we obtain a notion differential calculus and a natural linear algebra on the set of intervals.

**Introduction** The interval arithmetic, or interval analysis has been introduced to compute very quickly range bounds (for example if a data is given up to an incertitude). Now interval arithmetic is a computing system which permits to perform error analysis by computing mathematic bounds. The extensions of the areas of applications is important: non linear problems, PDE, inverse problems. It finds a large place of applications in controllability, automatism and robotic. The interval arithmetic is based on the following natural operations (called also Minkowski operations): if $X$ and $Y$ are bounded intervals of $\mathbb{R}$, then

$$X \diamond Y = \{x \diamond y \ / \ x \in X, \ y \in Y\},$$

where $\diamond$ denotes a binary operation such as $+, -, *$. Thus the set of intervals $\mathbb{IR}$ of $\mathbb{R}$ is a set provided with some binary operations but these operations do not give an algebraic structure on $\mathbb{IR}$. In many problems using interval arithmetic, there exists an informal transfers principle which permits, to associate with a real function $f$ a function define on the set of intervals $\mathbb{IR}$ which coincides with $f$ on the interval reduced to a point. But this transferred function is not unique. For example, if we consider the real function $f(x) = x^2 + x = x(x+1)$, we associate naturally the functions $\widetilde{f_1} : \mathbb{IR} \longrightarrow \mathbb{IR}$ given by $\widetilde{f_1}(X) = X(X+1)$ and $\widetilde{f_2}(X) = X^2 + X$. As $\mathbb{IR}$ is not algebraically structured, these two functions do not coincide. Usually this problem is removed considering the most interesting transfers. But the qualitative "interesting" depends of the studied model and it is not given by a formal process. There exists some properties of the inclusive function ( see [34]). In this work, we determine a natural extension $\overline{\mathbb{IR}}$ of $\mathbb{IR}$ provided with a vector space structure. The vectorial substraction $X \setminus Y$ does not correspond to the classical difference of intervals and the interval $\setminus X$ has no real interpretation. But these "negative" intervals have a computational role. If a problem conduce to a "negative" result, then this problem is "pervert" (see Lazare Carnot with his feeling on the natural negative number). We prove also, in this paper, that the vector space $\overline{\mathbb{IR}}$ is a Banach space, that is, a complete normed space. The interest of such a structure is that it permits to introduce a differential calculus and to use some important tools and the fixed point theorem.

The plan of this chapter is the following. In a first time we recall the semi-group structure on the set $\mathbb{IR}$ of intervals. By a classical process of completion, we endow this completed semi-group, denoted by $\overline{\mathbb{IR}}$, with a vector space structure. The norm given by $||X|| = l(X) + |c(X)|$ where $l(X)$ is the length of the interval $X$ and $c(X)$ the center of $X$ is complete and $(\overline{\mathbb{IR}}, ||.||)$ is a Banach space. We define the notion of differential function with values of $\overline{\mathbb{IR}}$. Next we extend the classical product to have a distributivity property. We end this chapter by giving some simple applications.

An interval is a bounded non empty connected closed subset of $\mathbb{R}$. The classical arithmetic operations on intervals are defined such that the result of the corresponding operation on elements belonging to operand intervals belongs to the resulting interval. That is, if $\diamond$ denotes one of the classical operations $+, -, *$, we have

$$[x^-, x^+] \diamond [y^-, y^+] = \{x \diamond y \ / \ x \in [x^-, x^+], \ y \in [y^-, y^+]\} \quad .$$

In particular we have

$$\begin{cases} [x^-, x^+] + [y^-, y^+] = [x^- + y^-, x^+ + y^+], \\ [x^-, x^+] - [y^-, y^+] = [x^- - y^+, x^+ - y^-] \end{cases}$$

and

$$[x^-, x^+] - [x^-, x^+] = [x^- - x^+, x^+ - x^-] \neq 0.$$

Let $\mathbb{IR}$ be the set of intervals. It is in one to one correspondence with the half plane of $\mathbb{R}^2$:

$$\mathcal{P}_1 = \{(a, b), a \leq b\}.$$

This set is closed for the addition and $\mathcal{P}_1$ is endowed with a regular semi-group structure. Let $\mathcal{P}_2$ be the half plane symmetric to $\mathcal{P}_1$ with respect to the first bisector $\Delta$ of equation $y - x = 0$. The substraction on $\mathbb{IR}$, which is not the symmetric operation of $+$, corresponds to the following operation on $\mathcal{P}_1$:

$$(a, b) - (c, d) = (a, b) + s_\Delta \circ s_0(c, d),$$

where $s_0$ is the symmetry with respect to 0, and $s_\Delta$ with respect to $\Delta$. The multiplication $*$ is not globally defined. Consider the following subset of $\mathcal{P}_1$:

$$\begin{cases} \mathcal{P}_{1,1} = \{(a, b) \in \mathcal{P}_1, a \geq 0, b \geq 0\}, \\ \mathcal{P}_{1,2} = \{(a, b) \in \mathcal{P}_1, a \leq 0, b \geq 0\}, \\ \mathcal{P}_{1,3} = \{(a, b) \in \mathcal{P}_1, a \leq 0, b \leq 0\}. \end{cases}$$

We have the following cases:

1) If $(a, b), (c, d) \in \mathcal{P}_{1,1}$ the product is written $(a, b) * (c, d) = (ac, bd)$.

The vectors $e_1 = (1, 1)$ and $e_2 = (0, 1)$ generate $\mathcal{P}_{1,1}$ that is any $(x, y)$ in $\mathcal{P}_{1,1}$, can be decomposed as

$$(x, y) = xe_1 + (y - x)e_2, \text{with } x > 0 \text{ and } y - x > 0.$$

The multiplication corresponds in this case to the following associative commutative algebra:

$$\begin{cases} e_1 e_1 = e_1, \\ e_1 e_2 = e_2 e_1 = e_2 e_2 = e_2. \end{cases}$$

2) Assume that $(a, b) \in \mathcal{P}_{1,1}$ and $(c, d) \in \mathcal{P}_{1,2}$ so $c \leq 0$ and $d \geq 0$. Thus we obtain $(a, b) * (c, d) = (bc, bd)$ and this product does not depend of $a$. Then we obtain the same result for any $a < b$. The product $(a, b) * (c, d) = (bc, bd)$ corresponds to

$$\begin{cases} e_1 e_1 = e_2 e_1 = e_1 \\ e_1 e_2 = e_2 e_2 = e_2 \end{cases}$$

This algebra is not commutative and it is different from the previous.

3) If $(a, b) \in \mathcal{P}_{1,1}$ and $(c, d) \in \mathcal{P}_{1,3}$ then $a \geq 0, b \geq 0$ and $c \leq 0, d \leq 0$ and we have $(a, b) * (c, d) = (bc, ad)$. Let $e_1 = (1, 1)$, $e_2 = (0, 1)$. This product corresponds to the following associative algebra:

$$\begin{cases} e_1 e_1 = e_1, \\ e_1 e_2 = e_2, \\ e_2 e_1 = e_1 - e_2. \end{cases}$$

This algebra is not associative because $(e_2 e_1) e_1 \neq e_2(e_1 e_1)$. We have similar results for the cases $(\mathcal{P}_{1,2}, \mathcal{P}_{1,2})$, $(\mathcal{P}_{1,}$ and $(\mathcal{P}_{1,3}, \mathcal{P}_{1,3})$.

An objective of this paper is to present an associative algebra which contains all these results.

## 4.1 The real vector space $\overline{\mathbb{IR}}$

### 4.1.1 The group $(\mathbb{IR},+)$

We recall briefly the construction proposed by Markov [46] to define a structure of abelian group. As $(\mathbb{IR},+)$ is a commutative and regular semi-group, the quotient set, denoted by $\overline{\mathbb{IR}}$, associated with the equivalence relations:

$$(x,y) \sim (z,t) \Longleftrightarrow x+t = y+z,$$

for all $x,y,z,t \in \mathbb{IR}$, is provided with a structure of abelian group for the natural addition:

$$\overline{(x,y)} + \overline{(z,t)} = \overline{(x+z,y+t)}$$

where $\overline{(x,y)}$ is the equivalence class of $(x,y)$. We denote by $\setminus\overline{(x,y)}$ the opposite of $\overline{(x,y)}$ . We have $\setminus\overline{(x,y)} = \overline{(y,x)}$. If $x=[a,a]$, $a \in \mathbb{R}$, then $\overline{(x,0)} = \overline{(0,-x)}$ where $-x = [-a,-a]$, and $\setminus\overline{(x,0)} = \overline{(0,x)}$. In this case, we identify $x=[a,a]$ with $a$ and we denote always by $\mathbb{R}$ the subset of intervals of type $[a,a]$. Naturally, the group $\overline{\mathbb{IR}}$ is isomorphic to the additive group $\mathbb{R}^2$ by the isomorphism $\overline{(([a,b],[c,d])} \rightarrow (a-c,b-d)$. We find the notion of generalized interval.

**Proposition 22** *Let $\mathcal{X} = \overline{(x,y)}$ be in $\overline{\mathbb{IR}}$. Thus*

*1.If $l(y) < l(x)$, there is an unique $A \in \mathbb{IR} \setminus \mathbb{R}$ such that $\mathcal{X} = \overline{(A,0)}$,*

*2. f $l(y) > l(x)$, there is an unique $A \in \mathbb{IR} \setminus \mathbb{R}$ such that $\mathcal{X} = \overline{(0,A)} = \setminus\overline{(A,0)}$,*

*3. If $l(y) = l(x)$, there is an unique $A = \alpha \in \mathbb{R}$ such that $\mathcal{X} = \overline{(\alpha,0)} = \overline{(0,-\alpha)}$.*

Any element $\mathcal{X} = \overline{(A,0)}$ with $A \in \mathbb{IR}-\mathbb{R}$ is said positive and we write $\mathcal{X} > 0$. Any element $\mathcal{X} = \overline{(0,A)}$ with $A \in \mathbb{IR}-\mathbb{R}$ is said negative and we write $\mathcal{X} < 0$. We write $\mathcal{X} \geq \mathcal{X}'$ if $\mathcal{X} \setminus \mathcal{X}' \geq 0$. For example if $\mathcal{X}$ and $\mathcal{X}'$ are positive, $\mathcal{X} \geq \mathcal{X}' \Longleftrightarrow l(\mathcal{X}) \geq l(\mathcal{X}')$.. The elements $\overline{(\alpha,0)}$ with $\alpha \in \mathbb{R}^*$ are neither positive nor negative.

### 4.1.2 Vector space structure on $\overline{\mathbb{IR}}$

In [46], one defines on the abelian group $\overline{\mathbb{IR}}$ , a structure of quasi linear space with the external multiplication given by $\forall \alpha,\beta,\gamma \in \mathbb{R}$ and $\forall a,b,c \in \overline{\mathbb{IR}}$, we have

$$\left\{ \begin{array}{l} \alpha * (\beta * \gamma) = (\alpha * \beta) * \gamma, \\ \gamma * (a+b) = \gamma * a + \gamma * b, \\ 1 * a = a, \\ (\alpha+\beta) * c = \alpha * c + \beta * c \text{ if } \alpha\beta > 0. \end{array} \right.$$

Our approach is a little bit different. We propose to construct a real vector space structure on the group $(\overline{\mathbb{IR}},+)$. We recall that if $A=[a,b] \in \mathbb{IR}$ and $\alpha \in \mathbb{R}^+$, the product $\alpha A$ is the interval $[\alpha a, \alpha b]$. We consider the external multiplication:

$$\cdot : \mathbb{R} \times \overline{\mathbb{IR}} \longrightarrow \overline{\mathbb{IR}}$$

defined, for all $A \in \mathbb{IR}$, by

$$\left\{ \begin{array}{ll} \alpha \cdot \overline{(A,0)} & = \overline{(\alpha A,0)} \ , \\ \alpha \cdot \overline{(0,A)} & = \overline{(0,\alpha A)} \ , \end{array} \right.$$

for all $\alpha > 0$. If $\alpha < 0$ we put $\beta = -\alpha$. So we put:

$$\left\{ \begin{array}{ll} \alpha \cdot \overline{(A,0)} & = \overline{(0,\beta A)}, \\ \alpha \cdot \overline{(0,A)} & = \overline{(\beta A,0)}. \end{array} \right.$$

We denote $\alpha\mathcal{X}$ instead of $\alpha \cdot \mathcal{X}$. This operation satisfies

1. For any $\alpha \in \mathbb{R}$ and $\mathcal{X} \in \overline{\mathbb{IR}}$ we have:

$$\left\{ \begin{array}{l} \alpha(\setminus\mathcal{X}) = \setminus(\alpha\mathcal{X}), \\ (-\alpha)\mathcal{X} = \setminus(\alpha\mathcal{X}). \end{array} \right.$$

2. For all $\alpha, \beta \in \mathbb{R}$, and for all $\mathcal{X}, \mathcal{X}' \in \overline{\mathbb{IR}}$, we have

$$\begin{cases} (\alpha+\beta)\mathcal{X} = \alpha\mathcal{X} + \beta\mathcal{X}, \\ \alpha(\mathcal{X}+\mathcal{X}') = \alpha\mathcal{X} + \alpha\mathcal{X}', \\ (\alpha\beta)\mathcal{X} = \alpha(\beta\mathcal{X}). \end{cases}$$

The two other equalities are defined in the multiplication of quasi linear space. So we have the result:

**Theorem 31** *The triplet* $(\overline{\mathbb{IR}}, +, \cdot)$ *is a real vector space and the vectors* $\mathcal{X}_1 = \overline{([0,1],0)}$ *and* $\mathcal{X}_2 = \overline{([1,1],0)}$ *of* $\overline{\mathbb{IR}}$ *determine a basis of* $\overline{\mathbb{IR}}$. *So* $\dim_{\mathbb{R}} \overline{\mathbb{IR}} = 2$.

*Proof.* We have the following decompositions:

$$\begin{cases} \overline{([a,b],0)} = (b-a)\mathcal{X}_1 + a\mathcal{X}_2, \\ \overline{(0,[c,d])} = (c-d)\mathcal{X}_1 - c\mathcal{X}_2. \end{cases}$$

The linear map

$$\varphi : \overline{\mathbb{IR}} \longrightarrow \mathbb{R}^2$$

defined by

$$\begin{cases} \varphi(\ \overline{([a,b],0)}\ ) = (b-a, a), \\ \varphi(\ \overline{(0,[c,d])}\ ) = (c-d, -c) \end{cases}$$

is a linear isomorphism and $\overline{\mathbb{IR}}$ is canonically isomorphic to $\mathbb{R}^2$.

**Remark.** Let $E$ be the subspace generated by $\mathcal{X}_2$. The vectors of $E$ correspond to the elements which have a non defined sign. Then the relation $\leq$ defined in the paragraph 1.2 gives an order relation on the quotient space $\overline{\mathbb{IR}}/E$.

### 4.1.3 A Banach structure on $\overline{\mathbb{IR}}$

Let us begin to define a norm on $\overline{\mathbb{IR}}$. Any element $\mathcal{X} \in \overline{\mathbb{IR}}$ is written $\overline{(A,0)}$ or $\overline{(0,A)}$. We define its length $l(\mathcal{X})$ as the length of $A$ and its center as $c(A)$ or $-c(A)$ in the second case.

**Theorem 32** *The map* $||\ ||: \overline{\mathbb{IR}} \longrightarrow \mathbb{R}$ *given by*

$$||\mathcal{X}|| = l(\mathcal{X}) + |c(\mathcal{X})|$$

*for any* $\mathcal{X} \in \overline{\mathbb{IR}}$ *is a norm.*

*Proof.* We have to verify the following axioms:

$$\begin{cases} 1)\ ||\mathcal{X}|| = 0 \Longleftrightarrow \mathcal{X} = 0, \\ 2)\ \forall \lambda \in \mathbb{R}\ ||\lambda\mathcal{X}|| = |\lambda| ||\mathcal{X}||, \\ 3)\ ||\mathcal{X} + \mathcal{X}'|| \leq ||\mathcal{X}|| + ||\mathcal{X}'||. \end{cases}$$

1) If $||\mathcal{X}|| = 0$, then $l(\mathcal{X}) = |c(\mathcal{X})| = 0$ and $\mathcal{X} = 0$.

2) Let $\lambda \in \mathbb{R}$. We have

$$||\lambda\mathcal{X}|| = l(\lambda\mathcal{X}) + |c(\lambda\mathcal{X})| = |\lambda| l(\mathcal{X}) + |\lambda||c(\mathcal{X})| = |\lambda| ||\mathcal{X}||.$$

3) We consider that $I$ refers to $\mathcal{X}$ and $J$ refers to $\mathcal{X}'$ thus $\mathcal{X} = \overline{(I,0)}$ or $= \overline{(0,I)}$. We have to study the two different cases:

i) If $\mathcal{X}+\mathcal{X}'=\overline{(I+J,0)}$ or $\overline{(0,I+J)}$, then

$$\begin{aligned}||\mathcal{X}+\mathcal{X}'|| &= l(I+J)+|c(I+J)| = l(I)+l(J)+|c(I)+c(J)| \leq l(I)+|c(I)|+l(J)+|c(J)| \\ &= ||\mathcal{X}||+||\mathcal{X}'||.\end{aligned}$$

ii) Let $\mathcal{X}+\mathcal{X}'=\overline{(I,J)}$. If $\overline{(I,J)}=\overline{(K,0)}$ then $K+J=I$ and

$$||\mathcal{X}+\mathcal{X}'|| = ||\overline{(K,0)}|| = l(K)+|c(K)| = l(I)-l(J)+|c(I)-c(J)|$$

that is

$$||\mathcal{X}+\mathcal{X}'|| \leq l(I)+|c(I)|-l(J)+|c(J)| \leq l(I)+|c(I)|+l(J)+|c(J)| = ||\mathcal{X}||+||\mathcal{X}'||.$$

So we have a norm on $\overline{\mathbb{IR}}$.

**Theorem 33** *The normed vector space $\overline{\mathbb{IR}}$ is a Banach space.*

*Proof.* In fact, all the norms on $\mathbb{R}^2$ are equivalent and $\mathbb{R}^2$ is a Banach space for any norm. The vector space $\overline{\mathbb{IR}}$ is isomorphic to $\mathbb{R}^2$. Thus it is complete.

**Remarks.**

1. To define the topology of the normed space $\overline{\mathbb{IR}}$, it is sufficient to describe the $\varepsilon$-neighborhood of any point $\chi_0 \in \overline{\mathbb{IR}}$ for $\varepsilon$ a positive infinitesimal number. We can give a geometrical representation, considering $\chi_0=\overline{([a,b],0)}$ represented by the point $(a,b)\in\mathbb{R}^2$. We assume that $\chi_0=\overline{([a,b],0)}$ and $\varepsilon$ an infinitesimal real number. Let $A_1,\cdots,A_4$ the points $A_1=(a-\varepsilon,b-\varepsilon)$, $A_2=(a+\frac{\varepsilon}{2},b-\frac{\varepsilon}{2})$, $A_3=(a+\varepsilon,b+\varepsilon)$, $A_4=(a-\frac{\varepsilon}{2},b+\frac{\varepsilon}{2})$. If $0<a<b$, then the $\varepsilon$-neighborhood of $\chi_0=\overline{([a,b],0)}$ is represented by the parallelograms whose vertices are $A_1,A_2,A_3,A_4$.

2. We can consider another equivalent norms on $\overline{\mathbb{IR}}$. For example

$$||\mathcal{X}|| = ||\setminus\mathcal{X}|| = Sup(|x|,|y|)$$

where $\mathcal{X}=\overline{([x,y],0)}$. But we prefer the initial one because it has a better geometrical interpretation.

## 4.2 A 4-dimensional associative algebra associated to $\overline{\mathbb{IR}}$

### 4.2.1 Classical product of intervals

We consider $X$,$Y\in\mathbb{IR}$. The multiplication of intervals is defined by

$$X\cdot Y = [\min(x^-y^-,x^-y^+,x^+y^-,x^+y^+),\max(x^-y^-,x^-y^+,x^+y^-,x^+y^+)].$$

Let $\mathcal{X}=\overline{(X,0)}$ and $\mathcal{X}'=\overline{(Y,0)}$ be in $\overline{\mathbb{IR}}$. We put

$$\mathcal{X}\mathcal{X}'=\overline{(XY,0)}.$$

For this product we have:

**Proposition 23** *For all $\mathcal{X}=\overline{(X,0)}$ and $\mathcal{X}'=\overline{(Y,0)}$ in $\overline{\mathbb{IR}}$, we have*

$$||\mathcal{X}\mathcal{X}'|| \leq ||\mathcal{X}||\,||\mathcal{X}'||.$$

*Proof.* If $\mathcal{X} = \overline{([x_1, x_2], 0)}$ then

$$\begin{cases} ||\mathcal{X}|| = \dfrac{3x_2 - x_1}{2} \text{ if } c(\mathcal{X}) > 0, \\ ||\mathcal{X}|| = \dfrac{x_2 - 3x_1}{2} \text{ if } c(\mathcal{X}) < 0. \end{cases}$$

Considering the different situations, we obtain

$$||\mathcal{X}||||\mathcal{X}'|| - ||\mathcal{X}\mathcal{X}'|| = \frac{3}{4} l(\mathcal{X}) l(\mathcal{X}')$$

or $\frac{1}{2}||\mathcal{X}||l(\mathcal{X}')$ or $\frac{1}{2}||\mathcal{X}'||l(\mathcal{X})$. These expressions are always positive. We have $||\mathcal{X}||||\mathcal{X}'|| = ||\mathcal{X}\mathcal{X}'||$ if $\mathcal{X}$ or $\mathcal{X}'$ are reduce to one point.

**Proposition 24** *We consider $\mathcal{X} = \overline{(X, 0)}$ and $\mathcal{X}' = \overline{(Y, 0)}$ in $\overline{\mathbb{IR}}$. We have*

$$X \subset Y \Rightarrow ||\mathcal{X}|| \leq ||\mathcal{X}'||.$$

*Proof.* Consider $X = [x_1, x_2]$ and $Y = [y_1, y_2]$.

*First case:* $y_1 \geq 0$. So $2||\mathcal{X}'|| = 3y_2 - y_1$. As $X \subset Y$, then $2||\mathcal{X}|| = 3x_2 - x_1$ and $||\mathcal{X}|| \leq ||\mathcal{X}'||$.

*Second case:* $y_1 < 0, y_2 > 0$. If $c(Y) \geq 0$, so $2||\mathcal{X}'|| = 3y_2 - y_1$. If $c(X) \geq 0$, from the first case $||\mathcal{X}|| \leq ||\mathcal{X}'||$. Otherwise $2||\mathcal{X}|| = x_2 - 3x_1$. Thus $||\mathcal{X}|| \leq ||\mathcal{X}'||$ if and only if $3y_2 - y_1 \geq x_2 - 3x_1$, that is $3(y_2 + x_1) \geq x_2 + y_1$ which is true.

If $c(Y) \leq 0$, then $2||\mathcal{X}'|| = y_2 - 3y_1$. If $c(X) \leq 0$, thus $2||\mathcal{X}|| = x_2 - 3x_1$ and $||\mathcal{X}|| \leq ||\mathcal{X}'||$. If $c(X) \geq 0$, $||\mathcal{X}|| \leq ||\mathcal{X}'||$ is equivalent to $y_2 - 3y_1 \geq 3x_2 - x_1$. But $c(Y) \leq 0$ implies $y_1 + y_2 \leq 0$ and $y_2 - 3y_1 \geq 4y_2$. Similarly $3x_2 - x_1 \leq 4x_2$, thus $y_2 - 3y_1 \geq 3x_2 - x_1$ because $x_2 \leq y_2$.

*Third case:* $y_1 < 0$, $y_2 < 0$. Similar computations give the result.

**Remark.** If $\mathcal{X} > 0$, i.e $\mathcal{X} = \overline{(X, 0)}$, and $\mathcal{X}' < 0$, i.e. $\mathcal{X}' = \overline{(0, Y)}$, so $\setminus \mathcal{X}' > 0$ and if $X \subset Y$ we deduce $||\mathcal{X}|| \leq || \setminus \mathcal{X}'|| = ||\mathcal{X}'||$.

### 4.2.2 Definition of $\mathcal{A}_4$

In introduction, we have observed that the semi-group $\overline{\mathbb{IR}}$ is identified to $\mathcal{P}_{1,1} \cup \mathcal{P}_{1,2} \cup \mathcal{P}_{1,3}$. Let us consider the following vectors of $\mathbb{R}^2$

$$\begin{cases} e_1 = (1, 1), \\ e_2 = (0, 1), \\ e_3 = (-1, 0), \\ e_4 = (-1, -1). \end{cases}$$

They correspond to the intervals $[1, 1], [0, 1], [-1, 0], [-1, -1]$. Any point of $\mathcal{P}_{1,1} \cup \mathcal{P}_{1,2} \cup \mathcal{P}_{1,3}$ admits the decomposition

$$(a, b) = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \alpha_4 e_4$$

with $\alpha_i \geq 0$.The dependance relations between the vectors $e_i$ are

$$\begin{cases} e_2 = e_3 + e_1 \\ e_4 = -e_1. \end{cases}$$

Thus there exists a unique decomposition of $(a, b)$ in a chosen basis such that the coefficients are non negative. These basis are $\{e_1, e_2\}$ for $\mathcal{P}_{1,1}$, $\{e_2, e_3\}$ for $\mathcal{P}_{1,2}$, $\{e_3, e_4\}$ for $\mathcal{P}_{1,3}$, Let us consider the free algebra of basis

$\{e_1, e_2, e_3, e_4\}$ whose products correspond to the Minkowski products. The multiplication table is

|  | $e_1$ | $e_2$ | $e_3$ | $e_4$ |
|---|---|---|---|---|
| $e_1$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ |
| $e_2$ | $e_2$ | $e_2$ | $e_3$ | $e_3$ |
| $e_3$ | $e_3$ | $e_3$ | $e_2$ | $e_2$ |
| $e_4$ | $e_4$ | $e_3$ | $e_2$ | $e_1$ |

This algebra is associative. Let us denote this 4-dimensional associative algebra by $\mathcal{A}_4$. If $x, y \in \mathcal{A}_4$, thus $x = \sum \alpha_i e_i$ and $y = \sum \beta_i e_i$ and the analytic expression of the product is

$$\begin{array}{rl} xy = & (\alpha_1\beta_1 + \alpha_4\beta_4)e_1 + (\alpha_1\beta_2 + \alpha_2\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 + \alpha_3\beta_4 + \alpha_4\beta_3)e_2 \\ & +(\alpha_1\beta_3 + \alpha_3\beta_1 + \alpha_2\beta_3 + \alpha_3\beta_2 + \alpha_2\beta_4 + \alpha_4\beta_2)e_3 + (\alpha_1\beta_4 + \alpha_4\beta_1)e_4. \end{array}$$

**Theorem 34** *The multiplication of intervals in the algebra $\mathcal{A}_4$ is distributive with respect the addition.*

**Example.** Let us consider the product

$$[2, 3]([-1, 3] + [2, 6]).$$

The classical operations give

$$[2, 3].[1, 9] = [2, 27]$$

and

$$[2, 3].[-1, 3] + [2, 3][2, 6] = [-3, 9] + [4, 18] = [1, 27]$$

this shows the non distributivity of the classical product. In $\mathcal{A}_4$ we have

$$\begin{cases} [2, 3] = 2e_1 + e_2, \\ [-1, 3] = 3e_2 + e_3, \\ [2, 6] = 2e_1 + 4e_2. \end{cases}$$

Thus

$$\begin{array}{rl} [2, 3].[-1, 3] + [2, 3][2, 6] & = (2e_1 + e_2)(3e_2 + e_3) + (2e_1 + e_2)(2e_1 + 4e_2) \\ & = (9e_2 + 3e_3) + (4e_1 + 14e_2) \\ & = 4e_1 + 23e_2 + 3e_3 \end{array}$$

and

$$\begin{array}{rl} [2, 3].([-1, 3] + [2, 3]) & = (2e_1 + e_2)((3e_2 + e_3) + (2e_1 + 4e_2)) \\ & = (2e_1 + e_2) + (2e_1 + 7e_2 + e_3) \\ & = 4e_1 + 14e_2 + 23e_3 + 2e_2 + 7e_2 + e_3 \\ & = 4e_1 + 23e_2 + 3e_3. \end{array}$$

The vector $4e_1 + 23e_2 + 3e_3 \in \mathcal{P}_{1,3}$. It is written

$$4e_1 + 23e_2 + 3e_3 = 4e_1 + 23e_2 + 3(e_2 - e_1) = e_1 + 26e_2.$$

This vector corresponds to $[1, 27]$. Thus we have

$$[2, 3].[-1, 3] + [2, 3][2, 6] = [2, 3]([-1, 3] + [2, 6]) = [1, 27].$$

This example shows how to pass from $\mathcal{A}_4$ to $\mathbb{IR}$.The difficulty results from the fact that the application $\varphi : \mathbb{IR} \rightarrow \mathcal{A}_4$ is not bijective. It is defined by

$$\begin{cases} x = [a, b] \in \mathcal{P}_{1,1}, \varphi(x) = ae_1 + (b - a)e_2 \quad (a \geq 0, b - a \geq 0) \\ x = [a, b] \in \mathcal{P}_{1,2}, \varphi(x) = -ae_3 + be_2 \quad (-a \geq 0, b \geq 0) \\ x = [a, b] \in \mathcal{P}_{1,3}, \varphi(x) = -be_4 + (b - a)e_3 \quad (-b \geq 0, b - a \geq 0). \end{cases}$$

Consider in $\mathcal{A}_4$ the linear subspace $F$ generated by the vectors $e_1 - e_2 + e_3, e_1 + e_4$. As

$$\begin{array}{l} (e_1 + e_4)(e_1 + e_4) = 2(e_1 + e_4) \\ (e_1 + e_4)(e_1 - e_2 + e_3) = e_1 + e_4 \\ (e_1 - e_2 + e_3)(e_1 - e_2 + e_3) = e_1, \end{array}$$

$F$ is not a subalgebra of $\mathcal{A}_4$. Let us consider the map

$$\overline{\varphi} : \mathbb{IR} \rightarrow \mathcal{A}_4/F$$

defined from $\varphi$ and the canonical projection on the quotient vector space $\mathcal{A}_4/F$. A vector $x = \sum \alpha_i e_i \in \mathcal{A}_4$ is equivalent to a vector of $\mathcal{A}_4$ with positive components if and only if

$$\alpha_2 + \alpha_3 \geq 0.$$

In this case, all the vectors equivalent to $x = \sum \alpha_i e_i$ with $\alpha_2 + \alpha_3 \geq 0$ correspond to the interval $[\alpha_1 - \alpha_3 - \alpha_4, \alpha_1 + \alpha_2 - \alpha_4]$ of $\mathbb{IR}$. Thus we have for any equivalent classes of $\mathcal{A}_4/F$ associated with $\sum \alpha_i e_i$ with $\alpha_2 + \alpha_3 \geq 0$ we have a preimage in $\mathbb{IR}$. The map $\overline{\varphi}$ is injective. In fact, two intervals belonging to pieces $\mathcal{P}_{1,i}, \mathcal{P}_{1,j}$ with $i \neq j$, have distinguish images. Now if $(a, b)$ and $(c, d)$ belong to the same piece, for example $\mathcal{P}_{1,1}$, thus

$$\overline{\varphi}(a, b) = \{(a + \lambda + \mu, b - a - \lambda, \lambda, \mu), \lambda, \mu \in \mathbb{R}.\}$$

If $\overline{\varphi}(c, d) = \overline{\varphi}(a, b)$, there are $\lambda, \mu \in \mathbb{R}$ such that $(c, d) = (a + \lambda + \mu, b - a - \lambda, \lambda, \mu)$. This gives $a = c, b = d$. We have the same results for all the other pieces.Thus $\overline{\varphi} : \mathbb{IR} \rightarrow \mathcal{A}_4/F$ is bijective on its image, that is the hyperplane of $\mathcal{A}_4/F$ corresponding to $\alpha_2 + \alpha_3 \geq 0$.

Practically the multiplication of two intervals will so be made: let $X, Y \in \mathbb{R}$. Thus $X = \sum \alpha_i e_i, Y = \sum \beta_i e_i$ with $\alpha_i, \beta_j \geq 0$ and we have the product

$$X \bullet Y = \overline{\varphi}^{-1}(\varphi(X).\varphi(Y))$$

this product is well defined because $\overline{\varphi(X).\varphi(Y)} \in Im\overline{\varphi}$. This product is distributive because

$$\begin{array}{ll} X \bullet (Y + Z) & = \overline{\varphi}^{-1}(\varphi(X).\varphi(Y + Z)) \\ & = \overline{\varphi}^{-1}(\varphi(X).(\varphi(Y) + \varphi(Z)) \\ & = \overline{\varphi}^{-1}(\varphi(X).\varphi(Y) + \varphi(X).\varphi(Z)) \\ & = X \bullet Y + X \bullet Z \end{array}$$

**Remark.** We have

$$\overline{\varphi}^{-1}(\varphi(X).\varphi(Y + Z)) \neq \overline{\varphi}^{-1}(\varphi(X)).\overline{\varphi}^{-1}(\varphi(Y + Z))).$$

We shall be careful not to return in $\mathbb{IR}$ during the calculations as long as the result is not found. Otherwise we find the classic problems of the distributivity.

We extend naturally the map $\varphi : \mathbb{IR} \rightarrow \mathcal{A}_4$ to $\overline{\mathbb{IR}}$ by

$$\left\{ \begin{array}{l} \varphi\overline{(A, 0)} = \varphi(A) \\ \varphi\overline{(0, A)} = -\varphi(A) \end{array} \right.$$

for every $A \in \mathbb{IR}$.

**Theorem 35** *The multiplication*

$$\mathcal{X}' \bullet \mathcal{X}'' = \overline{\varphi}^{-1}(\varphi(\mathcal{X}').\varphi(\mathcal{X}''))$$

*is distributive with respect the addition.*

*Proof.* This is a direct consequence of the previous computations.

### 4.2.3 The algebras $\mathcal{A}_n$ and an better result of the product

In this section, we compute the product of intervals using the product in $\mathcal{A}_4$ and we compare with the Minkowski product. Let $X = [a, b]$ and $Y = [c, d]$ two intervals.

**Lemma 6** *If $X$ and $Y$ are not in the same piece $\mathcal{P}_{1,i}$, then $X \bullet Y$ corresponds to the Minkowski product.*

*Proof.* i) If $X \in \mathcal{P}_{1,1}$ and $Y \in \mathcal{P}_{1,2}$ then $\varphi(X) = (a, b - a, 0, 0)$ and $\varphi(Y) = (0, d, -c, 0)$. Thus

$$\begin{aligned}\varphi(X)\varphi(Y) &= (ae_1 + (b-a)e_2)(de_2 - ce_3)\\ &= bde_2 - cbe_3\\ &= (0, bd, -cb, 0)\\ &= \varphi([cb, bd]).\end{aligned}$$

ii) If $X \in \mathcal{P}_{1,1}$ and $Y \in \mathcal{P}_{1,3}$ then $\varphi(X) = (a, b - a, 0, 0)$ and $\varphi(Y) = (0, 0, d - c, -d)$. Thus

$$\begin{aligned}\varphi(X)\varphi(Y) &= (ae_1 + (b-a)e_2)((d-c)e_3 - de_4)\\ &= (ad - bc)e_3 - ade_4\\ &= (0, 0, ad - cb, -ad)\\ &= \varphi([bc, ad]).\end{aligned}$$

iii) If $X \in \mathcal{P}_{1,2}$ and $Y \in \mathcal{P}_{1,3}$ then $\varphi(X) = (0, b, -a, 0)$ and $\varphi(Y) = (0, 0, d - c, -d)$. Thus

$$\begin{aligned}\varphi(X)\varphi(Y) &= (be_2 - ae_3)((d-c)e_3 - de_4)\\ &= ace_2 - bce_3\\ &= (0, ac, -cb, 0)\\ &= \varphi([bc, ad]).\end{aligned}$$

**Lemma 7** *If $X$ an $Y$ are both in the same piece $\mathcal{P}_{1,1}$ or $\mathcal{P}_{1,3}$, then the product $X \bullet Y$ corresponds to the Minkowski product.*

The proof is analogous to the previous.

Let us assume that $X = [a, b]$ and $Y = [c, d]$ belong to $\mathcal{P}_{1,2}$. Thus $\varphi(X) = (0, b, -a, 0)$ and $\varphi(Y) = (0, d, -c, 0)$. We obtain

$$XY = (be_2 - ae_3)(de_2 - ce_3) = (bd + ac)e_2 + (-bc - ad)e_3.$$

Thus

$$[a, b][c, d] = [bc + ad, bd + ac].$$

This result is greater that all the possible results associated with the Minkowski product. However, we have the following property:

**Proposition 25** ***Monotony property:*** *Let $\mathcal{X}_1, \mathcal{X}_2 \in \overline{\mathbb{IR}}$. Then*

$$\begin{cases} \mathcal{X}_1 \subset \mathcal{X}_2 \Longrightarrow \mathcal{X}_1 \bullet \mathcal{Z} \subset \mathcal{X}_2 \bullet \mathcal{Z} \text{ for all } \mathcal{Z} \in \overline{\mathbb{IR}}.\\ \overline{\varphi}(\mathcal{X}_1) \leq \overline{\varphi}(\mathcal{X}_2) \Longrightarrow \overline{\varphi}(\mathcal{X}_1 \bullet \mathcal{Z}) \leq \overline{\varphi}(\mathcal{X}_2 \bullet \mathcal{Z}) \end{cases}$$

The order relation on $\mathcal{A}_4$ that ones uses here is

$$\begin{cases} (x_1, x_2, 0, 0) \leq (y_1, y_2, 0, 0) \Longleftrightarrow y_1 \leq x_1 \text{ and } x_2 \leq y_2,\\ (x_1, x_2, 0, 0) \leq (0, y_2, y_3, 0) \Longleftrightarrow x_2 \leq y_2,\\ (0, x_2, x_3, 0) \leq (0, y_2, y_3, 0) \Longleftrightarrow x_3 \leq y_3 \text{ and } x_2 \leq y_2,\\ (0, 0, x_3, x_4) \leq (0, y_2, y_3, 0) \Longleftrightarrow x_3 \leq y_3,\\ (0, 0, x_3, x_4) \leq (0, 0, y_3, y_4) \Longleftrightarrow x_3 \leq y_3 \text{ and } y_4 \leq x_4. \end{cases}$$

*Proof.* Let us note that the second property is equivalent to the first. It is its translation in $\overline{\mathcal{A}_4}$. We can suppose that $\mathcal{X}_1$ and $\mathcal{X}_2$ are intervals belonging moreover to $\mathcal{P}_{1,2}$: $\varphi(\mathcal{X}_1) = (0, b, -a, 0), \varphi(\mathcal{X}_2) = (0, d, -c, 0)$. If $\varphi(\mathcal{Z}) = (z_1, z_2, z_3, z_4)$, then

$$\begin{cases} \overline{\varphi}(\mathcal{X}_1 \bullet \mathcal{Z}) = (0, bz_1 + bz_2 - az_3 - az_4, -az_1 + bz_3 - az_2 + bz_4, 0), \\ \overline{\varphi}(\mathcal{X}_2 \bullet \mathcal{Z}) = (0, dz_1 + dz_2 - cz_3 - cz_4, -cz_1 + dz_3 - cz_2 + dz_4, 0). \end{cases}$$

Thus

$$\overline{\varphi}(\mathcal{X}_1 \bullet \mathcal{Z}) \leq \overline{\varphi}(\mathcal{X}_2 \bullet \mathcal{Z}) \Longleftrightarrow \begin{cases} (b-d)(z_1+z_2) - (a-c)(z_3-z_4) \leq 0, \\ -(a-c)(z_1+z_2) + (b-d)(z_3 = z_4) \leq 0. \end{cases}$$

But $(b-d)$, $-(a-c) \leq 0$ and $z_2, z_3 \geq 0$. This implies $\overline{\varphi}(\mathcal{X}_1 \bullet \mathcal{Z}) \leq \overline{\varphi}(\mathcal{X}_2 \bullet \mathcal{Z})$.

We can refine our result of the product to come closer to the result of Minkowski. Consider the one dimensional extension $\mathcal{A}_4 \oplus \mathbb{R}e_5 = \mathcal{A}_5$, where $e_5$ is a vector corresponding to the interval $[-1, 1]$ of $\mathcal{P}_{1,2}$. The multiplication table of $\mathcal{A}_5$ is

| | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ |
|---|---|---|---|---|---|
| $e_1$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ |
| $e_2$ | $e_2$ | $e_2$ | $e_3$ | $e_3$ | $e_5$ |
| $e_3$ | $e_3$ | $e_3$ | $e_2$ | $e_2$ | $e_5$ |
| $e_4$ | $e_4$ | $e_3$ | $e_2$ | $e_1$ | $e_5$ |
| $e_5$ | $e_5$ | $e_5$ | $e_5$ | $e_5$ | $e_5$ |

.

The piece $\mathcal{P}_{1,2}$ is written $\mathcal{P}_{1,2} = \mathcal{P}_{1,2,1} \cup \mathcal{P}_{1,2,1}$ where $\mathcal{P}_{1,2,1} = \{[a, b], -a \leq b\}$ and $\mathcal{P}_{1,2,2} = \{[a, b], -a \geq b\}$. If $X = [a, b] \in \mathcal{P}_{1,2,1}$ and $Y = [c, d] \in \mathcal{P}_{1,2,2}$, thus

$$\varphi(X).\varphi(Y) = (0, b+a, 0, 0, -a).(0, 0, -c-d, 0, d) = (0, -(a+b)(c+d), 0, 0, a(c+d)+bd).$$

Thus we have

$$X \bullet Y = [-bd - ac - ad, -bc].$$

**Example** Let $X = [-2, 3]$ and $Y = [-4, 2]$. We have $X \in \mathcal{P}_{1,2,1}$ and $Y \in \mathcal{P}_{1,2,2}$. The product in $\mathcal{A}_4$ gives

$$X \bullet Y = [-16, 14].$$

The product in $\mathcal{A}_5$ gives

$$X \bullet Y = [-12, 10].$$

The Minkowski product is

$$[-2, 3].[-4, 2] = [-12, 8].$$

Thus the product in $\mathcal{A}_5$ is better.

**Conclusion.** Considering a partition of $\mathcal{P}_{1,2}$, we can define an extension of $\mathcal{A}_4$ of dimension $n$, the choice of $n$ depends on the approach wanted of the Minkowski product. For example, let us consider the vector $e_6$ corresponding to the interval $[-1, \frac{1}{2}]$. Thus the Minkowsky product gives $e_6.e_6 = e_7$ where $e_7$ corresponds to $[-\frac{1}{2}, 1]$. We obtain a 7-dimensional associative algebra whose table of multiplication is

| | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
| $e_2$ | $e_2$ | $e_2$ | $e_3$ | $e_3$ | $e_5$ | $e_6$ | $e_7$ |
| $e_3$ | $e_3$ | $e_3$ | $e_2$ | $e_2$ | $e_5$ | $e_7$ | $e_6$ |
| $e_4$ | $e_4$ | $e_3$ | $e_2$ | $e_1$ | $e_5$ | $e_7$ | $e_6$ |
| $e_5$ | $e_5$ | $e_5$ | $e_5$ | $e_5$ | $e_5$ | $e_5$ | $e_5$ |
| $e_6$ | $e_6$ | $e_6$ | $e_7$ | $e_7$ | $e_5$ | $e_7$ | $e_6$ |
| $e_7$ | $e_7$ | $e_7$ | $e_6$ | $e_6$ | $e_5$ | $e_6$ | $e_7$ |

.

**Example** Let $X = [-2,3]$ and $Y = [-4,2]$. The decomposition on the basis $\{e_1, \cdots, e_7\}$ with positive coefficients writes

$$X = e_5 + 2e_7, \quad Y = 2e_6.$$

Thus

$$X \bullet Y = (e_5 + 2e_7)(4e_6) = 4e_5 + 8e_6 = [-12, 8].$$

We obtain now the Minkowski product.

### 4.2.4 Algebraic study of $\mathcal{A}_4$

In $\mathcal{A}_4$ we consider the change of basis

$$\begin{cases} e'_1 = e_1 - e_2 \\ e'_i = e_i, i = 2,3 \\ e'_4 = e_4 - e_3. \end{cases}$$

This change of basis shows that $\mathcal{A}_4$ is isomorphic to $\mathcal{A}'_4$

| | $e_1$ | $e_2$ | $e_3$ | $e_4$ |
|---|---|---|---|---|
| $e_1$ | $e_1$ | 0 | 0 | $e_4$ |
| $e_2$ | 0 | $e_2$ | $e_3$ | 0 |
| $e_3$ | 0 | $e_3$ | $e_2$ | 0 |
| $e_4$ | $e_4$ | 0 | 0 | $e_1$ |

.

The unit of $\mathcal{A}'_4$ is the vector $e_1 + e_2$. This algebra is a direct sum of two ideals: $\mathcal{A}'_4 = I_1 + I_2$ where $I_1$ is generated by $e_1$ and $e_4$ and $I_2$ is generated by $e_2$ and $e_3$. It is not an integral domain, that is, we have divisors of 0. For example $e_1 \cdot e_2 = 0$.

We denote by $\mathcal{A}'^*_4$ the group of invertible elements. We compute this group. The cartesian expression of this product is, for $x = (x_1, x_2, x_3, x_4)$ and $y = (y_1, y_2, y_3, y_4)$ in $\mathcal{A}'_4$:

$$x \cdot y = (x_1y_1 + x_4y_4, x_2y_2 + x_3y_3, x_3y_2 + x_2y_3, x_4y_1 + x_1y_4).$$

We consider the equation

$$x \cdot y = (1, 1, 0, 0).$$

We obtain

$$\begin{cases} x_1y_1 + x_4y_4 = 1, \\ x_2y_2 + x_3y_3 = 1, \\ x_3y_2 + x_2y_3 = 0, \\ x_4y_1 + x_1y_4 = 0. \end{cases}$$

For a given vector $x$, we obtain a solution $y$ if and only if:

$$(x_1^2 - x_4^2)(x_2^2 - x_3^2) \neq 0.$$

**Proposition 26** *The multiplicative group $\mathcal{A}^*_4$ is the set of elements $x = (x_1, x_2, x_3, x_4)$ such that*

$$\begin{cases} x_4 \neq \pm x_1, \\ x_3 \neq \pm x_2. \end{cases}$$

*If $x \in \mathcal{A}^*_4$ we have:*

$$x^{-1} = \left( \frac{x_1}{x_1^2 - x_4^2}, \frac{x_2}{x_2^2 - x_3^2}, \frac{x_3}{x_2^2 - x_3^2}, \frac{x_4}{x_1^2 - x_4^2} \right).$$

## 4.3 Divisibility and an Euclidean division

We have computed the invertible elements of $\mathcal{A}'_4$. If $x = (x_1, x_2, x_3, x_4) \in \mathcal{A}'_4$ and if $\Delta = (x_1^2 - x_4^2)(x_2^2 - x_3^2) \neq 0$ then

$$x^{-1} = \left( \frac{x_1}{x_1^2 - x_4^2}, \frac{x_2}{x_2^2 - x_3^2}, \frac{x_3}{x_2^2 - x_3^2}, \frac{x_4}{x_1^2 - x_4^2} \right).$$

The elements associated to $\mathcal{X} = \overline{(K, 0)}$ are of the form

$$\begin{cases} (x_1, x_2, 0, 0) \text{ if } 0 < x_1 < x_2, \\ (0, x_2, -x_1, 0) \text{ if } x_1 < 0 < x_2, \\ (0, 0, -x_1, -x_2) \text{ if } x_1 < x_2 < 0, \end{cases}$$

and to $\mathcal{X} \in \overline{(0, K)}$

$$\begin{cases} (0, 0, x_1, x_2) \text{ if } 0 < x_1 < x_2, \\ (-x_1, 0, 0, x_2) \text{ if } x_1 < 0 < x_2, \\ (-x_1, -x_2, 0, 0) \text{ if } x_1 < x_2 < 0. \end{cases}$$

The inverse of $(x_1, x_2, 0, 0)$ with $0 < x_1 < x_2$ is $\left( \frac{1}{x_1}, \frac{1}{x_2}, 0, 0 \right)$.

The inverse of $(0, 0, -x_1, -x_2)$ with $x_1 < x_2 < 0$ is $\left( 0, 0, -\frac{1}{x_1}, -\frac{1}{x_2} \right)$.

The inverse of $(0, 0, x_1, x_2)$ with $0 < x_1 < x_2$ is $\left( 0, 0, \frac{1}{x_1}, \frac{1}{x_2} \right)$.

The inverse of $(-x_1, -x_2, 0, 0)$ with $x_1 < x_2 < 0$ is $\left( -\frac{1}{x_1}, -\frac{1}{x_2}, 0, 0 \right)$.

For $\mathcal{X} = (0, x_2, -x_1, 0)$ or $(-x_1, 0, 0, x_2)$ with $x_1 x_2 \le 0$, then $\Delta = 0$ and $\mathcal{X}$ is not invertible. Then if $\Delta \neq 0$ the inverse is always represented by an element of $\overline{\mathbb{IR}}$ thought $\psi$.

### 4.3.1 Division by an invertible element

We denote by $\overline{\mathbb{IR}}^+$ the subset $\overline{(X, 0)}$ with $X = [x_1, x_2]$ and $0 \le x_1$.

**Proposition 27** *Let $\mathcal{X} = \overline{(X, 0)}$ and $\mathcal{Y} = \overline{(Y, 0)}$ be in $\overline{\mathbb{IR}}^+$ with $X = [x_1, x_2]$, $Y = [y_1, y_2]$. If $\frac{y_2}{y_1} \ge \frac{x_2}{x_1}$ then there exists an unique $\mathcal{Z} = \overline{(Z, 0)} \in \overline{\mathbb{IR}}^+$ such that $\mathcal{Y} = \mathcal{X}\mathcal{Z}$.*

*Proof.* Let $\mathcal{Z}$ be defined by $c(\mathcal{Z}) = \frac{1}{2}\left( \frac{y_2}{x_2} + \frac{y_1}{x_1} \right)$ and $l(\mathcal{Z}) = \left( \frac{y_2}{x_2} - \frac{y_1}{x_1} \right)$. Then $l(\mathcal{Z}) \ge 0$ if and only if $\frac{y_2}{x_2} \ge \frac{y_1}{x_1}$ that is $\frac{y_2}{y_1} \ge \frac{x_2}{x_1}$. Thus we have $\mathcal{Y} = \mathcal{X}\mathcal{Z}$. In fact

$$(\overline{\varphi}(\mathcal{X}))^{-1} = \overline{\left( \frac{1}{x_1}, \frac{1}{x_2}, 0, 0 \right)} = \psi \left( \overline{(0, [-\frac{1}{x_1}, -\frac{1}{x_2}])} \right).$$

Thus

$$\overline{\varphi}(\mathcal{Y}) \cdot \overline{\varphi}(\mathcal{X})^{-1} = (y_1, y_2, 0, 0) \cdot \left( \frac{1}{x_1}, \frac{1}{x_2}, 0, 0 \right) = \left( \frac{y_1}{x_1}, \frac{y_2}{x_2}, 0, 0 \right).$$

As $\frac{y_1}{x_1} \le \frac{y_2}{x_2}$,

$$\psi(\overline{\varphi}(\mathcal{Y}) \cdot \overline{\varphi}(\mathcal{X})^{-1}) = \overline{\left( [\frac{y_1}{x_1}, \frac{y_2}{x_2}], 0 \right)}.$$

We can note also that
$$\overline{\left(0,[-\frac{1}{x_1},-\frac{1}{x_2}]\right)} \bullet \overline{([y_1,y_2],0)} = \overline{\left([\frac{y_1}{x_1},\frac{y_2}{x_2}],0\right)}.$$
Then the divisibility corresponds to the multiplication by the inverse element.

### 4.3.2 Division by a non invertible element

Let $X = [-x_1, x_2]$ with $x_1, x_2 > 0$. We have seen that $\varphi(X) = (0, x_2, x_1, 0)$ is not invertible in $\mathcal{A}_4$. For any $M = (y_1, y_2, y_3, y_4) \in \mathcal{A}_4$ we have
$$\varphi(X) \cdot M = (0, x_2y_2 + x_1y_3, x_1y_2 + x_2y_3, 0)$$
and this point represents a non invertible interval. Thus we can solve the equation $\mathcal{Y} = \mathcal{X} \bullet \mathcal{Z}$ for $\mathcal{X} = \overline{([-x_1, x_2], 0)}$ , $\mathcal{Y} = \overline{([-y_1, y_2], 0)}$ with $x_1, x_2 > 0$ and $y_1, y_2 > 0$. Putting $\varphi(\mathcal{Z}) = (z_1, z_2, z_3, z_4)$, we obtain
$$(0, y_2, y_1, 0) = (0, x_2, x_1, 0) \cdot (z_1, z_2, z_3, z_4),$$
that is
$$\left\{ \begin{array}{l} y_2 = x_2z_2 + x_1z_3, \\ y_1 = x_2z_3 + x_1z_2, \end{array} \right.$$
or
$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_1 \end{pmatrix} \begin{pmatrix} z_2 \\ z_3 \end{pmatrix}.$$
If $x_1^2 - x_2^2 \neq 0$,
$$\left\{ \begin{array}{l} z_2 = \dfrac{x_1y_1 - x_2y_2}{x_1^2 - x_2^2}, \\ z_3 = \dfrac{-x_2y_1 + x_1y_2}{x_1^2 - x_2^2}. \end{array} \right.$$
If $x_1^2 - x_2^2 = 0$ then $x_1 = x_2$ and the center of $X = [-x_1, x_1]$ is 0. Let us assume that $x_1 \neq x_2$. If $x_1^2 - x_2^2 < 0$ that is $x_1 < x_2$ then
$$\left\{ \begin{array}{l} x_1y_1 - x_2y_2 < 0, \\ x_1y_2 - x_2y_1 < 0, \end{array} \right.$$
and $\dfrac{x_1}{x_2} < \dfrac{y_2}{y_1}$, $\dfrac{x_1}{x_2} < \dfrac{y_1}{y_2}$. If $\alpha = \dfrac{x_1}{x_2} < 1$ we have $y_2 > \alpha y_1$, $y_1 > \alpha y_2$ then $y_2 > \alpha^2 y_2$ and $1 - \alpha^2 > 0$. This case admits solution.

**Proposition 28** *Let $\mathcal{X} = \overline{([-x_1, x_2], 0)}$ with $x_1, x_2 > 0$ and $x_1 < x_2$. Then for any $\mathcal{Y} = \overline{([-y_1, y_2], 0)}$ with $y_1, y_2 > 0$ and $\dfrac{x_1}{x_2} < \dfrac{y_2}{y_1}$, $\dfrac{x_1}{x_2} < \dfrac{y_1}{y_2}$, there is $\mathcal{Z} = \overline{([-z_1, z_2], 0)}$ such that $\mathcal{Y} = \mathcal{X} \bullet \mathcal{Z}$.*

Suppose now that $x_1^2 - x_2^2 > 0$ that us $x_1 > x_2$. In this case we have
$$\left\{ \begin{array}{l} x_1y_1 - x_2y_2 > 0, \\ x_1y_2 - x_2y_1 > 0, \end{array} \right.$$
that is $\dfrac{y_2}{y_1} < \dfrac{x_1}{x_2}$ and $\dfrac{y_1}{y_2} < \dfrac{x_1}{x_2}$.

**Proposition 29** *Let $\mathcal{X} = \overline{([-x_1, x_2], 0)}$ with $x_1, x_2 > 0$ and $x_1 > x_2$. For any $\mathcal{Y} = \overline{([-y_1, y_2], 0)}$ with $y_1, y_2 > 0$, $\dfrac{x_1}{x_2} > \dfrac{y_2}{y_1}$, $\dfrac{x_1}{x_2} > \dfrac{y_1}{y_2}$, there is $\mathcal{Z} = \overline{([-z_1, z_2], 0)}$ such that $\mathcal{Y} = \mathcal{X} \bullet \mathcal{Z}$.*

**Example.** $\mathcal{X} = \overline{([-4, 2], 0)}$, $\mathcal{Y} = \overline{([-2, 3], 0)}$. We have $\dfrac{x_2}{x_1} = \dfrac{1}{2}$, $\dfrac{x_1}{x_2} = 2$ and $\dfrac{3}{2} < 2 < 6$. Then $\mathcal{Z}$ exists and it is equal to $\mathcal{Z} = \overline{([-\frac{8}{12}, \frac{2}{12}], 0)}$.

### 4.3.3 An Euclidean division

Consider $\mathcal{X} = \overline{([x_1, x_2], 0)}$ and $\mathcal{Y} = \overline{([y_1, y_2], 0)}$ in $\overline{\mathbb{IR}}^+$. We have seen that $\mathcal{Y}$ is divisible by $\mathcal{X}$ as soon as $\frac{x_1}{x_2} \geq \frac{y_1}{y_2}$. We suppose now that $\frac{x_1}{x_2} < \frac{y_1}{y_2}$. In this case we have

**Theorem 36** *Let $\mathcal{X}$ and $\mathcal{Y}$ be in $\overline{\mathbb{IR}}^+$ with $\frac{x_1}{x_2} < \frac{y_1}{y_2}$. There is a unique pair $(\mathcal{Z},\mathcal{R})$ unique in $\overline{\mathbb{IR}}^+$ such that*

$$\left\{ \begin{array}{l} \mathcal{Y} = \mathcal{X} \bullet \mathcal{Z} + \mathcal{R}, \\ l(\mathcal{R}) = 0 \text{ and } c(\mathcal{R}) \text{ minimal.} \end{array} \right.$$

*This pair is given by*

$$\left\{ \begin{array}{l} \mathcal{Z} = \dfrac{y_2 - y_1}{x_2 - x_1} \overline{([1,1], 0)}, \\ \mathcal{R} = \dfrac{x_2 y_1 - x_1 y_2}{x_2 - x_1} \overline{([1,1], 0)}. \end{array} \right.$$

*Proof.* We consider $\mathcal{Z} = \overline{([z_1, z_2], 0)}$ with $z_1 > 0$. Then $\mathcal{Y} = \mathcal{X} \bullet \mathcal{Z} + \mathcal{R}$ gives

$$\mathcal{R} = \overline{([y_1, y_2], [z_1 x_1, z_2 x_2])}.$$

We have $\mathcal{R} \in \overline{\mathbb{IR}}^+$ if and only if $0 \leq y_1 - x_1 z_1 \leq y_2 - x_2 z_2$ that is

$$\left\{ \begin{array}{l} z_1 \leq \dfrac{y_1}{x_1}, \\ z_2 \leq \dfrac{y_2}{x_2}, \\ z_1 \geq \dfrac{y_1 - y_2 + x_2 z_2}{x_1}. \end{array} \right.$$

The condition $z_1 \leq z_2$ implies $\frac{y_1 - y_2 + x_2 z_2}{x_1} \leq z_2$ that is $z_2 \leq \frac{y_2 - y_1}{x_2 - x_1}$. Consider the case $z_2 = \frac{y_2 - y_1}{x_2 - x_1}$. Then $z_1 \geq \frac{y_1 - y_2 + x_2 z_2}{x_1} = \frac{y_2 - y_1}{x_2 - x_1} = z_2$ and $z_1 = z_2$. This case corresponds to

$$\left\{ \begin{array}{l} \mathcal{Z} = \dfrac{y_2 - y_1}{x_2 - x_1} \overline{([1,1], 0)}, \\ \mathcal{R} = \dfrac{x_2 y_1 - x_1 y_2}{x_2 - x_1} \overline{([1,1], 0)}. \end{array} \right.$$

Let us note that $y_1 x_2 - x_1 y_2 > 0$ is equivalent to $\frac{y_1}{y_2} > \frac{x_1}{x_2}$ which is satisfied by hypothesis. We have also for this solution $l(\mathcal{R}) = 0$ and $c(\mathcal{R}) = \frac{x_2 y_1 - x_1 y_2}{x_2 - x_1}$.

Conversely, if $l(\mathcal{R}) = 0$ then $y_1 - z_1 x_1 = y_2 - z_2 x_2$ and $z_1 = z_2 \frac{x_2}{x_1} + \frac{y_1 - y_2}{x_1}$. As $z_1 > 0$, we obtain $z_2 > \frac{y_1 - y_2}{x_1}$ and $z_1 \leq z_2$ implies

$$\frac{y_2 - y_1}{x_2} \leq z_2 \leq \frac{y_2 - y_1}{x_2 - x_1}.$$

But $c(\mathcal{R}) = y_1 - z_1 x_1 = y_2 - z_2 x_2$. Thus

$$\frac{x_2 y_1 - x_1 y_2}{x_2 - x_1} \leq c(\mathcal{R}) \leq y_1.$$

The norm is minimal when $c(\mathcal{R}) = \frac{x_2 y_1 - x_1 y_2}{x_2 - x_1}$.

**Example.** Let $\mathcal{X} = \overline{([1,4],0)}$ and $\mathcal{Y} = \overline{([1,3],0)}$. We have $\frac{x_1}{x_2} = \frac{1}{4} < \frac{y_1}{y_2} = \frac{1}{3}$. Thus $\mathcal{Z} = \frac{2}{3}\overline{([1,1],0)}$ and $\mathcal{R} = \frac{1}{3}\overline{([1,1],0)}$. The division writes

$$\overline{([1,3],0)} = \overline{([1,4],0)} \cdot \overline{([\frac{2}{3},\frac{2}{3}],0)} + \overline{([\frac{1}{3},\frac{1}{3}],0)}.$$

Suppose now that $\mathcal{X}$ and $\mathcal{Y}$ are not invertible, that is $\mathcal{X} = \overline{([-x_1,x_2],0)}$ and $\mathcal{Y} = \overline{([-y_1,y_2],0)}$ with $x_1$,$x_2,y_1,y_2$ positive. We have seen that $\mathcal{Y}$ is divisible by $\mathcal{X}$ as soon as

$$\left\{ \begin{array}{l} \frac{x_1}{x_2} > \frac{y_2}{y_1} \text{ and } \frac{x_1}{x_2} > \frac{y_1}{y_2}, \\ \text{or} \\ \frac{x_1}{x_2} < \frac{y_2}{y_1} \text{ and } \frac{x_1}{x_2} < \frac{y_1}{y_2}. \end{array} \right.$$

We suppose now that these conditions are not satisfied. For example we assume that

$$\frac{x_1}{x_2} > \frac{y_2}{y_1} \text{ and } \frac{x_1}{x_2} < \frac{y_1}{y_2}$$

(The other case is similar). If $\mathcal{Y} = \mathcal{X} \bullet \mathcal{Z} + \mathcal{R}$ then $\mathcal{R} = \overline{([-r_1,r_2],0)}$ with $r_1 \geq 0$ and with $r_2 \geq 0$ because $\overline{\varphi}(\mathcal{R}) = (0, r_2, r_1, 0)$. This shows that we can choose $\mathcal{Z}$ such that $\overline{\varphi}(\mathcal{R}) = (0, z_2, z_3, 0)$ and

$$\left\{ \begin{array}{l} z_2 = \dfrac{x_1(y_1 - r_1) - x_2(y_2 - r_2)}{x_1^2 - x_2^2}, \\ z_3 = \dfrac{x_1(y_2 - r_2) - x_2(y_1 - r_1)}{x_1^2 - x_2^2}, \end{array} \right.$$

with the condition $z_2 \geq 0$ and $z_3 \geq 0$. If $x_1 < x_2$ then this is equivalent to

$$\left\{ \begin{array}{l} \dfrac{x_1}{x_2} < \dfrac{y_2 - r_2}{y_1 - r_1}, \\ \dfrac{x_1}{x_2} < \dfrac{y_1 - r_1}{y_2 - r_2}. \end{array} \right.$$

If we suppose that $\mathcal{R} \leq \mathcal{Y}$, thus $0 < r_2 < y_2$ and $0 < r_1 < y_1$, we obtain

$$r_1 > \frac{x_2}{x_1} r_2 + \frac{-x_2 y_2 + x_1 y_1}{x_1} < r_1 < \frac{x_1}{x_2} r_2 + \frac{x_2 y_1 - x_1 y_2}{x_2}.$$

Then length $l(\mathcal{R}) = r_1 + r_2$ is minimal if and only if $r_2 = 0$ and in this case $r_1 = \frac{x_1 y_1 - x_2 y_2}{x_1}$. We obtain

$$\left\{ \begin{array}{l} z_2 = 0, \\ z_3 = \dfrac{y_2}{x_1}. \end{array} \right.$$

**Theorem 37** *Let $\mathcal{X} = \overline{([-x_1,x_2],0)}$ with $x_1, x_2 > 0$ and $x_1 > x_2$. If $\mathcal{Y} = \overline{([-y_1,y_2],0)}$ with $y_1, y_2 > 0$, satisfies $\frac{x_1}{x_2} > \frac{y_2}{y_1}$ and $\frac{x_1}{x_2} < \frac{y_1}{y_2}$, there is a unique pair $\mathcal{R}$,$\mathcal{Z}$ of non invertible elements such that*

$$\left\{ \begin{array}{l} l(\mathcal{R}) \textit{ minimal}, \\ \mathcal{R} < \mathcal{Y}. \end{array} \right.$$

*This pair is given by*

$$\left\{ \begin{array}{l} \mathcal{Z} = \overline{([-\frac{y_2}{x_1}, 0], 0)}, \\ \mathcal{R} = \overline{([-\frac{x_1 y_1 - x_2 y_2}{x_1}, 0], 0)}. \end{array} \right.$$

## 4.4 Applications

### 4.4.1 Differential calculus on $\overline{\mathbb{IR}}$

As $\overline{\mathbb{IR}}$ is a Banach space, we can describe a notion of differential function on it. Consider $\mathcal{X}_0 = \overline{(X_0, 0)}$ in $\overline{\mathbb{IR}}$ . The norm $||.||$ defines a topology on $\overline{\mathbb{IR}}$ whose a basis of neighborhoods is given by the balls $\mathcal{B}(X_0, \varepsilon) = \{X \in \overline{\mathbb{IR}}, ||\mathcal{X} \smallsetminus \mathcal{X}_0|| < \varepsilon\}$. Let us characterize the elements of $\mathcal{B}(X_0, \varepsilon)$. $\mathcal{X}_0 = \overline{(X_0, 0)} = \overline{([a, b], 0)}$.

**Proposition 30** *Consider* $\mathcal{X}_0 = \overline{(X_0, 0)} = \overline{([a, b], 0)}$ *in* $\overline{\mathbb{IR}}$ *and* $\varepsilon \simeq 0$, $\varepsilon > 0$. *Then every element of* $\mathcal{B}(X_0, \varepsilon)$ *is of type* $\mathcal{X} = \overline{(X, 0)}$ *and satisfies*

$$l(X) \in B_{\mathbb{R}}(l(X_0), \varepsilon_1) \text{ and } c(X) \in B_{\mathbb{R}}(c(X_0), \varepsilon_2)$$

*with* $\varepsilon_1, \varepsilon_2 \geq 0$ *and* $\varepsilon_1 + \varepsilon_2 \leq \varepsilon$, *where* $B_{\mathbb{R}}(x, a)$ *is the canonical open ball in* $\mathbb{R}$ *of center* $x$ *and radius* $a$.

*Proof. First case :* Assume that $\mathcal{X} = \overline{(X, 0)} = \overline{([x, y], 0)}$ . We have

$$\begin{aligned} \mathcal{X} \smallsetminus \mathcal{X}_0 &= \overline{(X, X_0)} = \overline{([x, y], [a, b])} \\ &= \begin{cases} \overline{([x - a, y - b], 0)} \text{ if } l(X) \geq l(X_0) \\ \overline{(0, [a - x, b - y])} \text{ if } l(X) \leq l(X_0) \end{cases} \end{aligned}$$

If $l(X) \geq l(X_0)$ we have

$$\begin{aligned} ||\mathcal{X} \smallsetminus \mathcal{X}_0|| &= (y - b) - (x - a) + \left| \frac{y - b + x - a}{2} \right| \\ &= l(X) - l(X_0) + |c(X) - c(X_0)|. \end{aligned}$$

As $l(X) - l(X_0) \geq 0$ and $|c(X) - c(X_0)| \geq 0$, each one of this term if less than $\varepsilon$. If $l(X) \leq l(X_0)$ we have

$$||\mathcal{X} \smallsetminus \mathcal{X}_0|| = l(X_0) - l(X) + |c(X_0) - c(X)|.$$

and we have the same result.

*Second case :* Consider $\mathcal{X} = \overline{(0, X)} = \overline{([x, y], 0)}$ . We have

$$\mathcal{X} \smallsetminus \mathcal{X}_0 = \overline{(0, X_0 + X)} = \overline{([x + a, y + b])}$$

and

$$||\mathcal{X} \smallsetminus \mathcal{X}_0|| = l(X_0) + l(X) + |c(X_0) + c(X)|.$$

In this case, we cannot have $||\mathcal{X} \smallsetminus \mathcal{X}_0|| < \varepsilon$ thus $X \notin \mathcal{B}(X_0, \varepsilon)$.

**Definition 38** *A function* $f : \overline{\mathbb{IR}} \longrightarrow \overline{\mathbb{R}}$ *is continuous at* $\mathcal{X}_0$ *if*

$$\forall \varepsilon > 0, \exists \eta > 0 \text{ such that } ||\mathcal{X} \smallsetminus \mathcal{X}_0|| < \eta \text{ implies } ||f(\mathcal{X}) \smallsetminus f(\mathcal{X}_0)|| < \varepsilon.$$

Consider $(\mathcal{X}_1, \mathcal{X}_2)$ the basis of $\overline{\mathbb{IR}}$ given in section 2. We have

$$f(\mathcal{X}) = f_1(\mathcal{X})\mathcal{X}_1 + f_2(\mathcal{X})\mathcal{X}_2 \text{ with } f_i : \overline{\mathbb{IR}} \longrightarrow \mathbb{R}.$$

If $f$ is continuous at $\mathcal{X}_0$ so

$$f(\mathcal{X}) \smallsetminus f(\mathcal{X}_0) = (f_1(\mathcal{X}) - f_1(\mathcal{X}_0))\mathcal{X}_1 + (f_2(\mathcal{X}) - f_2(\mathcal{X}_0))\mathcal{X}_2.$$

To simplify notations let $\alpha = f_1(\mathcal{X}) - f_1(\mathcal{X}_0)$ and $\beta = f_2(\mathcal{X}) - f_2(\mathcal{X}_0)$. If $||f(\mathcal{X}) \setminus f(\mathcal{X}_0)|| < \varepsilon$, and if we assume $f_1(\mathcal{X}) - f_1(\mathcal{X}_0) > 0$ and $f_2(\mathcal{X}) - f_2(\mathcal{X}_0) > 0$ (other cases are similar), then we have

$$l(\alpha\mathcal{X}_1 + \beta\mathcal{X}_2) = l\overline{([\beta, \alpha+\beta], 0)} < \varepsilon$$

thus $f_1(\mathcal{X}) - f_1(\mathcal{X}_0) < \varepsilon$. Similarly,

$$c(\alpha\mathcal{X}_1 + \beta\mathcal{X}_2) = c\overline{([\beta, \alpha+\beta], 0)} = \frac{\alpha}{2} + \beta < \varepsilon$$

and this implies that $f_2(\mathcal{X}) - f_2(\mathcal{X}_0) < \varepsilon$.

**Corollary 39** *$f$ is continuous at $\mathcal{X}_0$ if and only if $f_1$ and $f_2$ are continuous at $\mathcal{X}_0$.*

**Examples.**

- $f(\mathcal{X}) = \mathcal{X}$. This function is continuous at any point.
- $f(\mathcal{X}) = \mathcal{X}^2$. Consider $\mathcal{X}_0 = \overline{(X_0, 0)} = \overline{([a,b], 0)}$ and $\mathcal{X} \in \mathcal{B}(X_0, \varepsilon)$. We have

$$\begin{aligned} ||\mathcal{X}^2 \setminus \mathcal{X}_0^2|| &= ||(\mathcal{X} \setminus \mathcal{X}_0)(\mathcal{X} + \mathcal{X}_0)|| \\ &\leq ||\mathcal{X} \setminus \mathcal{X}_0|| ||\mathcal{X} + \mathcal{X}_0||. \end{aligned}$$

  Given $\varepsilon > 0$, let $\eta = \dfrac{\varepsilon}{||\mathcal{X} + \mathcal{X}_0||}$, thus if $||\mathcal{X} \setminus \mathcal{X}_0|| < \eta$, we have $||\mathcal{X}^2 \setminus \mathcal{X}_0^2|| < \varepsilon$ and $f$ is continuous.

- Consider $P = a_0 + a_1 X + \cdots + a_n X^n \in \mathbb{R}[\mathbb{X}]$. We define $f : \overline{\mathbb{IR}} \longrightarrow \overline{\mathbb{IR}}$ with $f(\mathcal{X}) = a_0\mathcal{X}_2 + a_1\mathcal{X} + \cdots + a_n^n \mathcal{X}^n$ where $\mathcal{X}^n = \mathcal{X} \cdot \mathcal{X}^{n-1}$ . From the previous example, all monomials are continuous, it implies that $f$ is continuous.

**Definition 40** *Consider $\mathcal{X}_0$ in $\overline{\mathbb{IR}}$ and $f : \overline{\mathbb{IR}} \longrightarrow \overline{\mathbb{IR}}$ continuous. We say that $f$ is differentiable at $\mathcal{X}_0$ if there is $g : \overline{\mathbb{IR}} \longrightarrow \overline{\mathbb{IR}}$ linear such as*

$$||f(\mathcal{X}) \setminus f(\mathcal{X}_0) \setminus g(\mathcal{X} \setminus \mathcal{X}_0)|| = o(||\mathcal{X} \setminus \mathcal{X}_0||).$$

### 4.4.2 Study of the function $q_2$

We consider the function $q_2 : \overline{\mathbb{IR}} \longrightarrow \overline{\mathbb{IR}}$ given by

$$q_2\overline{([a,b],0)} = \begin{cases} \overline{([a^2, b^2], 0)} \text{ if } 0 < a < b, \\ \overline{([b^2, a^2], 0)} \text{ if } a < b < 0, \\ \overline{([0, \sup(a^2, b^2)])} \text{ if } a < 0 < b. \end{cases}$$

and $q_2\overline{(0, [a,b])} = q_2\overline{([a,b], 0)}$. For any invertible element $\mathcal{X} \in \overline{\mathbb{IR}}$, we have $q_2(\mathcal{X}) = \mathcal{X} \bullet \mathcal{X}$. If $\mathcal{X}$ is not invertible, it writes $\mathcal{X} = \overline{([a,b], 0)}$ with $a < 0 < b$ ( we assume that $\mathcal{X}$ is of type $\overline{(K, 0)}$). In this case $\mathcal{X} \bullet \mathcal{X} = \overline{([2ab, a^2 + b^2], 0)}$ and $q_2 \subset \mathcal{X} \bullet \mathcal{X}$ .

**Proposition 31** *The function $q_2$ is continuous on $\overline{\mathbb{IR}}$.*

*Proof.* Let $\mathcal{X}_0 \in \overline{\mathbb{IR}}$. Assume that $\mathcal{X}_0 = \overline{([a,b], 0)}$ with $0 < a < b$. An $\eta$ -neighborhood is represented by the parallelogram $(A, B, C, D)$ with $A = (a - \frac{\eta}{2}, b + \frac{\eta}{2})$, $B = (a - \eta, b - \eta)$, $C = (a + \frac{\eta}{2}, b - \frac{\eta}{2})$, $D = (a + \eta, b + \eta)$. We have $q_2(\mathcal{X}_0) = \mathcal{X}_0^2 = \overline{([a^2, b^2], 0)}$. For any $\varepsilon > 0$ we consider the $\varepsilon$-neighborhood of $q_2(\mathcal{X}_0)$. it is represented by the parallelogram $(A_1, B_1, C_1, D_1)$ with $A_1 = (a^2 - \frac{\varepsilon}{2}, b^2 + \frac{\varepsilon}{2})$, $B_1 = (a^2 - \varepsilon, b^2 - \varepsilon)$, $C_1 = (a^2 + \frac{\varepsilon}{2}, b^2 - \frac{\varepsilon}{2})$, $D_1 = (a^2 + \varepsilon, b^2 + \varepsilon)$. If $\eta$ satisfy

$$\begin{cases} 2a\eta + \eta^2 < \frac{\varepsilon}{2}, \\ \eta^2 - 2a\eta > -\frac{\varepsilon}{2} \end{cases}$$

the for every point of the $\eta$-neighborhood of $\mathcal{X}_0$, the image $q_2(\mathcal{X})$ is contained in the $\varepsilon$ -neighborhood of $q_2(\mathcal{X}_0)$. If $a \neq 0$, as $\varepsilon$ is infinitesimal we have $\eta = \frac{\varepsilon}{8a}$. If $a = 0$, we have $\eta = \varepsilon$. Then $q_2$ is continuous at the point $\mathcal{X}_0$. Is $\mathcal{X}_0 = \overline{([a, b], 0)}$ with $a < b < 0$, taking $\eta = -\frac{\varepsilon}{8a}$ we prove in a similar way the continuity at $\mathcal{X}_0$. Assume that $\mathcal{X}_0 = \overline{([a, b], 0)}$ with $a < 0 < b$ then $q_2(\mathcal{X}_0) = \overline{([0, \sup(a^2, b^2)])}$. If $\mathcal{X} = \overline{([x, y], 0)}$ is an $\eta$-neighborhood of $\mathcal{X}_0$ with $q_2(\mathcal{X}) = \overline{([0, \sup((x+\eta)^2, (y+\eta)^2)])}$ then $a - \eta < x < a + \eta, b - \eta < y < b + \eta$ and we can find $\eta$ such that $\sup(a^2, b^2) - \frac{\varepsilon}{2} < \sup((x+\eta)^2, (y+\eta)^2) < \sup(a^2, b^2) + \frac{\varepsilon}{2}$. Thus $q_2$ is also continuous in this point. As $q_2\overline{(0, K)} = q_2\overline{(K, 0)}$, we have the continuity of any point.

**Theorem 41** *The function $q_2$ is not differentiable.*

*Proof.* The function $q_2$ is differentiable at the point $\mathcal{X}_0$ if there is a linear map $L$ such that

$$\lim_{||\mathcal{X} \setminus \mathcal{X}_0|| \to 0} \frac{||q_2(\mathcal{X}) \setminus q_2(\mathcal{X}_0) \setminus L(\mathcal{X} \setminus \mathcal{X}_0)||}{||\mathcal{X} \setminus \mathcal{X}_0||} = 0.$$

We consider $L$ be the linear function given by

$$L(\mathcal{X}) = 2\mathcal{X}_0 \bullet (\mathcal{X}).$$

We assume that $\mathcal{X}_0 = (\overline{[a, b], 0})$ with $0 < a < b$. If $\mathcal{X}$ is in an infinitesimal neighborhood of $\mathcal{X}_0$, then $\mathcal{X} = (\overline{[x, y], 0})$ with $0 < x < y$.

- If $0 < x - a < y - b$

$$\mathcal{X} \setminus \mathcal{X}_0 = (\overline{[x, y], [a, b]}) = (\overline{[x - a, y - b], 0})$$

Thus $L(\mathcal{X} \setminus \mathcal{X}_0) = 2(\overline{[a, b], 0}) \bullet (\overline{[x - a, y - b], 0}) = 2(\overline{[a(x - a), b(y - b), 0})$ and

$$\begin{aligned} q_2(\mathcal{X}) \setminus q_2(\mathcal{X}_0) \setminus L(\mathcal{X} \setminus \mathcal{X}_0) &= (\overline{[x^2, y^2], 0}) \setminus (\overline{[a^2, b^2], 0}) \setminus 2(\overline{[a(x - a), b(y - b), 0]}), \\ &= (\overline{[x^2 - a^2, y^2 - b^2], 0}) \setminus 2(\overline{[a(x - a), b(y - b), 0]}), \\ &= (\overline{[(x - a)^2, (y - b)^2], 0}). \end{aligned}$$

We deduce

$$\begin{aligned} ||q_2(\mathcal{X}) \setminus q_2(\mathcal{X}_0) \setminus L(\mathcal{X} \setminus \mathcal{X}_0)|| &= (y - b)^2 - (x - a)^2 + |\frac{(y - b)^2 + (x - a)^2}{2}|, \\ &= \frac{3(y - b)^2 - (x - a)^2}{2}. \end{aligned}$$

Thus

$$\frac{||q_2(\mathcal{X}) \setminus q_2(\mathcal{X}_0) \setminus L(\mathcal{X} \setminus \mathcal{X}_0)||}{||\mathcal{X} \setminus \mathcal{X}_0||} = \frac{3(y - b)^2 - (x - a)^2}{3(y - b) - (x - a)}.$$

Then $\frac{||q_2(\mathcal{X}) \setminus q_2(\mathcal{X}_0) \setminus L(\mathcal{X} \setminus \mathcal{X}_0)||}{||\mathcal{X} \setminus \mathcal{X}_0||} \leq \varepsilon$ is equivalent to

$$3(y - b)^2 - (x - a)^2 \leq \varepsilon(3(y - b) - (x - a)).$$

Then for every point of the $\varepsilon$-neighborhood of $\mathcal{X}_0$, the $\varepsilon$ inequality of the differentiability is satisfied. This shows that, if $q_2$ is differentiable at $\mathcal{X}_0$, then the differential is the linear function $L(\mathcal{X}) = 2\mathcal{X}_0 \bullet \mathcal{X}$.

- If $0 < a - x < y - b$. We find again the previous case.
- If $0 < y - b < a - x$, then

$$\mathcal{X} \setminus \mathcal{X}_0 = (\overline{[x,y],[a,b]}) = (\overline{[x-a,y-b],0}).$$

Thus $L(\mathcal{X} \setminus \mathcal{X}_0) = 2(\overline{[a,b],0}) \bullet (\overline{[x-a,y-b],0}) = 2(\overline{[b(x-a),b(y-b),0})$ and

$$\begin{aligned} q_2(\mathcal{X}) \setminus q_2(\mathcal{X}_0) \setminus L(\mathcal{X} \setminus \mathcal{X}_0) &= (\overline{[x^2,y^2],0}) \setminus (\overline{[a^2,b^2],0}) \setminus 2(\overline{[b(x-a),b(y-b)],0}), \\ &= (\overline{[x^2-a^2,y^2-b^2],0}) \setminus 2(\overline{[b(x-a),b(y-b)],0}), \\ &= (\overline{[(x-b)^2-(a-b)^2,(y-b)^2],0}). \end{aligned}$$

We deduce

$$\frac{||q_2(\mathcal{X}) \setminus q_2(\mathcal{X}_0) \setminus L(\mathcal{X} \setminus \mathcal{X}_0)||}{||\mathcal{X} \setminus \mathcal{X}_0||} = \frac{3(y-b)^2 - (x-b)^2 + (a-b)^2}{(y-b)+3(a-x)}.$$

Then $\dfrac{||q_2(\mathcal{X}) \setminus q_2(\mathcal{X}_0) \setminus L(\mathcal{X} \setminus \mathcal{X}_0)||}{||\mathcal{X} \setminus \mathcal{X}_0||} \leq \varepsilon$ is equivalent to

$$3(y-b)^2 - (x-b)^2 + (a-b)^2 \leq \varepsilon(y-b) + 3(a-x).$$

We see that the representation of $E$ doesn't contains any points of the representation of a $\eta$-neighborhood of $\mathcal{X}_0$ for all $\eta$. This gives a contradiction of the differentiability at $\mathcal{X}_0$.

**Remarks.**

1. In a following work, we will study the differentiability of the function $s : \overline{\mathbb{IR}} \longrightarrow \overline{\mathbb{IR}}$ define by $s[a,b] = [-b,-a]$. This function is used in numerical approach. It is different of the function $\chi \longrightarrow \setminus\chi$.
2. Some applications concerning linear problems in the vector spaces $\overline{\mathbb{IR}}$, such as reductions of matrices of intervals, eigenvalues, eigenspaces, are studied in [31].
3. Non-linear simplex methods using interval analysis and linear algebra of intervals is proposed in [37].

# Chapter 5

# Linear Algebra in the vector space of intervals $\overline{\mathbb{IR}}$

In the previous chapter, we have given an algebraic model to the set of intervals. Here, we apply this model in a linear frame. We define a notion of diagonalization of square matrices whose coefficients are intervals. But in this case, with respect to the real case, a matrix of order $n$ could have more than $n$ eigenvalues (the set of intervals is not factorial). We consider a notion of central eigenvalues permits to describe criterium of diagonalization. As application, we define a notion of Exponential mapping.

## 5.1 The module $gl(n, \overline{\mathbb{IR}})$

Let $gl(n, \overline{\mathbb{IR}})$ be the set of square matrices of order $n$ whose elements are in $\overline{\mathbb{IR}}$. A matrices of $gl(n, \overline{\mathbb{IR}})$ is denoted by

$$A = (\mathcal{X}_{ij})_{i,j=1,\cdots,n}$$

with $\mathcal{X}_{ij} = (K_{ij}, 0)$ or $(0, K_{ij})$. It is clear that $gl(n, \overline{\mathbb{IR}})$ is a real vector space. We define a product on it pouting

$$A \cdot B = (\mathcal{X}_{ij}) \cdot (Y_{ij}) = (Z_{ij})$$

with $Z_{ij} = \sum_{k=1}^{n} \mathcal{X}_{ik} \cdot Y_{kj}$. This last product being the associative product on $\overline{\mathbb{IR}}$. Thus $gl(n, \overline{\mathbb{IR}})$ is an associative algebra.

**Definition 42** *A matrix $A \in gl(n, \overline{\mathbb{IR}})$ is called invertible if its determinant, computed by the Cramer rule, is an invertible element in $\overline{\mathbb{IR}}$.*

Recall that the group $\overline{\mathbb{IR}}$ of invertible elements contain

$$\mathcal{X}_i = (K_i, 0) \text{ or } (0, K_i)$$

with $0 \notin K_i$. To compute the determinant, we use the classical formula of Cramer.

**Example 1.** Let us consider the matrix

$$M = \begin{pmatrix} [1,2] & [-1.3] \\ [-1,3] & [1,2] \end{pmatrix}$$

Thus

$$\begin{aligned}\det B_1 &= ([1,2],0)([1,2],0) \setminus ([-1,3],0)([-1,3],0) \\ &= ([1,4],0) \setminus ([-3,9],0) \\ &= ([0,[-4,5]) \\ &= \setminus([-4,5],0).\end{aligned}$$

As $([-4,5],0)$ is not an invertible element of $\overline{\mathbb{IR}}$, the matrix $B_1$ is not invertible.

**Example 2.** Now if

$$B_2 = \begin{pmatrix} [1,2] & [-1.3] \\ [-1,3] & [1,7] \end{pmatrix}$$

then, by the similar computation, we obtain

$$\det B_2 = \setminus([-7,-4],0)$$

and $B_2$ is invertible.

**Definition 43** *If $A$ is an invertible matrix on $gl(n,\overline{\mathbb{IR}})$, the inverse matrix $A^{-1}$ of $A$ is given by*

$$A \cdot A^{-1} = Id$$

*where*

$$Id = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

*with* $1 = ([1,1],0)$ *and* $0 = ([0,0],0)$.

The determination of $A^{-1}$ can be computed using the classical rules.

**Example.** If we consider the invertible matrix $B_2$, we obtain

$$B_2^{-1} = [\frac{1}{7},\frac{1}{4}]\begin{pmatrix} [1,7] & \setminus[-1.3] \\ \setminus[-1,3] & [1,2] \end{pmatrix}.$$

Let us verify that $B_2B_2^{-1} = Id$. Using the product on $\overline{\mathbb{IR}}$ we obtain

$$B_2B_2^{-1} = [\frac{1}{7},\frac{1}{4}]\begin{pmatrix} [1,2] & [-1.3] \\ [-1,3] & [1,7] \end{pmatrix} \cdot \begin{pmatrix} [1,7] & \setminus[-1.3] \\ \setminus[-1,3] & [1,2] \end{pmatrix}.$$

The coefficient in place $(1,1)$ is

$$a_{11} = [\frac{1}{7},\frac{1}{4}]([1,2][1,7] + [-1.3](\setminus[-1,3])).$$

¿From the definition of the product (see section 1), this element is

$$\begin{aligned}a_{11} &= (\frac{1}{7},\frac{1}{4},0,0)((1,2,0,0)(1,7,0,0) - (0,3,1,0)(0,3,1,0) \\ &= (\frac{1}{7},\frac{1}{4},0,0)((1,14,0,0) - (0,10,6,0)) \\ &= (\frac{1}{7},\frac{1}{4},0,0)(1,4,-6,0) \\ &= (\frac{1}{7},\frac{1}{4},0,0)(7,4,0,0) \\ &= (1,1,0,0)\end{aligned}$$

which corresponds to $[1,1]$. Similarly we have $a_{12} = a_{21} = (0,0,0,0)$ and $a_{22} = (1,1,0,0)$. Thus $B_2B_2^{-1} = Id$.

# 5.2 Diagonalization

## 5.2.1 Eigenvalues and central eigenvalues

Let $A$ be in $gl(n,\overline{\rm I\!R})$. An eigenvalue of $A$ is an element $\mathcal{X}\in\overline{\rm I\!R}$ such that there exists a vector $\mathcal{V}\neq 0\in\overline{\rm I\!R}^n$ with

$$A\cdot^t\mathcal{V}=\mathcal{X}\cdot^t\mathcal{V}.$$

Thus $\mathcal{X}$ is a root of the characteristically polynomial with coefficients in the ring $\overline{\rm I\!R}$

$$C_A(\mathcal{X})=\det(A-\mathcal{X}I)=0.$$

**Example.** Let

$$B_3=\begin{pmatrix}[1,2] & [1,2]\\ [1,3] & [2,5]\end{pmatrix}.$$

We have

$$B_3-\mathcal{X}I=\begin{pmatrix}[1,2]\setminus\mathcal{X} & [1,2]\\ [1,3] & [2,5]\setminus\mathcal{X}\end{pmatrix}$$

and

$$\begin{aligned}\det(B_3-\mathcal{X}I) &= ([1,2]\setminus\mathcal{X})([2,5]\setminus\mathcal{X})-[1,3][1,2]\\ &= [2,10]-\mathcal{X}[2,5]-\mathcal{X}[1,2]+(\setminus\mathcal{X})(\setminus\mathcal{X})-[1,6]\\ &= (\setminus\mathcal{X})(\setminus\mathcal{X})-\mathcal{X}[3,7]+[1,4].\end{aligned}$$

Let $\mathcal{X}=([x,y],0)$. It is represented in $\mathcal{A}_4$ by $(x,y,0,0)$ or $(0,y,x,0)$ or $(0,0,x,y)=-(x,y,0,0)$.

**First case**: $\det(B_3-\mathcal{X}I)=(x^2,y^2,0,0)-(3x,7y,0,0)+(1,4,0,0)=(x^2-3x+1,y^2-7y+4,0,0)$. Then $\det(B_3-\mathcal{X}I)=0$ implies

$$\begin{cases}x^2-3x+1=0,\\ y^2-7y+4=0,\end{cases}$$

that is

$$\begin{cases}x=\dfrac{3\pm\sqrt{5}}{2},\\ y=\dfrac{7\pm\sqrt{33}}{2}.\end{cases}$$

We obtain

$$\begin{cases}\mathcal{X}_1=([\dfrac{3+\sqrt{5}}{2},\dfrac{7+\sqrt{33}}{2}],0),\\ \mathcal{X}_2=([\dfrac{3-\sqrt{5}}{2},\dfrac{7+\sqrt{33}}{2}],0),\\ \mathcal{X}_3=([\dfrac{3-\sqrt{5}}{2},\dfrac{7-\sqrt{33}}{2}],0).\end{cases}$$

**Second case:** $\det(B_3-\mathcal{X}I)=(0,y^2+x^2,2xy,0)-(0,7y,7x,0)+(1,4,0,0)=(1,y^2+x^2-7y+4,2xy-7x,0)$. Then $\det(B_3-\mathcal{X}I)=0$ implies

$$\begin{cases}1-2xy+7x=0,\\ y^2+x^2-7y+4=0.\end{cases}$$

This gives

$$4y^4-56y^3+261y^2-455y+197=0.$$

We have the following solutions

$$(x;y)=\{(-2,8;3,32),(2,9;3,67),(-0,17;0,63),(0,17;6,37)\}.$$

We obtain the eigenvalues

$$\begin{cases} \mathcal{X}_4 = ([-2,8,\ 3.32],0), \\ \mathcal{X}_5 = ([-0.17,\ 0.63],0). \end{cases}$$

**Third case**: $\det(B_3 - \mathcal{X}I) = (x^2,y^2,0,0) + (3x,7y,0,0) + (1,4,0,0) = (x^2+3x+1, y^2+7y+4,0,0)$. Then $\det(B_3 - \mathcal{X}I) = 0$ implies

$$\begin{cases} x = \dfrac{-3 \pm \sqrt{5}}{2}, \\ y = \dfrac{-7 \pm \sqrt{33}}{2}, \end{cases}$$

then

$$\mathcal{X}_6 = ([\frac{-3-\sqrt{5}}{2}, \frac{-7+\sqrt{33}}{2}],0).$$

We obtain six eigenvalues.

**Remark.** To compute the interval-eigenvalues of a matrix $A$, we have to find the roots of the characteristically polynomial of $A$. But this polynomial is with coefficients in $\overline{\mathbb{IR}}$ (or $\mathcal{A}_4$) and this set is not a field neither a factorial ring. Then it is natural to meet some special results (e.g if we consider the second degree polynomial $X^2-1$ with coefficients in $\frac{\mathbb{Z}}{8\mathbb{Z}}$ which is not factorial, it admits four roots,$1,3,5,7$.) In our example we finds 6 roots. Now if we consider the real matrix whose coefficients are the centers of interval-coefficients of $B_3$, that is

$$c_{B_3} = \begin{pmatrix} 1.5 & 1.5 \\ 2 & 3.5 \end{pmatrix}$$

then the eigenvalues of $c_{B_3}$ are 4.5 and 0.5 which are closed to the center of $\mathcal{X}_1$ and $\mathcal{X}_3$. We call these eigenvalues, the central eigenvalues.

**Definition 44** *Let $A$ be a matrix in $gl(n,\overline{\mathbb{IR}})$. Lat $A_c$ be the real matrix whose elements are the center of the intervals of $A$. We say that an eigenvalue of $A$ is a central eigenvalue if its center is (close to) an eigenvalue of $A_c$.*

**Remark.** The determination of negative eigenvalues that is of type $(0,K)$ is similar. Nevertheless we have to consider only matrices with positive entries thus we studies only the positive eigenvalues. The negative eigenvalues do not correspond to physical entities.

### 5.2.2 Eigenvectors, eigenspaces

Now we will look the problem of reduction of an interval matrix. Recall that the characteristically polynomial is with coefficient in a non factorial ring. This is the biggest change with respect the classical real linear algebra.

**Definition 45** *Let $A$ a square matrix with coefficients in $\overline{\mathbb{IR}}$. If $\mathcal{X}$ is an eigenvalue of $A$, then every vector $\mathcal{V} \in \overline{\mathbb{IR}}^n$ satisfying $A^t\mathcal{V} = \mathcal{X}^t\mathcal{V}$ is an eigenvector associated with $\mathcal{X}$.*

Let $E_{\mathcal{X}}$ be the set

$$E_{\mathcal{X}} = \{\mathcal{V} \in \overline{\mathbb{IR}}^n \text{ such that } A^t\mathcal{V} = \mathcal{X}^t\mathcal{V}\}.$$

Then $E_{\mathcal{X}}$ is a $\mathbb{R}$-subspace of $\overline{\mathbb{IR}}^n$ where $n$ is the order of the matrix $A$. It is also a $\overline{\mathbb{IR}}$ submodule of $\overline{\mathbb{IR}}^n$.

**Proposition 32** *Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be two distinguish eigenvalues of $A$. Then $E_{\mathcal{X}_1} \cap E_{\mathcal{X}_2} = \{0\}$.*

*Proof.* Let $\mathcal{V}$ be in $E_{\mathcal{X}_1} \cap E_{\mathcal{X}_2}$. We have

$$\begin{array}{rcl} A^t\mathcal{V} & = & \mathcal{X}_1\mathcal{V}, \\ A^t\mathcal{V} & = & \mathcal{X}_2\mathcal{V}. \end{array}$$

This $\mathcal{X}_1\mathcal{V} \setminus \mathcal{X}_2\mathcal{V} = (\mathcal{X}_1 \setminus \mathcal{X}_2)\mathcal{V} = 0$. As $\overline{\mathbb{IR}}$ is without zero divisor, we have $\mathcal{X}_1 \setminus \mathcal{X}_2 = 0$ or $\mathcal{V} = 0$. We deduce $E_{\mathcal{X}_1} \cap E_{\mathcal{X}_2} = \{0\}$.

**Proposition 33** *Let $C_A(\mathcal{X})$ be the characteristically polynomial of $A$. If the real polynomial $C_{C_A}(\mathcal{X})$ associated with the central matrix of $A$ is a product of factor of degree 1, then $C_A(\mathcal{X})$ admits a factorization on* $\overline{\mathbb{IR}}$

We have seen that $C_A(\mathcal{X})$ can be have more than degree($C_A(\mathcal{X})$ roots. If $\mathcal{X}_1, \cdots, \mathcal{X}_n$ are the central roots, we have the decomposition

$$C_A(\mathcal{X}) = a_n \prod_{i=1}^{n} (\mathcal{X} \setminus \mathcal{X}_i).$$

**Example.** If we consider the matrix

$$B_3 = \begin{pmatrix} [1,2] & [1,2] \\ [1,3] & [2,5] \end{pmatrix}.$$

then $C_{B_3}(\mathcal{X})$ admits $\mathcal{X}_1, \cdots, \mathcal{X}_6$ as positive roots. The central eigenvalues are $\mathcal{X}_1$ and $\mathcal{X}_3$ and we have

$$\det(B_3 \setminus \mathcal{X}I) = (\mathcal{X} \setminus \mathcal{X}_1)(\mathcal{X} \setminus \mathcal{X}_3).$$

If we consider the roots $\mathcal{X}_2 = ([\frac{3-\sqrt{5}}{2}, \frac{7+\sqrt{33}}{2}], 0)$, and if we assume that $C_{B_3s}(\mathcal{X}) = (\mathcal{X} \setminus \mathcal{X}_2)(\mathcal{X} \setminus Y)$, we obtain

$$Y = (3, 7, \frac{3-\sqrt{5}}{2}, \frac{7+\sqrt{33}}{2}) = (\frac{3+\sqrt{5}}{2}, \frac{7-\sqrt{33}}{2}, 0, 0)$$

which does not correspond to a positive eigenvalue.

**Theorem 46** *For any n-uple of roots $(\mathcal{X}_1, \cdots, \mathcal{X}_n)$ such that $C_A(\mathcal{X}) = a_n \prod_{i=1}^{n} (\mathcal{X} \setminus \mathcal{X}_i)$, and if for any $i = 1, \cdots, n$ the dimension of $E_{\mathcal{X}_i}$ coincides with the multiplicity of $\mathcal{X}_i$, then we have the vectorial decomposition $\overline{\mathbb{IR}}^n = \oplus_{i \in I} E_{\mathcal{X}_i}$ where the roots $\mathcal{X}_i, i \in I$ are pairwise distinguish.*

**Example.** Let us compute the eigenspaces of $B_3$ associated to the central eigenvalues.

- $\mathcal{X}_1 = ([\frac{3+\sqrt{5}}{2}, \frac{7+\sqrt{33}}{2}], 0)$.

  Let $V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \in \overline{\mathbb{IR}}$. Then

  $$(A - \mathcal{X}_1 I) \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = 0$$

  is equivalent to

  $$\begin{pmatrix} (0, [\frac{1+\sqrt{5}}{2}, \frac{3+\sqrt{33}}{2}])V_1 & ([1,2], 0)V_2 \\ ([1.3], 0)V_1 & (0, [\frac{-1+\sqrt{5}}{2}, \frac{-3+\sqrt{33}}{2}])V_2 \end{pmatrix} = 0$$

that is

$$\begin{cases} \nwarrow[\frac{1+\sqrt{5}}{2}, \frac{3+\sqrt{33}}{2}]V_1 + [1,2]V_2 = 0, \\ [1.3]V_1 \nwarrow [\frac{-1+\sqrt{5}}{2}, \frac{-3+\sqrt{33}}{2}]V_2 = 0. \end{cases}$$

This gives

$$V_2 = \frac{[\frac{1+\sqrt{5}}{2}, \frac{3+\sqrt{33}}{2}]V_1}{[1,2]}.$$

If we choose $V_1 = ([1,1],0)$ we have

$$\begin{aligned} V_2 &= \frac{[\frac{1+\sqrt{5}}{2}, \frac{3+\sqrt{33}}{2}]}{[1,2]} \\ &= [\frac{-1+\sqrt{5}}{2}, \frac{-3+\sqrt{33}}{2}] \bullet (\nwarrow[-1, \tfrac{-1}{2}]) \\ &= \nwarrow([-\tfrac{3+\sqrt{33}}{2}, -\tfrac{1+\sqrt{5}}{4}]) \end{aligned}$$

Thus the $\mathcal{X}_1$-eigenvectors are of the form

$$V = \begin{pmatrix} ([1,1],0) \\ \nwarrow([-\frac{3+\sqrt{33}}{2}, -\frac{1+\sqrt{5}}{4}],0) \end{pmatrix}.$$

**Remark.** We can choose $V_1$ such that all the coordinates of $V$ are positive. For example if $V_1 = [1,2]$ then $V = \begin{pmatrix} ([1,2],0) \\ ([\frac{1+\sqrt{5}}{2}, \frac{3+\sqrt{33}}{2}],0) \end{pmatrix}$

- $\mathcal{X}_3 = ([\frac{3-\sqrt{5}}{2}, \frac{7-\sqrt{33}}{2}],0)$.

  Let $V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \in \overline{\mathbb{IR}}$. Then

$$(A - \mathcal{X}_1 I)\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = 0$$

is equivalent to

$$\begin{pmatrix} ([\frac{-1+\sqrt{5}}{2}, \frac{-3+\sqrt{33}}{2}],0)V_1 & ([1,2],0)V_2 \\ ([1.3],0)V_1 & ([\frac{1+\sqrt{5}}{2}, \frac{3+\sqrt{33}}{2}],0)V_2 \end{pmatrix} = 0$$

that is

$$\begin{cases} [\frac{-1+\sqrt{5}}{2}, \frac{-3+\sqrt{33}}{2}]V_1 + [1,2]V_2 = 0, \\ [1.3]V_1 + [\frac{1+\sqrt{5}}{2}, \frac{-3+\sqrt{33}}{2}]V_2 = 0. \end{cases}$$

This gives

$$V_2 = \frac{\nwarrow([\frac{-1+\sqrt{5}}{2}, \frac{-3+\sqrt{33}}{2}])V_1}{[1,2]}.$$

If we choose $V_1 = ([1,1],0)$ we have

$$\begin{aligned} V_2 &= \frac{\searrow[\frac{-1+\sqrt{5}}{2}, \frac{-3+\sqrt{33}}{2}]}{[1,2]} \\ &= \searrow[\frac{-1+\sqrt{5}}{2}, \frac{-3+\sqrt{33}}{2}] \bullet (\searrow[-1, \frac{-1}{2}]) \\ &= ([\tfrac{3-\sqrt{33}}{2}, \tfrac{1-\sqrt{5}}{4}]) \end{aligned}$$

Thus the $\mathcal{X}_3$-eigenvectors are of the form

$$V = \begin{pmatrix} ([1,1],0) \\ ([\frac{3-\sqrt{33}}{2}, \frac{1-\sqrt{5}}{4}],0) \end{pmatrix}.$$

## 5.3 The Exponential map

We define the exponential map

$$Exp : gl(n, \overline{\mathbb{IR}}) \longrightarrow gl(n, \overline{\mathbb{IR}})$$

in a classical way by series expansions. If the matrix $A$ is diagonalizable, then

$$D = P^{-1}AP$$

is diagonal and $Exp(A)$ is a diagonal matrix whose diagonal element are the exponential of the eigenvalues.
**Example.** Let

$$B_3 = \begin{pmatrix} [1,2] & [1,2] \\ [1,3] & [2,5] \end{pmatrix}.$$

The central eigenvalues are

$$\begin{cases} \mathcal{X}_1 = ([\frac{3+\sqrt{5}}{2}, \frac{7+\sqrt{33}}{2}],0), \\ \mathcal{X}_3 = ([\frac{3-\sqrt{5}}{2}, \frac{7-\sqrt{33}}{2}],0). \end{cases}$$

and we have

$$D = \begin{pmatrix} \mathcal{X}_1 & 0 \\ 0 & \mathcal{X}_3 \end{pmatrix}$$

with

$$P = \begin{pmatrix} ([1,1],0) & ([1,1],0) \\ \searrow([-\frac{3+\sqrt{33}}{2}, -\frac{1+\sqrt{5}}{4}],0) & ([\frac{3-\sqrt{33}}{2}, \frac{1-\sqrt{5}}{4}],0) \end{pmatrix}.$$

We deduce

$$Exp(B_3) = P. \begin{pmatrix} ([exp(\frac{3+\sqrt{5}}{2}), exp(\frac{7+\sqrt{33}}{2}],0)) & 0 \\ 0 & ([exp(\frac{3-\sqrt{5}}{2}), exp(\frac{7-\sqrt{33}}{2}],0)) \end{pmatrix} .P^{-1}$$

In a forthcoming paper, we apply this calculus to solve linear differential system.

# Chapter 6

# The arithmetic of infinitesimal intervals

## 6.1 Introduction

The intervals arithmetic is used in many domains. For example, in the geometrical conception of a robot.It is necessary to look if the set of parameters does not contain some singularities. If we have to consider some local minimal points, thus the system depends weakly of initial data. The problem of conditioning of a robot can be approached by intervals arithmetic. A second important area of applications are problems where one are obliged to take into account of some uncertainty. These appear when the parameters are not in a reality given by real number (e.g the temperature, the degree of humidity..). Then the parameter have to be substituted to an infinitesimal intervals containing all the possible values of the parameter. This appear too in a computer calculus. A real number is not represented by a element of the real field but by an interval. If the calculations are long or recurrent, it implies an accumulation of mistakes In this chapter we present a study of infinitesimal intervals based :

1. On a non standard approach of infinitesimal numbers.

2. On the algebraic model of the set of intervals developed in the previous chapters.

## 6.2 Infinitesimal numbers

### 6.2.1 What are infinitesimal numbers

In 1964, A.Robinson [57] proposed a non archimedean extension of field, denoted by $\mathbb{R}^*$, of the field $\mathbb{R}$ of real numbers permitting to obtain a notion of infinitesimal numbers from their natural property

$$\alpha \simeq 0 \Longleftrightarrow |\alpha| < x \text{ for any } x \in \mathbb{R}$$

where $\simeq 0$ means infinitesimal. Such an element $\alpha \simeq 0$ belongs to $\mathbb{R}^* - \mathbb{R}$. We note that this relation on $\mathbb{R}$ has no sense. In fact, the relation $|\alpha| < x$ for any $x \in \mathbb{R}$ implies $\alpha = 0$. This is a consequence of the archimedean property of $\mathbb{R}^*$. And $\mathbb{R}$ doesn't contain infinitesimal elements except zero, infinitesimal can be understood in the natural sense, a number smallest that all the classical numbers. With the Robinson extension, one sees that infinitesimals belong to $\mathbb{R}^* - \mathbb{R}$. The elements of $\mathbb{R}^*$ which belong to $\mathbb{R}$ are the standard numbers, that is numbers constructed by standard process. Moreover, $\mathbb{R}^*$ appears as the smallest extension of the field $\mathbb{R}$ which contains infinitesimal numbers. Thus, all writings concerned by the classic infinitesimal calculus express themselves very merely in Robinson's extension. One can hope therefore that the use of the infinitesimal numbers facilitates the calculations. For example, classically, we have to use three quantificators to express that $f$ is derivable at $x_0$ with derivative equal to $a$:

$$\forall \varepsilon > 0, \exists \eta, \forall x, \ \mid x - x_0 \mid < \eta \Rightarrow \mid \frac{f(x) - f(x_0)}{x - x_0} \mid < \varepsilon.$$

Using infinitesimal numbers, the same identity writes:

$$\mid \frac{f(x_0 + \varepsilon) - f(x_0)}{\varepsilon} \mid \simeq a.$$

### 6.2.2 Characteristically properties of $\mathbb{R}^*$

1. $\mathbb{R}^*$ is a non-archimedean field that is

$$\exists a, b \in \mathbb{R}^*, \forall n \in \mathbb{N}, na < b$$

containing $\mathbb{R}$ as a subfield.

2) $\mathbb{R}^*$ is a valued field. It exists a valuation $v$ on $\mathbb{R}^*$, that is a map

$$v : \mathbb{R}^* \longrightarrow G \cup \infty$$

where $G$ is an totaly ordered abelian group, satisfying

$$\begin{cases} v(xy) = v(x) + v(y), \\ v(x + y) \geq \min(v(x), v(y)), \\ v(x) = \infty \Longleftrightarrow x = 0. \end{cases}$$

We denote by $\mathcal{L}$ the associated valuation ring, that is

$$\mathcal{L} = \{x \in \mathbb{R}^*; v(x) \geq 0\}.$$

It is a local ring admitting an unique maximal ideal $\mathfrak{m}$

$$\mathfrak{m} = \{x \in \mathbb{R}^*; v(x) > 0\}.$$

3) Let $\mathfrak{m}$ be the maximal ideal of the local ring $\mathfrak{L}$. The quotient ring $\frac{\mathcal{L}}{\mathfrak{m}}$ is a field ( because $\mathfrak{m}$ is maximal) isomorphic to $\mathbb{R}$ satisfying

$$x \in \mathbb{R}^* - \mathcal{L} \Longleftrightarrow x^{-1} \in \mathfrak{m}.$$

If we interpreter the elements of $\mathbb{R}^* - \mathcal{L}$ as infinitely large elements, the previous relation means that $x$ is an infinitely large element if and only is the converse $x^{-1}$ is infinitesimal. For the reasons $\mathcal{L}$ is called the ring of limited numbers (limited is synonymous with not infinitely large) and $\mathfrak{m}$ the ideal of infinitesimal numbers.

**Remark.** E. Nelson, in 1976, proposed another approach of infinitesimal numbers, called IST and based on a conservative extension of the theory of sets from the axioms of Zermelo-Fraenkel. In this framework the infinitesimals are elements of $\mathbb{R}$ and the classical real numbers are expressed by a predicate "standard". This approach is undoubtedly attractive. But contrary to what claim The adepts to this approach, this theory is relatively complicated. Proof is that all articles using IST, devotes a third of their contents to recall what the IST is. We prefer to stay here in the algebraic approach proposed by Robinson.

### 6.2.3 Operations in the field $\mathbb{R}^*$

Practically, in $\mathbb{R}^*$ we have two types of elements: the elements of $\mathbb{R}$, called standard and the elements of $\mathbb{R}^* - \mathbb{R}$ called of course non standard. Among the elements of $\mathbb{R}^*$ we have

- the elements of $\mathcal{L}$, called limited or not infinitely large (all standard element is limited).
- the elements of $\mathfrak{m}$ called infinitesimal.
- the elements of $\mathbb{R}^* - \mathcal{L}$ called infinitely large.

Thus we find again the classical Leibniz rules:

- limited + limited = limited
- limited × limited = limited
- infinitesimal × limited = infinitesimal
- infinitesimal + infinitesimal = infinitesimal
- 1/infinitesimal = infinitely large
- infinitely large × (limited not infinitesimal) = infinitely large.

### 6.2.4 Relation between $\mathbb{R}^*$ and $\mathbb{R}$

The most important property that joins the field $\mathbb{R}$ and its extension $\mathbb{R}^*$ is the following : let us consider a non infinitely large element $\alpha \in \mathbb{R}^*$, that is $\alpha \in \mathcal{L}$. Then, there is a unique element $a \in \mathbb{R}$ such that

$$\alpha - a \in \mathfrak{m}$$

that is this difference is an infinitesimal number. It is clear that, if $\alpha \in \mathbb{R}$, thus $a = \alpha$. This number $a$ is classically denoted by $^\circ\alpha$. The map

$$^\circ : \mathcal{L} \rightarrow \mathbb{R}$$

is a surjective $\mathbb{R}$-linear map and a ring homomorphism.

If $f$ is a real-valued function of a real variable$x \in \mathbb{R}$, defined on an open interval $]a, b[$, on passing in $\mathbb{R}^*$, $f(x)$ is extended to a function which is defined for all number $x \in \mathbb{R}^*$ such that $a < x < b$. Usually we denote by the same letter the initial function and its transferred. For example, a function $f(x)$ is continuous at the point $x_0 \in \mathbb{R}$, if and only if the transferred function satisfies

$$f(x) \simeq f(x_0), \quad \forall x \simeq x_0, \ x \in \mathbb{R}^*,$$

where $x \simeq x_0$ means $x - x_0 \in \mathfrak{m}$.

## 6.3 Arithmetic of halos

The notion of halo corresponds to a general infinitesimal neighborhood of a point $a \in \mathbb{R}$. In this section we shall construct an arithmetic on the set of halos analogous to the arithmetic of intervals.

**Definition 47** *Let $a$ in $\mathbb{R}$. The halo of $a$ denoted by $h(a)$, is a subset of $\mathbb{R}^*$ defined by*

$$h(a) = a + \mathfrak{m}$$

*where $\mathfrak{m}$ is the maximal ideal of infinitesimals.*

Thus for any point $\alpha \in h(a)$ we have

$$^{\circ}\alpha = a$$

and

$$\mathcal{L} = \bigcup_{a \in \mathbb{R}} h(a).$$

Let $\mathcal{H}$ be the subset of $\mathcal{P}(\mathbb{R}^*)$ whose elements are the halos of elements of $\mathbb{R}$. We define of $\mathcal{H}$ the following operation

$$\begin{cases} h(a) + h(b) = h(a+b), \ \forall a, b \in \mathbb{R}, \\ \lambda h(a) = h(\lambda a), \ \forall a, \lambda \in \mathbb{R}. \end{cases}$$

This addition is associative and commutative. It has an identity element $h(0) = \mathfrak{m}$ because $h(a)+h(0) = h(a)$. Then we have

**Proposition 34** *The set $\mathcal{H}$ is real vector space.*

As $h(a) = ah(1)$, this space is of dimension 1. If we put $||h(a)|| = |a|$, we endow the space $\mathcal{H}$ of a normed complete vector space.

We can define also an internal multiplication on $\mathcal{H}$ , putting :

$$h(a)h(b) = h(ab).$$

This multiplication is distributive and the vector space with this multiplication is an algebra (and even a field ) isomorphic to $\mathbb{R}$.

**Proposition 35** *The external multiplication*

$$(\lambda, h(a)) \in \mathcal{L} \times \mathcal{H} \longmapsto h(^{\circ}\lambda a).$$

*endows the abelian group $\mathcal{H}$ with a $\mathcal{L}$-module structure.*

This module is finitely generated and of rank 1. In fact any element $h(a)$ is written $h(a) = ah(1)$. But this module is not free. In fact we have $\varepsilon h(1) = h(0) = \mathfrak{m}$. For any $n \in N$, $n \neq 0$, the module

$$\mathcal{H}^n = \{h(a), \ a \in \mathbb{R}^n\}.$$

is isomorphic to the cartesian product $n$ times of $\mathcal{H}$ is a finitely generated $\mathcal{L}$-module of rank $n$. But this module is not free. This structure have been studied in [58] and called neutrices.

## 6.4 Infinitesimal intervals

### 6.4.1 The module $\overline{\mathcal{I}}$

The notion of halo of a point $a$ in $\mathbb{R}$ corresponds to a general notion of infinitesimal neighborhood. That amounts to give a real number $a$ up to an infinitesimal incertitude but in some problems the data are given with a precise incertitude. In this case the real datum belong to an interval of the type $[a-\varepsilon, a+\varepsilon]$ with $\varepsilon$ infinitesimal, $\varepsilon \geq 0$. Thus we have to define an arithmetic of the sets of such intervals.

Let $a \in \mathbb{R} \subset \mathbb{R}^*$. We denote by $I_a(\varepsilon)$ the interval $[a-\epsilon, a+\epsilon]$ in $\mathbb{R}^*$ with $\epsilon \in \mathfrak{m}$ , and by $\mathcal{I}$ the set

$$\mathcal{I} = \{I_a(\epsilon), a \in \mathbb{R}, \epsilon \in \mathfrak{m}\}.$$

We put

$$I_a(\epsilon_1) + I_b(\epsilon_2) = I_{a+b}(\epsilon_1 + \epsilon_2).$$

This operation is associative commutative and $I_0(0)$ is an unit. Thus $\mathcal{I}$ is a semi-group. Following the approach developed in the previous chapter, we construct a natural vectorial structure on the symmetrized $\overline{\mathcal{I}}$ of the regular semi-group $(\mathcal{I},+)$. Recall that $\overline{\mathcal{I}}$ is the quotient set associated with the equivalence relation

$$(I_a(\epsilon_1), I_b(\epsilon_2)) \sim (I_c(\epsilon_3), I_d(\epsilon_4)) \Longleftrightarrow I_a(\epsilon_1) + I_d(\epsilon_4) = I_b(\epsilon_2) + I_c(\epsilon_3).$$

We denote by

$$\overline{(I_a(\epsilon_1), I_b(\epsilon_2))}$$

the equivalent class of

$$(I_a(\epsilon_1), I_b(\epsilon_2)).$$

In particular the class of $\overline{(0,0)}$ is

$$\overline{(0,0)} = \{(I_a(\epsilon), I_a(\epsilon)), a \in \mathbb{R}, \epsilon \in \mathfrak{m}\}$$

where 0 is the interval $[0,0]$. In this context we can define the opposite of $\overline{(I_a(\epsilon_1), I_b(\epsilon_2))}$ by

$$\smallsetminus \overline{(I_a(\epsilon_1), I_b(\epsilon_2))} = \overline{(I_b(\epsilon_2), I_a(\epsilon_1))}.$$

This gives

$$\overline{(I_a(\epsilon_1), I_b(\epsilon_2))} \smallsetminus \overline{(I_c(\epsilon_3), I_d(\epsilon_4))} = \overline{(I_a(\epsilon_1) + I_d(\epsilon_4), (I_c(\epsilon_3) + I_b(\epsilon_2))}.$$

This defines a structure of abelian group on $\overline{\mathcal{I}}$ and every element of $\overline{\mathcal{I}}$ writes $\overline{(I_a(\epsilon), 0)}$ or $\overline{(0, I_a(\epsilon))}$.

**Theorem 48** *The group $\overline{\mathcal{I}}$ is provided with a structure of $\mathbb{R}$-vector space and of $\mathcal{L}$-module.*

*Proof.* We have seen that $\overline{\mathcal{I}}$ is an abelian group for the addition. Let $\lambda \in \mathcal{L}$. We put

$$\begin{cases} 1) \text{ If } \lambda > 0, \begin{cases} \lambda\overline{(I_a(\epsilon), 0)} = \overline{(I_{\lambda a}(\lambda\epsilon), 0)} \\ \lambda\overline{(0, I_a(\epsilon))} = \overline{(0, I_{\lambda a}(\lambda\epsilon))} \end{cases}, \\ 2) \text{ If } \lambda < 0, \begin{cases} \lambda\overline{(I_a(\epsilon), 0)} = \overline{(0, I_{-\lambda a}(-\lambda\epsilon))} \\ \lambda\overline{(0, I_a(\epsilon))} = \overline{(I_{-\lambda a}(-\lambda\epsilon), 0)} \end{cases}. \end{cases}$$

We deduce

$$(-1)\overline{(I_a(\epsilon), 0)} = \overline{(0, I_a(\epsilon))} = \smallsetminus \overline{(I_a(\epsilon), 0)}.$$

The external multiplication verifies the axioms of modules. If we restrict the scalar $\lambda$ to $\mathbb{R}$ (recall that $\mathbb{R}$ in included in $\mathcal{L}$), we obtain the structure of $\mathbb{R}$-vector space on $\overline{\mathcal{I}}$. But the dimension of this real vector space is not finite.

### 6.4.2 Decomposition of an infinitesimal of $\mathbb{R}^{*n}$.

**Theorem 49** *Let $(\epsilon_1, \cdots, \epsilon_n)$ be $n$ element of $\mathfrak{m}$. Then there are $k$ linear independent vectors in $\mathbb{R}^n$, $k \leq n$, such that*

$$(\epsilon_1, \cdots, \epsilon_n) = \alpha_1 V_1 + \alpha_1\alpha_2 V_2 + \cdots + \alpha_1 \cdots \alpha_k V_k$$

*with $\alpha_i \in \mathfrak{m} - \{0\}$. This decomposition is unique if we assume that the frame $\{V_1, \cdots, V_k\}$ is an orthonormed frame.*

Let's examine the particular case $n = z$. The decomposition of $(\epsilon_1, \epsilon_2)$ writes

$$(\epsilon_1, \epsilon_2) = \alpha_1 V_1 + \alpha_1\alpha_2 V_2$$

with $\alpha_i \simeq 0$. If $V_2 \neq 0$, then $(V_1, V_2)$ are linearly independent. If we put $V_1 = (v_1, v_2)$ and $V_2 = (w_1, w_2)$, the decomposition is equivalent to

$$\begin{cases} \epsilon_1 = \alpha_1 v_1 + \alpha_1\alpha_2 w_1, \\ \epsilon_2 = \alpha_1 v_2 + \alpha_1\alpha_2 w_2. \end{cases}$$

If $v_1v_2 \neq 0$, thus the quotient $\frac{\epsilon_1}{\epsilon_2}$ is a an infinitesimal number and

$$\circ(\frac{\epsilon_1}{\epsilon_2}) = \frac{v_1}{v_2}.$$

In this case, we shall said that $\epsilon_1$ and $\epsilon_2$ are equivalent. If $v_1v_2 = 0$, for example if $v_2 = 0$, then

$$\left\{ \begin{array}{l} \epsilon_1 = \alpha_1 v_1 + \alpha_1\alpha_2 w_1, \\ \epsilon_2 = \alpha_1\alpha_2 w_2. \end{array} \right.$$

In this case

$$\frac{\epsilon_2}{\epsilon_1} = \frac{\alpha_1\alpha_2 w_2}{\alpha_1 v_1 + \alpha_1\alpha_2 w_1} = \alpha_2 \frac{w_2}{v_1 + \alpha_2 w_1}$$

and $\circ(\frac{\epsilon_2}{\epsilon_1}) = 0$. Thus $\frac{\epsilon_2}{\epsilon_1}$ is infinitesimal.

### 6.4.3 Equivalent infinitesimal intervals

If $I_a(\epsilon) \in \mathcal{I}$, we denote by $J_a(\epsilon)$ the class in $\overline{\overline{\mathcal{I}}}$ of $(I_a(\epsilon), 0)$. Thus $\diagdown J_a(\epsilon)$ is the class of $(0, I_a(\epsilon))$.

**Definition 50** *Let $J_a(\epsilon_1)$ and $J_a(\epsilon_2)$ be in $\overline{\overline{\mathcal{I}}}$. They are called equivalent if the decomposition*

$$(\epsilon_1, \epsilon_2) = \alpha_1 V_1 + \alpha_1\alpha_2 V_2$$

*with $\alpha_i \in m - \{0\}$ and $V_1 = (v_1, v_2)$ satisfies $v_1v_2 \neq 0$.*

This relation is an equivalence relation. We denote by $\widetilde{\mathcal{I}}$ the quotient set and $\widetilde{J}_a(\epsilon)$ the class of $J_a(\epsilon)$. Thus

$$\widetilde{J}_a(\epsilon) = \{(\overline{[a-\epsilon_1, a+\epsilon_1], 0}), \epsilon_1 = \rho_1\epsilon + \epsilon\epsilon' \text{ with } \rho_1 \in \mathbb{R} - \{0\}\}.$$

**Proposition 36** *The quotient set $\widetilde{\mathcal{I}}$ is a $\mathbb{R}-$vector space and a $\mathcal{L}-$module.*

*Proof:* In fact the addition is given by $\widetilde{J}_a(\epsilon_1) + \widetilde{J}_b(\epsilon_2) = \widetilde{J}_{a+b}(\epsilon_1 + \epsilon_2)$. The unitary element is $\widetilde{J}_0(0)$. If $b \in \mathbb{R}$, we put

$$b\widetilde{J}_a(\epsilon) = \left\{ \begin{array}{l} \widetilde{J}_{ba}(\epsilon) \text{ if } b \geq 0, \\ \diagdown \widetilde{J}_{ba}(\epsilon) \text{ if } b < 0. \end{array} \right.$$

Thus $\widetilde{\mathcal{I}}$ is a $\mathbb{R}-$vector space. If $\rho \in \mathcal{L}$, we put

$$\rho\widetilde{J}_a(\epsilon) = \left\{ \begin{array}{l} \widetilde{J}_{\circ\rho a}(\epsilon) \text{ if } \rho > 0 \text{ and } \rho \notin \mathfrak{m}, \\ \widetilde{J}_0(\rho\epsilon) \text{ if } \rho > 0 \text{ and } \rho \in \mathfrak{m}. \end{array} \right.$$

If $\rho < 0$, we use the signs rule. This product provides $\widetilde{\mathcal{I}}$ with a $\mathcal{L}$-module structure.

### 6.4.4 A multiplication in $\mathcal{I}$

To define a multiplication distributive with respect the addition, we use the product of $\mathcal{A}_4$ (see Chapter 4).

Assume that $\varepsilon \leq 0,\ \varepsilon \in \mathfrak{m}$.

- If $a > 0$, thus

$$[a-\varepsilon, a+\varepsilon] = (a-\varepsilon)e_1 + 2\varepsilon e_2.$$

- If $a < 0$, thus

$$[a-\varepsilon, a+\varepsilon] = (-a-\varepsilon)e_4 + 2\varepsilon e_3.$$

A direct computation of the multiplication gives

1. If $a > 0,\ b > 0,$

$$\begin{aligned}[a-\varepsilon_1, a+\varepsilon_1][b-\varepsilon_2, b+\varepsilon_2] &= ((a-\varepsilon_1)e_1 + 2\varepsilon_1 e_2)((b-\varepsilon_2)e_1 + 2\varepsilon_2 e_2)\\ &= (a-\varepsilon_1)(b-\varepsilon_2)e_1 + 2(a\varepsilon_2 + b\varepsilon_1)e_2\\ &= [ab - a\varepsilon_2 - b\varepsilon_1 + \varepsilon_1\varepsilon_2, ab + a\varepsilon_2 + b\varepsilon_1 + \varepsilon_1\varepsilon_2].\end{aligned}$$

2. If $a > 0,\ b < 0,$

$$\begin{aligned}[a-\varepsilon_1, a+\varepsilon_1][b-\varepsilon_2, b+\varepsilon_2] &= ((a-\varepsilon_1)e_1 + 2\varepsilon_1 e_2)((-b-\varepsilon_2)e_4 + 2\varepsilon_2 e_3)\\ &= (a-\varepsilon_1)(-b-\varepsilon_2)e_4 + 2(a\varepsilon_2 - b\varepsilon_1)e_3\\ &= [ab - a\varepsilon_2 + b\varepsilon_1 + \varepsilon_1\varepsilon_2, ab + a\varepsilon_2 - b\varepsilon_1 - \varepsilon_1\varepsilon_2].\end{aligned}$$

3. If $a < 0,\ b < 0,$

$$\begin{aligned}[a-\varepsilon_1, a+\varepsilon_1][b-\varepsilon_2, b+\varepsilon_2] &= ((-a-\varepsilon_1)e_4 + 2\varepsilon_1 e_3)((-b-\varepsilon_2)e_4 + 2\varepsilon_2 e_3)\\ &= (-a-\varepsilon_1)(-b-\varepsilon_2)e_4 + 2(-a\varepsilon_2 - b\varepsilon_1)e_3\\ &= [ab + a\varepsilon_2 + b\varepsilon_1 + \varepsilon_1\varepsilon_2, ab - a\varepsilon_2 - b\varepsilon_1 + \varepsilon_1\varepsilon_2].\end{aligned}$$

Each one of these results do not belong to $\mathcal{I}$.

**The product of first approximation.** Let us consider the decomposition of $(\varepsilon_1, \varepsilon_2)$:

$$(\epsilon_1, \epsilon_2) = \alpha_1 V_1 + \alpha_1\alpha_2 V_2$$

with $\alpha_i \simeq 0$ and if $V_2 \neq 0$, then the vectors $(V_1, V_2)$ are linearly independent. If we put $V_1 = (v_1, v_2)$ and $V_2 = (w_1, w_2)$, the decomposition is equivalent to

$$\begin{cases} \epsilon_1 = \alpha_1 v_1 + \alpha_1\alpha_2 w_1,\\ \epsilon_2 = \alpha_1 v_2 + \alpha_1\alpha_2 w_2.\end{cases}$$

As we assume that $\varepsilon_i \geq 0$ for $i = 1$ and 2, the infinitesimal elements $\alpha_i$ also are positif. Likewise, the components of the vectors $V_i$ are not negative. Let us examinate each one of the previous case

1. If $a > 0,\ b > 0,$

$$a\varepsilon_2 + b\varepsilon_1 = \alpha_1(av_2 + bv_1) + \alpha_1\alpha_2(aw_2 + bw_1)$$

and $\rho = av_2 + bv_1 = \langle (v_1, v_2), (b, a)\rangle > 0$ where $\langle V, W\rangle$ is the classical inner product on $\mathbb{R}^2$.

**Definition 51** *If $a > 0,\ b > 0$, the product of first approximation of the infinitesimal intervals $[a-\varepsilon_1, a+\varepsilon_1]$ and $[b-\varepsilon_2, b+\varepsilon_2]$ is*

$$[a-\varepsilon_1, a+\varepsilon_1].[b-\varepsilon_2, b+\varepsilon_2] = [ab - \alpha_1\rho, ab + \alpha_1\rho]$$

*where $\rho = av_2 + bv_1 = \langle (v_1, v_2), (b, a)\rangle$.*

2. If $a > 0,\ b < 0,$

$$a\varepsilon_2 - b\varepsilon_1 = \alpha_1(av_2 - bv_1) + \alpha_1\alpha_2(aw_2 - bw_1)$$

and $\rho = av_2 - bv_1 = \langle (v_1, v_2), (-b, a)\rangle > 0.$

**Definition 52** *If $a > 0,\ b < 0$, the product of first approximation of the infinitesimal intervals $[a-\varepsilon_1, a+\varepsilon_1]$ and $[b-\varepsilon_2, b+\varepsilon_2]$ is*

$$[a-\varepsilon_1, a+\varepsilon_1].[b-\varepsilon_2, b+\varepsilon_2] = [ab - \alpha_1\rho, ab + \alpha_1\rho]$$

*where $\rho = av_2 - bv_1 = \langle (v_1, v_2), (-b, a)\rangle$.*

3. If $a<0,\ b<0$,

$$-a\varepsilon_2 - b\varepsilon_1 = \alpha_1(-av_2 - bv_1) + \alpha_1\alpha_2(-aw_2 - bw_1)$$

and $\rho = -av_2 - bv_1 = \langle (v_1, v_2), (-b, -a)\rangle > 0$.

**Definition 53** *If $a<0,\ b<0$, the product of first approximation of the infinitesimal intervals $[a-\varepsilon_1, a+\varepsilon_1]$ and $[b-\varepsilon_2, b+\varepsilon_2]$ is*

$$[a-\varepsilon_1, a+\varepsilon_1].[b-\varepsilon_2, b+\varepsilon_2] = [ab - \alpha_1\rho, ab + \alpha_1\rho]$$

*where $\rho = -av_2 - bv_1 = \langle (v_1, v_2), (-b, -a)\rangle$.*

**The product of second approximation.**

If the first approximation is too coarse, that is if the result gives one interval whose infinitesimal length is too imprecise, it will be necessary to describe this length to an infinitesimal of order 2. We will precise this notion. If the length of the interval is an infinitesimal $\varepsilon$, this length will be given to an infinitesimal of order 2 if it is given by an infinitesimal $\varepsilon + \varepsilon\varepsilon'$ with $\varepsilon'$ infinitesimal. For each of the three previous cases, we are going to describe that produced of order 2.

1. If $a>0,\ b>0$,

$$\begin{aligned}[a-\varepsilon_1, a+\varepsilon_1][b-\varepsilon_2, b+\varepsilon_2] &= ((a-\varepsilon_1)e_1 + 2\varepsilon_1 e_2)((b-\varepsilon_2)e_1 + 2\varepsilon_2 e_2)\\ &= (a-\varepsilon_1)(b-\varepsilon_2)e_1 + 2(a\varepsilon_2 + b\varepsilon_1)e_2\\ &= [ab - a\varepsilon_2 - b\varepsilon_1 + \varepsilon_1\varepsilon_2, ab + a\varepsilon_2 + b\varepsilon_1 + \varepsilon_1\varepsilon_2].\end{aligned}$$

But

$$\begin{cases}\epsilon_1 = \alpha_1 v_1 + \alpha_1\alpha_2 w_1,\\ \epsilon_2 = \alpha_1 v_2 + \alpha_1\alpha_2 w_2.\end{cases}$$

Thus

$$\begin{aligned}a\varepsilon_2 + b\varepsilon_1 + \varepsilon_1\varepsilon_2 &= a(\alpha_1 v_2 + \alpha_1\alpha_2 w_2) + b(\alpha_1 v_1 + \alpha_1\alpha_2 w_1) + (\alpha_1 v_1 + \alpha_1\alpha_2 w_1)(\alpha_1 v_2 + \alpha_1\alpha_2 w_2)\\ &= \alpha_1(av_2 + bv_1) + \alpha_1\alpha_2(aw_2 + bw_1) + \alpha_1^2 v_1 v_2 + \alpha_1^2\theta\end{aligned}$$

with $\theta \simeq 0$. Thus, if we forgot the infinitesimal $\alpha_1^2\theta$, we have

$$a\varepsilon_2 + b\varepsilon_1 + \varepsilon_1\varepsilon_2 = \alpha_1\rho + \alpha_1\alpha_2\langle W, (b,a)\rangle + \alpha_1^2 v_1 v_2.$$

We have to compare the infinitesimals $\alpha_1\alpha_2$ and $\alpha_1^2$ that is $_1$ and $\alpha_2$. For this we use the decomposition:

$$\begin{cases}\alpha_1 = \beta_1 v_3 + \beta_1\beta_2 w_3,\\ \alpha_2 = \beta_1 v_4 + \beta_1\beta_2 w_4\end{cases}$$

with $\beta_i \simeq 0$ for $i = 1$ and 2, and the vectors $V_2 = (v_3, v_4)$ and $W_2 = (w_3, w_4)$ are independent vectors of $\mathbb{R}^2$. Thus

$$\begin{cases}\alpha_1\alpha_2 = \beta_1^2 v_3 v_4 + \beta_1^2\theta_1,\\ \alpha_1^2 = \beta_1^2 v_3^2 + \beta_1^2\theta_2,\end{cases}$$

with $\theta_i \simeq 0$ for $i = 1$ and 2. We deduce

$$\begin{cases}\begin{aligned}a\varepsilon_2 + b\varepsilon_1 + \varepsilon_1\varepsilon_2 &= \alpha_1\rho + \alpha_1(\alpha_1 v_1 v_2 + \alpha_2\langle W, (b,a)\rangle)\\ &= \alpha_1\rho + \alpha_1(\beta_1 v_1 v_2 v_3 + \beta_1 v_4\langle W, (b,a)\rangle)\\ &= \alpha_1\rho + \alpha_1\beta_1(v_1 v_2 v_3 + v_4\langle W, (b,a)\rangle)\end{aligned}\end{cases}$$

**Definition 54** *If $a>0,\ b>0$, the product of second approximation of the infinitesimal intervals $[a-\varepsilon_1, a+\varepsilon_1]$ and $[b-\varepsilon_2, b+\varepsilon_2]$ is*

$$[a-\varepsilon_1, a+\varepsilon_1].[b-\varepsilon_2, b+\varepsilon_2] = [ab - \epsilon, ab + \epsilon]$$

*where*

$$\epsilon = \alpha_1\rho + \alpha_1\beta_1(v_1 v_2 v_3 + v_4\langle W, (b,a)\rangle).$$

2. If $a > 0,\ b < 0,$

$$\begin{aligned}[a-\varepsilon_1, a+\varepsilon_1][b-\varepsilon_2, b+\varepsilon_2] &= ((a-\varepsilon_1)e_1 + 2\varepsilon_1 e_2)((-b-\varepsilon_2)e_4 + 2\varepsilon_2 e_3) \\ &= [ab - a\varepsilon_2 + b\varepsilon_1 + \varepsilon_1\varepsilon_2, ab + a\varepsilon_2 - b\varepsilon_1 - \varepsilon_1\varepsilon_2].\end{aligned}$$

If we use the decomposition of the infinitesimal vector $(\alpha_1, \alpha_2)$, we obtain

**Definition 55** *If $a > 0,\ b < 0$, the product of second approximation of the infinitesimal intervals $[a-\varepsilon_1, a+\varepsilon_1]$ and $[b-\varepsilon_2, b+\varepsilon_2]$ is*

$$[a-\varepsilon_1, a+\varepsilon_1].[b-\varepsilon_2, b+\varepsilon_2] = [ab-\epsilon, ab+\epsilon]$$

*where*

$$\epsilon = \alpha_1(av_2 - bv_1) + \alpha_1\beta_1(-v_1v_2v_3 + v_4(aw_2 - bw_1).$$

3. If $a < 0,\ b < 0,$

$$\begin{aligned}[a-\varepsilon_1, a+\varepsilon_1][b-\varepsilon_2, b+\varepsilon_2] &= ((-a-\varepsilon_1)e_4 + 2\varepsilon_1 e_3)((-b-\varepsilon_2)e_4 + 2\varepsilon_2 e_3) \\ &= (-a-\varepsilon_1)(-b-\varepsilon_2)e_4 + 2(-a\varepsilon_2 - b\varepsilon_1)e_3 \\ &= [ab + a\varepsilon_2 + b\varepsilon_1 + \varepsilon_1\varepsilon_2, ab - a\varepsilon_2 - b\varepsilon_1 + \varepsilon_1\varepsilon_2].\end{aligned}$$

In this case, we can define the product by

**Definition 56** *If $a < 0,\ b < 0$, the product of second approximation of the infinitesimal intervals $[a-\varepsilon_1, a+\varepsilon_1]$ and $[b-\varepsilon_2, b+\varepsilon_2]$ is*

$$[a-\varepsilon_1, a+\varepsilon_1].[b-\varepsilon_2, b+\varepsilon_2] = [ab-\epsilon, ab+\epsilon]$$

*where*

$$\epsilon = \alpha_1(-av_2 - bv_1) + \alpha_1\beta_1(v_1v_2v_3 - v_4(aw_2 - bw_1).$$

### 6.4.5 A distributive product in $\overline{\mathcal{I}}$

The extension of the product of first or second approximation of $\mathcal{I}$ to $\overline{\mathcal{I}}$ makes itself like the extension of the distributive product of $\mathbb{IR}$ to $\overline{\mathbb{IR}}$.

We have define the product of second approximation. It is clear that the process of decomposition of an infinitesimal vector permits to define a product until an arbitrary approximation.

## 6.5 A global product

Let $\widetilde{J}(a, \epsilon_1)$ and $\widetilde{J}(b, \epsilon_2) \in \widetilde{\mathcal{I}}$. We assume, in first time, that correspond to $\overline{(J(a, \epsilon_1), 0)}$ and $\overline{(J(b, \epsilon_2), 0)}$.

**First case:** $a > 0$, $b > 0$. We put

$$\widetilde{J}(a, \varepsilon_1) \bullet \widetilde{J}(b, \varepsilon_2) = \widetilde{J}(ab, a\varepsilon_2 + b\varepsilon_1).$$

We note that if $(\varepsilon_1, \varepsilon_2) = \alpha_1 V_1 + \alpha_1\alpha_2 V_2$, then as $\varepsilon_1$ and $\varepsilon_2$ are positive infinitesimals, the component $v_1$ and $v_2$ are positive and not zero. We deduce that

$$a\varepsilon_2 + b\varepsilon_1 = a(\alpha_1 v_2 + \alpha_1\alpha_2 w_2) + b(\alpha_1 v_1 + \alpha_1\alpha_2 w_1)$$

where $V_2 = (w_1, w_2)$.Thus

$$a\varepsilon_2 + b\varepsilon_1 = \alpha_1(av_2 + bv_1) + \alpha_1\alpha_2^{'}$$

with $\alpha_2^{'} \in \mathfrak{m}$. As $av_2 + bv_1 \neq 0$ $(a > 0, b > 0, v_1 \geq 0, v_2 \geq 0, v_1 \neq 0)$, then $a\varepsilon_2 + b\varepsilon_1$ are equivalent to $\alpha_1$ and

$$\widetilde{J}(ab, a\varepsilon_2 + b\varepsilon_1) = \widetilde{J}(ab, \alpha_1).$$

**Second case** : $a > 0$, $b < 0$.If $u \in [a - \varepsilon_1, a + \varepsilon_1]$ and $[b - \varepsilon_2, b + \varepsilon_2]$ then we can write $u = a + \rho_1$, $v = b + \rho_2$ with $-\varepsilon_i \leq \rho_i \leq \varepsilon_i$ and $i = 1, 2$. We deduce

$$uv = (a + \rho_1)(b + \rho_2) = ab + a\rho_2 + b\rho_1 + \rho_1\rho_2$$

and

$$-a\varepsilon_2 + b\varepsilon_1 \leq a\rho_2 + b\rho_1 \leq a\varepsilon_2 - b\varepsilon_1.$$

Then, as $a$ and $-b$ are positive, the infinitesimal $-a\varepsilon_2 + b\varepsilon_1 \leq a\rho_2 + b\rho_1 \leq a\varepsilon_2 - b\varepsilon_1$ up to an infinitesimal equivalent to $\varepsilon_1\varepsilon_2$. We put

$$\widetilde{J}(a, \varepsilon_1) \bullet \widetilde{J}(b, \varepsilon_2) = \widetilde{J}(ab, a\varepsilon_2 - b\varepsilon_1) = \widetilde{J}(ab, \alpha_1).$$

**Third case :** $a < 0$, $b < 0$. The study realized in the first case permits to write

$$\widetilde{J}_{11}(-a, \varepsilon_1) \bullet \widetilde{J_2}(-b, \varepsilon_2) = \widetilde{J}(ab, \epsilon_1 + \epsilon_2).$$

**Fourth case**:$a = 0$. If $b \neq 0$ we put

$$\widetilde{J}(0, \varepsilon_1) \bullet \widetilde{J}(b, \varepsilon_2) = \widetilde{J}(0, b\varepsilon_1).$$

If $b = 0$ then

$$\widetilde{J}(0, \varepsilon_1) \bullet \widetilde{J}(0, \varepsilon_2) = \widetilde{J}(0, \alpha_1\alpha_2)$$

if $v_1v_2 \neq 0$, or

$$\widetilde{J}(0, \varepsilon_1) \bullet \widetilde{J}(0, \varepsilon_2) = \widetilde{J}(0, \alpha_1^2\alpha_2).$$

To end the definition of this product we put

$$\widetilde{J}(a, \varepsilon_1) \bullet (\diagdown \widetilde{J}(b, \varepsilon_2)) = \diagdown(\widetilde{J}(a, \varepsilon_1) \bullet \widetilde{J}(b, \varepsilon_2)) = (\diagdown \widetilde{J}(a, \varepsilon_1)) \bullet \widetilde{J}(b, \varepsilon_2)$$

and

$$(\diagdown \widetilde{J}(a, \varepsilon_1)) \bullet (\diagdown \widetilde{J}(b, \varepsilon_2)) = \widetilde{J}(a, \varepsilon_1) \bullet \widetilde{J}(b, \varepsilon_2).$$

**Proposition 37** *This product is commutative and associative. Moreover $\widetilde{J}(1, 0)$ is an unit for this product.*

Let us examine the distributivity of this product with respect to the addition. Let $\widetilde{J}_1, \widetilde{J}_2, \widetilde{J}_3 \in \widetilde{\mathcal{I}}$. We put $\widetilde{J}_1 = \widetilde{J}(a, \varepsilon_1), \widetilde{J}_2 = \widetilde{J}(b, \varepsilon_2), \widetilde{J}_3 = \widetilde{J}(c, \varepsilon_3)$.

1) If $a, b, c$ are positive, then

$$\widetilde{J}_1 \bullet (\widetilde{J}_2 + \widetilde{J}_3) = \widetilde{J}_1 \bullet \widetilde{J}_2 + \widetilde{J}_1 \bullet \widetilde{J}_3.$$

2) Assume that $a > 0, b > 0$ and $c < 0$. Then

$$\widetilde{J}_2 + \widetilde{J}_3 = \widetilde{J}(b + c, \varepsilon_2 + \varepsilon_3).$$

If $b + c > 0$, then

$$\begin{aligned} \widetilde{J}_1 \bullet (\widetilde{J}_2 + \widetilde{J}_3) &= \widetilde{J}(a(b + c), (b + c)\varepsilon_1 + a(\varepsilon_2 + \varepsilon_3)), \\ \widetilde{J}_1 \bullet \widetilde{J}_2 + \widetilde{J}_1 \bullet \widetilde{J}_3 &= \widetilde{J}(ab, b\varepsilon_1 + a\varepsilon_2) + \widetilde{J}_1 \bullet \widetilde{J}_3. \end{aligned}$$

But $\widetilde{J}_1 \bullet \widetilde{J}_3 = \widetilde{J}(ac, a\varepsilon_3 - c\varepsilon_1)$, thus

$$\widetilde{J}_1 \bullet \widetilde{J}_2 + \widetilde{J}_1 \bullet \widetilde{J}_3 = \widetilde{J}(ab, \varepsilon_1(b-c) + a(\varepsilon_2 + \varepsilon_3).$$

The decomposition of the infinitesimal vector $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ writes:

$$(\varepsilon_1, \varepsilon_2, \varepsilon_3) = \beta_1 U_1 + \beta_1\beta_2 U_2 + \beta_1\beta_2\beta_3 U_3$$

with $\beta_i \simeq 0$ and $\{U_1, U_2, U_3\}$ a standard orthonormed frame. As $b - c > 0$, then $(b+c)\varepsilon_1 + a(\varepsilon_2 + \varepsilon_3)$ and $(b-c)\varepsilon_1 + a(\varepsilon_2 + \varepsilon_3)$ are equivalent to $\beta_1$. We deduce that

$$\widetilde{J}_1 \bullet (\widetilde{J}_2 + \widetilde{J}_3) = \widetilde{J}_1 \bullet \widetilde{J}_2 + \widetilde{J}_1 \bullet \widetilde{J}_3.$$

If $b + c < 0$, then

$$\widetilde{J}_1 \bullet (\widetilde{J}_2 + \widetilde{J}_3) = \widetilde{J}(a(b+c), a(\varepsilon_2 + \varepsilon_3) - (b+c)\varepsilon_1).$$

In this case we also $a(\varepsilon_2 + \varepsilon_3) - (b+c)\varepsilon_1$ and $(b-c)\varepsilon_1 + a(\varepsilon_2 + \varepsilon_3)$ are equivalent to $\beta_1$, thus

$$\widetilde{J}_1 \bullet (\widetilde{J}_2 + \widetilde{J}_3) = \widetilde{J}_1 \bullet \widetilde{J}_2 + \widetilde{J}_1 \bullet \widetilde{J}_3.$$

3) Assume that $a > 0, b < 0$ and $c < 0$. Then

$$\widetilde{J}_2 + \widetilde{J}_3 = \widetilde{J}(b+c, \varepsilon_2 + \varepsilon_3)$$

and

$$\begin{aligned}
\widetilde{J}_1 \bullet (\widetilde{J}_2 + \widetilde{J}_3) &= \widetilde{J}(a(b+c), -(b+c)\varepsilon_1 + a(\varepsilon_2 + \varepsilon_3)), \\
\widetilde{J}_1 \bullet \widetilde{J}_2 + \widetilde{J}_1 \bullet \widetilde{J}_3 &= \widetilde{J}(ab, -b\varepsilon_1 + a\varepsilon_2) + \widetilde{J}(ac, -c\varepsilon_1 + a\varepsilon_2), \\
&= \widetilde{J}(ab + ac, -(b+c)\varepsilon_1 + a(\varepsilon_2 + \varepsilon_3), \\
&= \widetilde{J}_1 \bullet (\widetilde{J}_2 + \widetilde{J}_3).
\end{aligned}$$

4) We obtain the same results for $a < 0$.

**Theorem 57** *The product $\bullet$ is distributive with respect the addition:*

$$\widetilde{J}_1 \bullet (\widetilde{J}_2 + \widetilde{J}_3) = \widetilde{J}_1 \bullet \widetilde{J}_2 + \widetilde{J}_1 \bullet \widetilde{J}_3$$

*for all $\widetilde{J}_1,\ \widetilde{J}_2,\ \widetilde{J}_3 \in \overline{\mathcal{I}}$.*

*Consequence:* the triple $(\overline{\mathcal{I}}, +, \bullet)$ is an (infinite dimensional) associative algebra.

# Chapter 7

# Annexe: On Poisson algebras

## 7.1 Poisson structures on $\mathbb{C}[X_1, \ldots, X_n]$ and exterior calculus

### 7.1.1 Poisson bracket and differential forms

Let $\mathcal{A}_n$ be the commutative associative algebra $\mathbb{C}[X_1, \ldots, X_n]$. A Poisson structure on $\mathcal{A}_n$ is given by a bivector

$$\mathcal{P} = \sum_{1 \leq i < j \leq n} P_{ij} \partial_i \wedge \partial_j$$

where $\partial_i = \frac{\partial}{\partial X_i}$ and $P_{ij} \in \mathcal{A}_n$, satisfying

$$[\mathcal{P}, \mathcal{P}]_S = 0$$

where $[,]_S$ is the Schouten's bracket. If $\mathcal{A}_n$ is endowed with a Poisson structure $\mathcal{P}$, the multiplication given by

$$\{P, Q\} = \mathcal{P}(P, Q),$$

for any $P, Q \in \mathcal{A}_n$ is a Lie bracket satisfying the Leibniz identity

$$\{PQ, R\} = P\{Q, R\} + Q\{P, R\}$$

for any $P, Q, R \in \mathcal{A}_n$.
We denote by $S_{p,q}$ the set of $(p, q)$-shuffles where a $(p, q)$-shuffle is a permutation $\sigma$ in the symmetric group $S_{p+q}$ of degree $p + q$ such that $\sigma(1) < \sigma(2) < \cdots < \sigma(p)$ and $\sigma(p + 1) < \sigma(p + 2) < \cdots < \sigma(q)$. Given a bivector $\mathcal{P}$ we consider the $(n - 2)$-exterior form

$$\Omega = \sum_{\sigma \in S_{2,n-2}} (-1)^{\epsilon(\sigma)} P_{\sigma(1)\sigma(2)} dX_{\sigma(3)} \wedge \cdots \wedge dX_{\sigma(n)}$$

where $\epsilon(\sigma))$ is the signature of the permutation $\sigma$.
We assume that $n > 3$. Let $\alpha_{i_1,\cdots,i_{n-3}}$ the pfaffian form given by

$$\alpha_{i_1,\cdots,i_{n-3}}(Y) = \Omega(\partial_{i_1}, \partial_{i_2}, \ldots, \partial_{i_{n-3}}, Y)$$

where $Y = \sum Y_i \partial_i$, $Y_i \in \mathcal{A}_n$. If $n = 3$ we put $\alpha = \Omega$.

**Theorem 58** *A bivector $\mathcal{P}$ on $\mathcal{A}_n$ satisfies $[\mathcal{P}, \mathcal{P}]_S = 0$ if and only if*

$$d\alpha_{i_1,\cdots,i_{n-3}} \wedge \Omega = 0$$

*for every $i_1, \cdots, i_{n-3}$ such that $1 \leq i_1 < \cdots < i_{n-3} \leq n$.*

*Proof.* The integrability condition $[\mathcal{P},\mathcal{P}]_S = 0$ writes

$$\sum_{r=1}^{n} P_{ri}\partial_r P_{jk} + P_{rj}\partial_r P_{ki} + P_{rk}\partial_r P_{ij} = 0$$

for any $1 \leq i,j,k \leq n$. But

$$\alpha_{i_1,\cdots,i_{n-3}} = \sum (-1)^N P_{jk} dX_l$$

the summand concerning the triples $(j,k,l)$ where $(j,k,i_1,\ldots,l,\ldots i_{n-3})$ is a permutation of $S_{2,n-2}$ and $N = \varepsilon(\sigma) + p - 3$. Then

$$d\alpha_{i_1,\cdots,i_{n-3}} = \sum (-1)^N dP_{jk} \wedge dX_l$$

and $d\alpha_{i_1,\cdots,i_{n-3}} \wedge \Omega = 0$ corresponds to $[\mathcal{P},\mathcal{P}]_S = 0$.

### 7.1.2 Lichnerowicz-Poisson cohomology

We denote by $\mathcal{A}_{\mathcal{P}}$ the algebra $\mathcal{A}_n = \mathbb{C}[X_1,\ldots,X_n]$ provided with the Poisson structure $\mathcal{P}$. Let $\chi^k(\mathcal{A}_{\mathcal{P}})$ be the vector space of $k$-biderivations that is of $k$-skew linear maps on $\mathcal{A}$ satisfying

$$\varphi(P_1Q_1,P_2,\ldots,P_k) = P_1\varphi(Q_1,P_2,\ldots,P_k) + Q_1\varphi(P_1,P_2,\ldots,P_k)$$

for all $Q_1,P_1,\ldots,P_k \in \mathcal{A}$. For $k=0$ we put $\chi^0(\mathcal{A}_{\mathcal{P}}) = \mathcal{A}_{\mathcal{P}}$. Let $\delta^k$ be the linear map

$$\delta^k : \chi^k(\mathcal{A}_{\mathcal{P}}) \longrightarrow \chi^{k+1}(\mathcal{A}_P)$$

given by

$$\begin{array}{l}\delta^k\varphi(P_1,P_2,\ldots,P_{k+1}) = \sum_{i=1}^{k+1}(-1)^{i-1}\{P_i,\varphi(P_1,\ldots,\widehat{P_i},\ldots P_{k+1})\} \\ +\sum_{1\leq i<j\leq k+1}(-1)^{i+j}\varphi(\{P_i,P_j\},P_1,\ldots,\widehat{P_i},\ldots,\widehat{P_j},\ldots P_{k+1})\end{array}$$

where $\widehat{P_i}$ means that the term $P_i$ does not appear. We have $\delta^{k+1}\circ\delta^k = 0$ and the Lichnerowicz-Poisson cohomology corresponds to the complex $(\chi^k(\mathcal{A}_{\mathcal{P}}),\delta^k)_k$. Let us note that $\chi^k(\mathcal{A}_{\mathcal{P}})$ is trivial as soon as $k > n$. A description of the cocycle $\delta^k\varphi$ is presented in [**?**] in the 3-dimensional case using the vector calculus. We will describe these formulae using exterior calculus for the dimension greater or equal to 3. Let us begin with some notations :

- To any element $P \in \mathcal{A} = \chi^0(\mathcal{A}_{\mathcal{P}})$, we associate the $n$-exterior form

$$\Phi_n(P) = PdX_1 \wedge \ldots \wedge dX_n.$$

- To any $\varphi \in \chi^k(\mathcal{A}_{\mathcal{P}})$, we associate the $n-k$ exterior form

$$\Phi_{n-k}(\varphi) = \sum_{\sigma\in S_{k,n-k}} (-1)^{\varepsilon(\sigma)}\varphi(X_{\sigma(1)}\ldots X_{\sigma(k)})dX_{\sigma(k+1)}\wedge\ldots\wedge dX_{\sigma(n)}.$$

- To any $\varphi \in \chi^n(\mathcal{A}_{\mathcal{P}})$ we associate the function $\Phi_0(\varphi) = \varphi$.

Finally, if $\theta$ is an $k$-exterior form and $Y = \sum Y_i\partial_i$ is a vector field with $Y_i \in \mathcal{A}$ then the inner product $i(Y)$ of $Y$ by $\theta$ is the $k-1$-exterior form given by

$$i(Y)\theta(Z_1,\cdots,Z_{k-1}) = \theta(Y,Z_1,\cdots,Z_{k-1})$$

for every vector field $Z_1,\cdots,Z_k$.

**Theorem 59** *Suppose $n = 3$. Then we have*

1. *For all $P \in \mathcal{A}_{\mathcal{P}}$,*

$$\Phi_2(\delta^0 P) = \Omega \wedge dP.$$

2. *For all $f \in \chi^1(\mathcal{A}_{\mathcal{P}})$,*

$$\begin{array}{rl}\Phi_1(\delta^1 f) & = -i(\partial_1, \partial_2)[\Omega \wedge d(i(\partial_3)\Phi_2(f)) + d(i(\partial_3)\Omega) \wedge \Phi_2(f)] \\ & +i(\partial_1, \partial_3)[\Omega \wedge d(i(\partial_2)\Phi_2(f)) + d(i(\partial_2)\Omega) \wedge \Phi_2(f)] \\ & -i(\partial_2, \partial_3)[\Omega \wedge d(i(\partial_1)\Phi_2(f)) + d(i(\partial_1)\Omega) \wedge \Phi_2(f)\end{array}$$

*where $i(X, Y)$ denotes the composition $i(X) \circ i(Y)$.*

3. *For all $\varphi \in \chi^2(\mathcal{A}_{\mathcal{P}})$,*

$$\Phi_0(\delta^2 \varphi) = i(\partial_1, \partial_2, \partial_3)(d\Omega \wedge \Phi_1(\varphi) + \Omega \wedge d\Phi_1(\varphi)).$$

*Proof* If $n = 3$ we have

$$\Omega = P_{12} dX_3 - P_{13} dX_2 + P_{23} dX_1$$

and $\alpha = \Omega$. Then the integrability of $\mathcal{P}$ is equivalent to the integrability condition $\Omega \wedge d\Omega = 0$. The theorem results of a direct computation and of the following general formula, which writes in the general case :

$$\forall \varphi \in \chi^k(\mathcal{A}_{\mathcal{P}}),\ \varphi(P_1, \ldots, P_k) = \sum_{1 \leq i_1 \leq \ldots \leq i_k \leq n} \varphi(\partial_{i_1, \ldots, \partial_{i_k}}) \partial_{i_1} P_1 \ldots \partial_{i_k} P_k.$$

**Application.** We consider the Poisson algebra $\mathcal{A}_1 = (\mathbb{C}[X_1, X_2, X_3], \mathcal{P})$ where $\mathcal{P}$ is given by

$$\left\{ \begin{array}{l} \mathcal{P}(X_1, X_2) = X_2 \\ \mathcal{P}(X_1, X_3) = 2X_3 \\ \mathcal{P}(X_2, X_3) = 0. \end{array} \right.$$

Then we have

$$\dim H^0(\mathcal{A}_1) = 1,\ \dim H^1(\mathcal{A}_1) = 3,\ \dim H^2(\mathcal{A}_1) = 2,\ H^3(\mathcal{A}_1) = \{0\}.$$

In this case $\Omega = X_2 dX_3 - 2X_3 dX_2$ and $d\Omega = 3dX_2 \wedge dX_3$. We will look, for example, $H^2(\mathcal{A}_1)$ . Let $\varphi \in \chi^2(\mathcal{A}_1)$. Then $\Phi_0(\delta^2 \varphi) = 0$ implies $d\Omega \wedge \Phi_1(\varphi) + \Omega \wedge d\Phi_1(\varphi)) = 0$, that is

$$\begin{array}{l} X_2(\partial_1 \varphi(X_1, X_3) + \partial_2 \varphi(X_2, X_3)) + 2X_3(-\partial_1 \varphi(X_1, X_2) + \partial_3 \varphi(X_2, X_3)) \\ +3\varphi(X_2, X_3) = 0 \end{array}$$

Now, if $f \in \chi^1(\mathcal{A}_1)$ then

$$\begin{array}{rl}\Phi_1(\delta f) = & [X_2(-\partial_2 f(X_2) - \partial_1 f(X_1)) - 2X_3(\partial_3 f(X_2)) + f(X_2)]dX_3 \\ & -[2X_3(\partial_1 f(X_1) + \partial_3 f(X_3)) + X_2(\partial_2 f(X_3)) - 2f(X_3)]dX_2 \\ & -[X_2(-\partial_1 f(X_3)) - 2X_3(\partial_1 f(X_2))]dX_1.\end{array}$$

Comparing these two relations we obtain that $H^2(\mathcal{A}_1)$ is generated by the two cocycles corresponding to

$$\left\{ \begin{array}{l} \Phi_1(\varphi_1) = X_3 dX_2 \\ \Phi_1(\varphi_2) = X_2^2 dX_2 \end{array} \right.$$

Now let us consider the general case. Let $\mathcal{A} = \mathbb{C}[X_1, \ldots, X_n]$ be provided with the Poisson structure $\mathcal{P}$.

**Theorem 60** *Let $\phi \in \chi^k(\mathcal{A}_{\mathcal{P}})$. Then, we have*

$$\Phi_{n-k-1}(\delta^k \phi) = \epsilon \sum_{\sigma \in S_{k+1, n-k-1}} i(\partial_{\sigma(1)}, \cdots, \partial_{\sigma(k+1)})[d(i(\partial_{\sigma(k+2)}, \cdots, \partial_{\sigma(n)})\Omega) \wedge \Phi_{n-k}(\phi)$$

$$+\Omega \wedge d(i(\partial_{\sigma(k+2)}, \cdots, \partial_{\sigma(n)})\Phi_{n-k}(\phi))]$$

*where $\epsilon(n, k) = (-1)^{\frac{(n-k)(n-k+1)}{2}}$.*

*Proof.* To simplify the length of the formulae, we will write $d_i$ in front of $dX_i$. We have seen that for every $P \in \mathcal{A}_\mathcal{P}$ we have $\delta^0 P = \Omega \wedge dP$. But

$$\Phi_{n-1}(\delta P) = \{P, X_1\} d_2 \wedge \cdots \wedge d_n - \{P, X_2\} d_1 \wedge d_3 \cdots \wedge d_n + \cdots +$$
$$+(-1)^{n-2}\{P, X_{n-1}\} d_1 \wedge \cdots \wedge d_{n-2} \wedge d_n + (-1)^{n-1}\{P, X_n\} d_1 \wedge \cdots \wedge d_{n-1}.$$

with

$$\{P, X_i\} d_1 \wedge \cdots \wedge \hat{d_i} \wedge \cdots \wedge d_n = \sum_{j=1}^{i-1} P_{ji} \partial_j P d_1 \wedge \cdots \wedge \hat{d_i} \wedge \cdots \wedge d_n$$
$$- \sum_{j=i+1}^{n} P_{ji} \partial_j P d_1 \wedge \cdots \wedge \hat{d_i} \wedge \cdots \wedge d_n$$

But

$i(\partial_1)[\Omega \wedge d(i(\partial_2, \cdots, \partial_n)\Phi_n(P)) + d(i(\partial_2, \cdots, \partial_n)\Omega) \wedge \Phi_n(P)]$

$= i(\partial_1)[\Omega \wedge d(i(\partial_2, \cdots, \partial_n)\Phi_n(P)] = (-1)^{\frac{n(n-1)}{2}} i(\partial_1)[\Omega \wedge dP \wedge d_1]$

$= -(-1)^{\frac{n(n-1)}{2}} \sum_{i=2}^{n} P_{1i} \partial_i P d_2 \wedge \cdots \wedge d_n = (-1)^{\frac{n(n-1)}{2}} \Phi_{n-1}(P)(\partial_2, \cdots, \partial_n).$

Similary

$i(\partial_j)[\Omega \wedge d(i(\partial_1, \cdots, \hat{\partial_j}, \cdots, \partial_n)\Phi_n(P)) + d(i(\partial_1, \cdots, \hat{\partial_j}, \cdots, \partial_n)\Omega) \wedge \Phi_n(P)]$

$= i(\partial_j)[\Omega \wedge d(i(\partial_1, \cdots, \hat{\partial_j}, \cdots, \partial_n)\Phi_n(P)) = (-1)^{j-1+\frac{n(n-1)}{2}} i(\partial_j)[\Omega \wedge dP \wedge dX_j]$

$= (-1)^{j-1+\frac{n(n-1)}{2}} i(\partial_j)(\sum_{l=1}^{l=j-1}(P_{1j} \partial_l P d_2 \wedge \cdots \wedge d_n - \sum_{l=j+1}^{l=n} P_{jl} \partial_l P d_2 \wedge \cdots \wedge d_n$

$= (-1)^{j-1+\frac{n(n-1)}{2}} i(\partial_j)(\sum_{l=1}^{l=j-1} P_{1j} \partial_l P - \sum_{l=j+1}^{l=n} P_{jl} \partial_l P) d_1 \wedge \cdots \wedge d_n$

$= (-1)^{\frac{n(n-1)}{2}} (\sum_{l=1}^{l=j-1} P_{1j} \partial_l P - \sum_{l=j+1}^{l=n} P_{jl} \partial_l P) d_1 \wedge \cdots \wedge \hat{d_j} \cdots \wedge d_n$

$= (-1)^{\frac{n(n-1)}{2}} \{P, X_i\} d_1 \wedge \cdots \wedge \hat{d_i} \wedge \cdots \wedge d_n.$

We deduce

$$\Phi_{n-1}(\delta^1 f) = (-1)^{\frac{n(n-1)}{2}} \sum_{j=1}^{n} (-1)^{j-1} i(\partial_j)[\Omega \wedge d(i(\partial_1, \cdots, \hat{\partial_j}, \cdots, \partial_n)\Phi_n(P))$$

which proves the theorem for $k = 1$. The proof is similar for any $k$.

**Application.** We consider the $n$-dimensional complex Lie algebra defined by the brackets

$$[X_1, X_i] = (i-1)X_i$$

for $i = 2, ..., n$.. Let $\mathcal{P}$ the Poisson bracket on $\mathbb{C}[X_1, ..., X_n]$ given by $\mathcal{P}(X_i, X_j) = [X_i, X_j]$. Let $\chi_2^k(\mathcal{A}_\mathcal{P})$ the subspace of $\chi^k(\mathcal{A}_\mathcal{P})$ whose elements are homogeneous of degree 2. We denote by $H_2^2(\mathcal{A}_\mathcal{P}) = Z_2^2/B_2^2$ the corresponding subspace of $H_2^2(\mathcal{A}_\mathcal{P})$. Let $N = \frac{n(n-1)}{2}$.

- If $n$ is even, then

$$dim B_2^2 = N + (N-1) + ... + N - n/2 + 1 = n^2(2n-1)/4.$$

- If $n$ is odd, then

$$dim B_2^2 = N + (N-1) + (N-1) + ... + (N - (n-1)/2) = (n-1)(2n^2 + n + 1)/4.$$

In fact, if $f \in \chi_2^1(\mathcal{A}_\mathcal{P})$, then $f(X_i) = P_i$ and $P_i$ is homogeneous of degree 2:

$$P_i = \Sigma a^i_{i_1, ..., i_n} X_1^{i_1} X_2^{i_2} ... X_n^{i_n}$$

with $i_1 + ....i_n = 2$. We deduce :

1. In the expansion of $\delta f(X_1, X_{2l})$ we find $N - l$ independent coefficients of $P_{2l}$. The coefficients which do not appear are:

$$a^{2l}_{1,0,0,...,0,1,0,...,0},\quad a^{2l}_{0,1,0,...,0,1,0,...,0},\quad ..., a^{2l}_{0,0,..,1,1,0,...,0}$$

(the second 1 are in the places $2l, 2l-1, ..., l+1$.

2. In the expansion of $\delta f(X_1, X_{2l+1})$ we find $N - l - 1$ independent coefficients of $P_{2l+1}$. The coefficients which do not appear are:

$$a^{2l+1}_{1,0,0,...,0,1,0,...,0},\quad a^{2l+1}_{0,1,0,...,0,1,0,...,0},\quad ..., a^{2l+1}_{0,0,..,0,2,0,...,0}$$

(the second 1 being in place $2l+1, 2l, ..., l+2$ and in the last case the 2 is in place $l+1$).

3. And $\delta f(X_i, X_j)$ for $i \geq 2$ and $j > i$ is defined by the $(n-2)$ coefficients $a^i_{1,0,0,...,0,1,0,..,0}$.

Now we are to able to find the generators of $H^2_2(\mathcal{A}_\mathcal{P})$. We can choose $\phi \in \chi^2_2$ such that

$$\begin{cases} \phi(X_1, X_2) = 0 \\ \phi(X_1, X_3) = a^{1,3}_{1,3} X_1 X_3 + a^{2,2}_{1,3} X_2^2 \\ \cdots \\ \phi(X_1, X_{2l}) = a^{1,2l}_{1,2l} X_1 X_{2l} + a^{2,2l-1}_{1,2l} X_2 X_{2l-1} + ... + a^{l,l+1}_{1,2l} X_l X_{l+1} \\ \phi(X_1, X_{2l+1}) = a^{1,2l+1}_{1,2l+1} X_1 X_{2l+1} + a^{2,2l}_{1,2l+1} X_2 X_{2l} + ... + a^{l,l}_{1,2l+1} X_l^2 \\ \cdots \\ \phi(X_1, X_n) = a^{1,n}_{1,n} X_1 X_n + a^{2,n-1}_{1,n} X_2 X_{n-1} + ... \\ \phi(X_i, X_j) = A_{i,j} \end{cases}$$

where $A_{i,j}$ is a degree 2 homogeneous polynomial without monomial of type $X_1 X_k$ and $X_i X_j$. If we solve $\Phi_{n-2}(\delta\phi) = 0$ we obtain the generators of $H^2_2(\mathcal{A}_\mathcal{P})$. They are given by

$$\begin{cases} \phi(X_1, X_2) = 0 \\ \phi(X_1, X_3) = a^{2,2}_{1,3} X_2^2 \\ \cdots \\ \phi(X_1, X_{2l}) = a^{2,2l-1}_{1,2l} X_2 X_{2l-1} + ... + a^{l,l+1}_{1,2l} X_l X_{l+1} \\ \phi(X_1, X_{2l+1}) = a^{2,2l}_{1,2l+1} X_2 X_{2l} + ... + a^{l+1,l+1}_{1,2l+1} X_{l+1}^2 \\ \cdots \\ \phi(X_1, X_n) = a^{2,n-1}_{1,n} X_2 X_{n-1} + ... + a^{m,m+1}_{1,n} X_m X_{m+1}, \quad \text{if } n = 2m \\ \phi(X_i, X_j) = A_{i,j}. \end{cases}$$

the last term being $a^{m+1,m+1}_{1,n} X^2_{m+1}$, if $n = 2m+1$.

For example:

- if $n = 2$ , $dim H^2_2(A, A) = 1$,
- if $n = 3$, $dim H^2_2(A, A) = 3$,
- if $n = 4$, $dim H^2_2(A, A) = 8$,
- if $n = 5$, $dim H^2_2(A, A) = 16$.

## 7.2 Poisson structures of degree 2 on $\mathbb{C}[X_1, X_2, X_3]$

Let $\mathcal{P}$ be a Poisson structure on $\mathcal{A} = \mathbb{C}[X_1, X_2, X_3]$. It writes $\mathcal{P} = \Sigma P_{ij} \partial_i \wedge \partial_j$ with $P_{ij} \in \mathbb{C}[X_1, X_2, X_3]$. We will say that $\mathcal{P}$ is of degree $k$ if $k = max(d^\circ P_{ij})$. It is homogeneous if all the $P_{ij}$ are homogeneous with the same degree. The homogeneous Poisson structure of degree less than 2 are well known. We will look in this section the non homogeneous case with $k \leq 2$. In this case $\mathcal{P}$ writes

$$\mathcal{P} = \mathcal{P}_0 + \mathcal{P}_1 + \mathcal{P}_2$$

where $\mathcal{P}_i$ is homogeneous of degree $i$. The associate form $\Omega$ is decomposed in homogeneous part $\Omega = \Omega_0 + \Omega_1 + \Omega_2$. As $\Omega$ is en integrable Pfaffian form, we obtain

$$\Omega_0 \wedge d\Omega_0 = 0$$
$$\Omega_2 \wedge d\Omega_2 = 0$$
$$\Omega_0 \wedge d\Omega_1 + \Omega_1 \wedge d\Omega_0 = 0$$
$$\Omega_0 \wedge d\Omega_2 + \Omega_2 \wedge d\Omega_0 + \Omega_1 \wedge d\Omega_1 = 0$$
$$\Omega_1 \wedge d\Omega_2 + \Omega_2 \wedge d\Omega_1 = 0.$$

If $k = 0$, then $\Omega_1 = \Omega_2 = 0$ and $\Omega$ is isomorphic to one of the following form

$$\Omega^1 = 0$$
$$\Omega^2 = \Omega_0^2 = dX_3.$$

If $k = 1$, then $\Omega_2 = 0$ and $\Omega$ is isomorphic to one of the following form

$$\begin{array}{l}
\Omega^3 = \Omega_1^3 = X_3 dX_3 \\
\Omega^4 = \Omega_1^4 = X_2 dX_3 + X_3 dX_2 + X_1 dX_1 \\
\Omega^5 = \Omega_1^5 = X_2 dX_3 - \alpha X_3 dX_2 \\
\Omega^6 = \Omega_1^6 = (X_2 + X_3) dX_3 - X_3 dX_2 \\
\Omega^7 = \Omega_0^2 + \Omega_1^7 = dX_3 - X_3 dX_2 \\
\Omega^8 = \Omega_0^2 + \Omega_1^8 = X_3 dX_3 - dX_2 \\
\Omega^9 = \Omega_0^9 + \Omega_1^9 = X_2 dX_3 + X_3 dX_2 + dX_1
\end{array}$$

The Poisson structures associated to $\Omega^1, \Omega_1^i$, $i = 3, 4, 5, 6$ correspond to the classification of 3-dimensional complex Lie algebras. For $\Omega_1^2, \Omega_1^7, \Omega_1^8$ and $\Omega_1^9$, they are associated to 4-dimensional complex Lie algebra with 1-dimensional center contained in the derived subalgebra.

Let us suppose now that $k = 2$. If $\Omega_0 = \Omega_1 = 0$, then $\Omega_2$ is homogeneous and the classification is given in ([**?**]). Thus we will assume that $\Omega_0$ or $\Omega_1$ is nontrivial. In this section we will describe the corresponding classification up a graded linear isomorphism of the graded algebra $\mathcal{A} = \oplus_{n \geq 0} V_n$ where $V_n$ is the space of degree $n$ homogeneous polynomials.

**Definition 61** *We call equivalence of order* 2*, any linear isomorphism*

$$f : \oplus_{n \geq 0} V_n \rightarrow \oplus_{n \geq 0} V_n$$

*satisfying*

- $f(V_1) \subset V_1 \oplus V_2$
- $f(V_0) = V_0$
- $f \mid_{\oplus_{n \geq 2} V_n} = Id$

Such a mapping is written

$$\begin{array}{rcl}
f(X_i) & = & \sum a_i^j X_j + \sum b_i^{jk} X_j X_k \\
f(X_i X_j) & = & X_i X_j
\end{array}$$

If the Poisson bracket on $\mathcal{A}$ induces a Lie algebra structure on $V_1$ (that is $\Omega_1 \wedge d\Omega_1 = 0$) we will impose that $\pi_1 \circ f$ is a Lie automorphism of $V_1$, where $\pi_1$ is the projection on $V_1$. We define a new Poisson structure on $\mathcal{A}$ putting $Y_i = f(X_i)$ and

$$\{Y_i, Y_j\} = f(\{f(X_i), f(X_j)\}).$$

These two Poisson structure are called equivalent.

*First case :* $\Omega_0 = 0$.

The Poisson structure is given by $\Omega = \Omega_1 + \Omega_2$ with

$$\begin{cases} \Omega_1 \wedge d\Omega_1 = 0 \\ \Omega_1 \wedge d\Omega_2 + \Omega_2 \wedge d\Omega_1 = 0 \\ \Omega_2 \wedge d\Omega_2 = 0. \end{cases}$$

Thus $\Omega_1 = \Omega_1^i$ , $i = 1, 3, 4, 5, 6$ and $V_1$ is a Lie algebra. Let

$$\Omega_2 = A_3 dX_3 - A_2 dX_2 + A_3 dX_1$$

be with

$$\begin{array}{rcl} A_1 & = & a_1X_1^2 + a_2X_2^2 + a_3X_3^2 + a_4X_1X_2 + a_5X_1X_3 + a_6X_2X_3 \\ A_2 & = & b_1X_1^2 + b_2X_2^2 + b_3X_3^2 + b_4X_1X_2 + b_5X_1X_3 + b_6X_2X_3 \\ A_3 & = & c_1X_1^2 + c_2X_2^2 + c_3X_3^2 + c_4X_1X_2 + c_5X_1X_3 + c_6X_2X_3. \end{array}$$

If $d\Omega_2 = 0$ then we have to solve $\Omega_2 \wedge d\Omega_1 = 0$ with $\Omega_1$ $\Omega_1^i$ , $i = 1, 3, 4, 5, 6$. For $i = 1, 3, 4$ we have $d\Omega_1 = 0$ and $d\Omega_2 = 0$ is equivalent to

$$\begin{array}{rcl} A_1 & = & a_1X_1^2 + a_2X_2^2 + a_3X_3^2 + a_4X_1X_2 + a_5X_1X_3 + a_6X_2X_3 \\ A_2 & = & b_1X_1^2 + b_2X_2^2 - \dfrac{a_6}{2}X_3^2 + b_4X_1X_2 - a_4X_1X_3 - 2a_2X_2X_3 \\ A_3 & = & c_1X_1^2 - \dfrac{b_4}{2}X_2^2 + \dfrac{a_5}{2}X_3^2 - 2b_1X_1X_2 + 2a_1X_1X_3 + a_4X_2X_3. \end{array}$$

For $i = 5$ we have $\Omega_1 = X_2dX_3 - \alpha X_3dX_2$ thus $d\Omega_1 = (1+\alpha)dX_2 \wedge dX_3$ with $1 + \alpha \neq 0$. Thus $\Omega_2 \wedge d\Omega_1 = 0$ implies $A_3 = 0$. For $i = 6$, we have $d\Omega_1 = 2dX_2 \wedge dX_3$ and $\Omega_2 \wedge d\Omega_1 = 0$ implies also $A_3 = 0$.

Let us assume that $d\Omega_2 \neq 0$. If $\Omega_1 = \Omega_1^3, \Omega_1^5$, then $\Omega_2$ have to satisfy

$$\begin{cases} \Omega_1 \wedge d\Omega_2 = 0 \\ \Omega_2 \wedge d\Omega_2 = 0 \end{cases}.$$

This implies $Pd\Omega_2 = \Omega_1 \wedge \Omega_2$ where $P$ is an homogeneous polynomial of degree 2. We will solve this equation in each of these cases after a simplification of $\Omega_2$ by equivalence.

- $\Omega_1 = \Omega_1^3 = X_3dX_3$ .

Let us consider the equivalence of degree 2 given by $Y_i = X_i$ for $i = 1, 2$ and $Y_3 = X_3 + B$ with $B \in V_2$. We obtain the equivalent Poisson structure

$$\begin{cases} \{X_1, X_2\} = X_3 - B \\ \{X_1, X_3\} = \{X_1, B\} \\ \{X_2, X_3\} = \{X_2, B\} \end{cases}$$

Thus we can suppose that $\Omega = X_3dX_3 - A_2dX_2 + A_3dX_1$ with $A_2, A_3 \in V_2$. Let us note that an equivalence of degree 1 permits to simplify one coefficient to $A_2$ or $A_3$. The equation $\Omega_1 \wedge \Omega_2 = Pd\Omega_2$ is written

$$\begin{cases} -\partial_1 A_2 - \partial_2 A_3 = 0 \\ P\partial_3 A_3 = -X_3 A_3 \\ P\partial_3 A_2 = -X_3 A_2. \end{cases}$$

If $X_3$ is not a factor of $P$, then $\partial_3 A_2 = \alpha X_3$ and $\partial_3 A_3 = \beta X_3$. If $\alpha = \beta = 0$, then $A_2 = A_3 = 0$ and $\Omega_2 = 0$. Let us assume that $\alpha \neq 0$. Then

$$A_2 = b_1 X_1^2 + b_2 X_2^2 + b_3 X_3^2 + b_4 X_1 X_2$$

with $\alpha = 2b_3$. If $\beta \neq 0$, then $P = -(\frac{1}{b_3})A_2 = -(\frac{1}{\beta})A_3$. As we can simplify one coefficient of $A_2$ or $A_3$, this hypothesis can be eliminated. Then $\beta = 0$ this gives $A_3 = 0$. The first relation implies $-\partial_1 A_2 = 0$ that is

$$A_2 = b_2 X_2^2 + b_3 X_3^2.$$

We deduce the following Poisson structure given by

$$\Omega^8 = X_3 dX_3 - (b_2 X_2^2 + b_3 X_3^2) dX_2.$$

Let us assume now that $P = X_3 Q$ where $Q$ is a degree 1 homogeneous polynomial. This gives

$$\begin{cases} -\partial_1 A_2 - \partial_2 A_3 = 0 \\ Q\partial_3 A_3 = -A_3 \\ Q\partial_3 A_2 = -A_2. \end{cases}$$

If $b_3 \neq 0$,then we can consider that $b_3 = 1$ and

$$A_2 = (2X_3 + b_5 X_1 + b_6 X_2)(\frac{1}{2}X_3 + \frac{b_5}{4}X_1 + \frac{b_6}{4}X_2).$$

We deduce

$$A_3 = (\frac{1}{2}X_3 + \frac{b_5}{4}X_1 + \frac{b_6}{4}X_2)\partial_3 A_3.$$

As we can suppose that the coefficient of $X_3^2$ inv$A_3$ vanishes, thus $A_3 = 0$ and $\partial_1 A_2 = 0$. This gives

$$A_2 = (2X_3 + b_6 X_2)(\frac{1}{2}X_3 + \frac{b_6}{4}X_2) = \frac{b_6^2}{4}X_2^2 + X_3^2 + b_6 X_2 X_3 = (\frac{b_6}{2}X_2 + X_3)^2.$$

**Proposition 38** *Every Poisson structure on $\mathbb{C}[X_1, X_2, X_3]$ whose linear part is given by $\Omega_1 = X_3 dX_3$ is equivalent to*

$$\begin{cases} (\mathcal{P}_1) = \begin{cases} \{X_1, X_2\} = X_3 \\ \{X_1, X_3\} = b_2 X_2^2 + b_3 X_3^2 \\ \{X_2, X_3\} = 0. \end{cases} \\ (\mathcal{P}_2) = \begin{cases} \{X_1, X_2\} = X_3 \\ \{X_1, X_3\} = (aX_2 + X_3)^2 \\ \{X_2, X_3\} = 0. \end{cases} \\ (\mathcal{P}_3) = \begin{cases} \{X_1, X_2\} = X_3 \\ \{X_1, X_3\} = b_1 X_1^2 + b_2 X_2^2 + b_4 X_1 X_2 \\ \{X_2, X_3\} = c_1 X_1^2 - \frac{b_4}{2} X_2^2 - 2b_1 X_1 X_2. \end{cases} \end{cases}$$

The last case corresponds to $d\Omega_2 = 0$.

- $\Omega_1 = \Omega_1^4 = X_2 dX_3 + X_3 dX_2 + X_1 dX_1$ .

If we consider $\Omega_2 = 0$, the change of basis given by $Y_1 = X_1 + B, Y_2 = X_2, Y_3 = X_3$ gives a structure of degree 2 corresponding to $A_3 = -B$ equivalent to the structure associated to $\Omega = \Omega_1$. Thus we can suppose in $\Omega_2$ that $A_3 = 0$ that is $\Omega_2 = A_1 dX_3 - A_2 dX_2$. By hypothesis, $Pd\Omega_2 = \Omega_1 \wedge \Omega_2$ with $P \in V_2$. But

$$\Omega_1 \wedge \Omega_2 = -X_1 A_2 dX_1 \wedge dX_2 + X_1 A_1 dX_1 \wedge dX_3 + (A_2 X_2 + A_1 X_3) dX_2 \wedge dX_3$$

this gives

$$\begin{cases} P\partial_1 A_2 = X_1 A_2 \\ P\partial_1 A_1 = X_1 A_1 \\ P(\partial_2 A_1 + \partial_3 A_2) = (A_2 X_2 + A_1 X_3) \end{cases}$$

This polynomial system can be easily solved. We obtain the following solutions:

$$\begin{aligned}
A_1 &= \frac{\alpha_1}{2}X_1^2 + \frac{\alpha_2}{2\alpha_1}(\alpha_1 - a_6)X_2^2 + \frac{\alpha_1}{2\alpha_2}(\alpha_1 - a_6)X_3^2 + a_6X_2X_3 \\
A_2 &= \frac{\alpha_2}{\alpha_1}A_1
\end{aligned}$$

$$\begin{aligned}
A_1 &= 0 \\
A_2 &= \frac{\alpha_2}{2}X_1^2 + b_2X_2^2 + \alpha_2X_2X_3
\end{aligned}$$

$$\begin{aligned}
A_1 &= a_4X_1X_2 \\
A_2 &= -a_4X_1X_3
\end{aligned}$$

Then we have

**Proposition 39** *Every Poisson structure on $\mathbb{C}[X_1, X_2, X_3]$ whose linear part is given by $\Omega_1 = X_2dX_3 + X_3dX_2 + X_1dX_1$ is equivalent to*

$$\left\{\begin{array}{l}
(\mathcal{P}_4) = \left\{\begin{array}{l} \{X_1, X_2\} = X_2 + \frac{\alpha_1}{2}X_1^2 + \frac{\alpha_2}{2\alpha_1}(\alpha_1 - a_6)X_2^2 + \frac{\alpha_1}{2\alpha_2}(\alpha_1 - a_6)X_3^2 + a_6X_2X_3 \\ \{X_1, X_3\} = -X_3 + \frac{\alpha_2}{\alpha_1}(\frac{\alpha_1}{2}X_1^2 + \frac{\alpha_2}{2\alpha_1}(\alpha_1 - a_6)X_2^2 + \frac{\alpha_1}{2\alpha_2}(\alpha_1 - a_6)X_3^2 + a_6X_2X_3) \\ \{X_2, X_3\} = X_1. \end{array}\right. \\
(\mathcal{P}_5) = \left\{\begin{array}{l} \{X_1, X_2\} = X_2 \\ \{X_1, X_3\} = -X_3 + \frac{\alpha_2}{2}X_1^2 + b_2X_2^2 + \alpha_2X_2X_3 \\ \{X_2, X_3\} = X_1. \end{array}\right. \\
(\mathcal{P}_6) = \left\{\begin{array}{l} \{X_1, X_2\} = X_2 + a_4X_1X_2 \\ \{X_1, X_3\} = -X_3 - a_4X_1X_3 \\ \{X_2, X_3\} = X_1. \end{array}\right. \\
(\mathcal{P}_7) = \left\{\begin{array}{l} \{X_1, X_2\} = X_2 + a_2X_2^2 + a_3X_3^2 + a_6X_2X_3 \\ \{X_1, X_3\} = -X_3 + b_2X_2^2 - \frac{a_6}{2}X_3^2 - 2a_2X_2X_3 \\ \{X_2, X_3\} = X_1. \end{array}\right.
\end{array}\right.$$

The last case corresponds to $d\Omega_2 = 0$.

- $\Omega_1 = \Omega_1^5 = X_2dX_3 - \alpha X_3dX_2$

Let us assume that $\alpha \neq 0$ and $\alpha \neq -1$. The equivalence given by $Y_2 = X_2 + B_2, Y_i = X_i$ for $i = 1, 3$ and $B_2 \in V_2$ shows that the structure corresponding to $\Omega = \Omega_1$ is equivalent to a structure of degree 2 given by

$$\begin{array}{l}
A_1 = a_2X_2^2 + a_3X_3^2 + \frac{c_6}{\alpha}X_1X_2 + c_3X_1X_3 \\
A_2 = 0 \\
A_3 = c_3X_3^2 + c_5X_1X_3 + c_6X_2X_3
\end{array}$$

Thus we can assume that in $\Omega_2$ we have $c_3 = c_5 = c_6 = a_2 = a_3 = a_6 = 0$. The new equivalence of degree 2 given by $Y_3 = X_3 + B_3, Y_i = X_i$ for $i = 1, 2$ and $B_3 \in V_2$ gives a Poisson structure of degree 2 equivalent to the structure of degree 1 with

$$\begin{array}{l}
A_1 = 0 \\
A_2 = b_2X_2^2 + b_3X_3^2 - c_2X_1X_2 + \frac{c_6}{\alpha}X_1X_3 \\
A_3 = c_2X_2^2 + c_4X_1X_2 + c_6X_2X_3
\end{array}$$

Thus we can assume that

$$\Omega_2 = (a_1X_1^2 + a_4X_1X_2 + a_5X_1X_3)dX_1 + (b_1X_1^2 + b_4X_1X_2 + b_5X_1X_3)dX_2 + c_1X_1^2dX_3.$$

Now $\Omega_1 \wedge d\Omega_2 + \Omega_2 \wedge d\Omega_1 = 0$ implies

$$\Omega_2 = a_4X_1X_2dX_1 + alphaa_4X_1X_3dX_2.$$

If $\alpha = -1$ then $d\Omega_1 = 0$ and $\Omega_1 \wedge d\Omega_2 = 0$ implies

$$\Omega_2 = (A_1 = a_4X_1X_2 + a_6X_2X_3)dX_1 + (-a_4X_1X_3 + b_6X_2X_3)dX_2 + c_1X_1^2dX_3.$$

The equation $\Omega_2 \wedge d\Omega_2 = 0$ implies $c_1b_6 = c_1a_6 = a_4b_6 = a_4a_6 = 0$.

If $\alpha = 0$, by equivalence of degree 2 we can assume that

$$\Omega_2 = a_3X_3^2dX_1 + (b_1X_1^2 + b_3X_3^2 + b_5X_1X_3)dX_2 + (c_2X_2^2 + c_3X_3^2 + c_4X_1X_2 + c_6X_2X_3)dX_3.$$

But $\Omega_1 \wedge d\Omega_2 + \Omega_2 \wedge d\Omega_1 = 0$ and $\Omega_2 \wedge d\Omega_2 = 0$ implies $X_2(\partial_1A_2 + \partial_2A_3) + A_3 = 0$ that is $c_3 = 0$ and $a_3c_4 = a_3c_6 = b_3c_4 = b_3c_6 = 0$.

**Proposition 40** *Every Poisson structure on $\mathbb{C}[X_1, X_2, X_3]$ whose linear part is given by $\Omega_1 = X_2dX_3 - \alpha X_3dX_2$ is equivalent to*

$$\begin{cases}
(\mathcal{P}_8) = \begin{cases} \{X_1,X_2\} = X_2 + a_4X_1X_2 \\ \{X_1,X_3\} = \alpha X_3 + \alpha a_4X_1X_3 \qquad \alpha \neq -1, 0 \\ \{X_2,X_3\} = 0. \end{cases} \\
(\mathcal{P}_9, \alpha = -1) = \begin{cases} \{X_1,X_2\} = X_2 + a_6X_2X_3 \\ \{X_1,X_3\} = -X_3 + b_6X_1X_3 \\ \{X_2,X_3\} = 0. \end{cases} \\
(\mathcal{P}_{10}, \alpha = -1) = \begin{cases} \{X_1,X_2\} = X_2 + a_4X_1X_2 \\ \{X_1,X_3\} = -X_3 - a_4X_1X_3 \\ \{X_2,X_3\} = c_1X_1^2. \end{cases} \\
(\mathcal{P}_{11}, \alpha = 0) = \begin{cases} \{X_1,X_2\} = X_2 \\ \{X_1,X_3\} = b_1X_1^2 + b_5X_1X_3 \\ \{X_2,X_3\} = c_2X_2^2 + c_4X_1X_2 + c_6X_2X_3.. \end{cases} \\
(\mathcal{P}_{12}, \alpha = 0) = \begin{cases} \{X_1,X_2\} = X_2 + a_3X_3^2 \\ \{X_1,X_3\} = b_1X_1^2 + b_3X_3^2 + b_5X_1X_3 \\ \{X_2,X_3\} = c_2X_2^2. \end{cases}
\end{cases}$$

• $\Omega_1 = \Omega_1^6 = (X_2 + X_3)dX_3 - X_3dX_2$. By equivalence of degree 2, we can assume that $A_1 = 0, c_5 = 0$ and $b_4 = 0$. The equation $\Omega_1 \wedge d\Omega_2 + \Omega_2 \wedge d\Omega_1 = 0$ implies $c_1 = c_4 = b_1 = 0, c_6 = -b_5 = 2c_2$. The equation $\Omega_2 \wedge d\Omega_2 = 0$ implies $c_2 = 0$ and $b_2c_3 = b_6c_3 = 0$. Then we have

**Proposition 41** *Every Poisson structure on $\mathbb{C}[X_1, X_2, X_3]$ whose linear part is given by $\Omega_1 = (X_2 + X_3)dX_3 - X_3dX_2$ is equivalent to*

$$\begin{cases}
(\mathcal{P}_{13}) = \begin{cases} \{X_1,X_2\} = X_2 + X_3 \\ \{X_1,X_3\} = X_3 + b_2X_2^2 + b_3X_3^2 + b_6X_2X_3 \\ \{X_2,X_3\} = 0. \end{cases} \\
(\mathcal{P}_{14}) = \begin{cases} \{X_1,X_2\} = X_2 + X_3 \\ \{X_1,X_3\} = X_3 + b_3X_3^2 \\ \{X_2,X_3\} = c_3X_3^2. \end{cases}
\end{cases}$$

*Second case* :$\Omega_0 \neq 0$.

Here the Poisson structure is given by $\Omega = \Omega_0 + \Omega_1 + \Omega_2$. As $\Omega_0$ is of degree 0, the $d\Omega_0 = 0$. We have

$$\begin{cases} \Omega_0 \wedge d\Omega_1 = 0 \\ \Omega_0 \wedge d\Omega_2 + \Omega_1 \wedge d\Omega_1 = 0 \\ \Omega_1 \wedge d\Omega_2 + \Omega_2 \wedge d\Omega_1 = 0 \\ \Omega_2 \wedge d\Omega_2 = 0. \end{cases}$$

The form $\Omega_0 \oplus \Omega_1$ provided the vector space $V_0 \oplus V_1$ with a linear Poisson bracket. Then $V_0 \oplus V_1$ is a ie algebra such that $V_0$ is in the center. This implies $\Omega_1 \wedge d\Omega_1 = 0$. The form $\Omega_2 = A_1 dX_3 - A_2 dX_2 + A_3 dX_1$ satisfies:

$$\begin{cases} \Omega_0 \wedge d\Omega_2 = 0 \\ \Omega_1 \wedge d\Omega_2 + \Omega_2 \wedge d\Omega_1 = 0 \\ \Omega_2 \wedge d\Omega_2 = 0. \end{cases}$$

•. $\Omega = \Omega^7 = \Omega_0^2 + Omega_1^7 = dX_3 - X_3 dX_2$. By equivalence we can assume that $a_3 = a_5 = b_3 = c_5 = 0$. The equation $\Omega_0 \wedge d\Omega_2 = 0$ implies $b_4 = -2c_2, c_4 = -2b_1, c_6 = -b_5$, $\Omega_1 \wedge d\Omega_2 + \Omega_2 \wedge d\Omega_1 = 0$ implies$c_1 = c_2 = c_3 = c_4 = 0, a_1 = a_4 = 0$, and $\Omega_2 \wedge d\Omega_2 = 0$ gives $b_5 b_2 = b_5 a_2 = b_5 a_6 = 0$.

**Proposition 42** *Every Poisson structure on $\mathbb{C}[X_1, X_2, X_3]$ given by $\Omega = dX_3 - X_3 dX_2 + \Omega_2$ is equivalent to*

$$\begin{cases} (\mathcal{P}_{15}) = \begin{cases} \{X_1, X_2\} = 1 + a_2 X_2^2 + a_6 X_2 X_3 \\ \{X_1, X_3\} = X_3 + b_2 X_2^2 + b_6 X_2 X_3 \\ \{X_2, X_3\} = 0. \end{cases} \\ (\mathcal{P}_{16}) = \begin{cases} \{X_1, X_2\} = 1 \\ \{X_1, X_3\} = X_3 + b_5 X_1 X_3 + b_6 X_2 X_3 \\ \{X_2, X_3\} = -b_5 X_2 X_3. \end{cases} \end{cases}$$

• $\Omega = \Omega^8 = \Omega_0^2 + \Omega_1^8 = X_3 dX_3 - dX_2$. We can assume that $A_2 = b_2 X_2^2 + b_4 X_1 X_2$. As $d\Omega_1 = 0$, the system is reduced to $\Omega_0 \wedge d\Omega_2 = \Omega_1 \wedge d\Omega_2 = 0$. This gives $c_4 = c_6 = a_4 = 0$ and $b_4 + 2c_2 = a_5 - 2c_3 = 2a_1 - c_5 = 0$. Thus $\Omega_2 \wedge d\Omega_2 = 0$ is equivalent to $(2a_2 X_2 + a_6 X_3) A_3 = 0$.

**Proposition 43** *Every Poisson structure on $\mathbb{C}[X_1, X_2, X_3]$ given by $\Omega = X_3 dX_3 - dX_2$ is equivalent to*

$$\begin{cases} (\mathcal{P}_{17}) = \begin{cases} \{X_1, X_2\} = 1 + a_1 X_1^2 + a_2 X_2^2 + a_3 X_3^2 + a_5 X_1 X_3 + a_6 X_2 X_3 \\ \{X_1, X_3\} = X_2 + b_2 X_2^2 + b_4 X_1 X_2 \\ \{X_2, X_3\} = 0. \end{cases} \\ (\mathcal{P}_{18}) = \begin{cases} \{X_1, X_2\} = 1 + a_1 X_1^2 + a_3 X_3^2 + a_5 X_1 X_3 \\ \{X_1, X_3\} = X_2 + b_5 X_1 X_3 + b_6 X_2 X_3 \\ \{X_2, X_3\} = c_1 X_1^2 - \frac{b_4}{2} X_2^2 + \frac{a_5}{2} X_3^2 + 2a_1 X_1 X_3. \end{cases} \end{cases}$$

• $\Omega = \Omega^9 = \Omega_0^2 + \Omega_1^9 = X_2 dX_3 + X_3 dX_2 + dX_1$. By equivalence we can assume $b_5 = b_2 = a_3 = a_5 = c_2 = c_5 = 0$. As $d\Omega_1 = 0$, the equation $\Omega_0 \wedge d\Omega_2 = \Omega_1 \wedge d\Omega_2 = 0$ implies $b_6 + 2a_2 = a_6 + 2b_3 = a_4 = a_1 = b_1 = b_4 = c_3 = 0$ . In this case $\Omega_2 \wedge d\Omega_2 = 0$ is equivalent to $c_6(X_2 A_2 + X_3 A_1) = 0$.

**Proposition 44** *Every Poisson structure on $\mathbb{C}[X_1, X_2, X_3]$ given by $\Omega = X_2 dX_3 + X_3 dX_2 + dX_1$ is equivalent to*

$$\begin{cases} (\mathcal{P}_{19}) = \begin{cases} \{X_1, X_2\} = X_2 + a_2 X_2^2 + a_6 X_2 X_3 \\ \{X_1, X_3\} = -X_3 - \frac{a_6}{2} X_3^2 - 2a_2 X_2 X_3 \\ \{X_2, X_3\} = 1 + c_1 X_1^2 \end{cases} \\ (\mathcal{P}_{20}) = \begin{cases} \{X_1, X_2\} = X_2 \\ \{X_1, X_3\} = -X_3 \\ \{X_2, X_3\} = 1 + c_1 X_1^2. \end{cases} \end{cases}$$

## 7.3 Poisson algebras associated to rigid Lie algebras

### 7.3.1 Rigid Lie algebras

We fix a basis of $\mathbb{C}^n$. With respect to this basis, a multiplication $\mu$ of a $n$-dimensional complex Lie algebra is determined by its structure constants $C_{ij}^k$ which satisfy the Jacobi polynomial conditions. We denote by

$L_n$ the algebraic variety $\mathbb{C}[\mathbb{C}_{ij}^k]/I$ where $I$ is the ideal generated by the polynomials:

$$\begin{cases} C_{ij}^k + C_{ji}^k = 0 \\ \sum_{l=1}^n C_{ij}^l C_{lk}^s + C_{jk}^l C_{li}^s + C_{ki}^l C_{li}^s = 0 \end{cases}$$

for all $1 \leq i,j,k,s \leq n$. Then every multiplication $\mu$ of a $n$-dimensional complex Lie algebra is identified to one point of $L_n$. We have a natural action of the algebraic group $Gl(n,\mathbb{C})$ on $L_n$ whose orbits correspond to the classes of isomorphic multiplication:

$$\mathcal{O}(\mu) = \left\{ f^{-1} \circ \mu \circ (f \times f), \quad f \in Gl(n,\mathbb{C}) \right\}.$$

Let $\mathfrak{g} = (\mathbb{C}^n, \mu)$ be a $n$-dimensional complex Lie algebra. We denote always by $\mu$ the corresponding point of $L_n$.

**Definition 62** *The Lie algebra $\mathfrak{g}$ is called rigid if its orbit $\mathcal{O}(\mu)$ is open (for the Zariski topology) in $L_n$.*

Amongst rigid complex Lie algebras, there are all the simple and semi-simple Lie algebras, all the Borel algebras and parabolic Lie algebras. We know also the classification of rigid Lie algebra up the dimension 8 ([?] ), the classification of solvable rigid Lie algebras whose nilradical is filiform ([?]). But to day we do not know any rigid nilpotent Lie algebras. Recall two interesting tools to look the rigidity or not of a given Lie algebra :

**Theorem 63** *Let $\mathfrak{g} = (\mathbb{C}^n, \mu)$ a $n$-dimensional complex Lie algebra. Then*

*1. $\mathfrak{g}$ is rigid if and only if any valued deformation $\mathfrak{g}'$ where the structure constants are in a valuation ring $R$ is ($K^*$)-isomorphic to $\mathfrak{g}$ where $K^*$ is the fraction field of $R$.*

*2. If $H^2(\mathfrak{g},\mathfrak{g}) = 0$, then $\mathfrak{g}$ is rigid.*

The notion of valued deformation, which extends of natural way the classical notion of Gerstenhaber deformations also called formal deformations is developed in [23]

The second part of this theorem is the Nijenhuis-Richardson theorem. But its converse is not true. There exists solvable rigid Lie algebras with $H^2(\mathfrak{g},\mathfrak{g}) \neq 0$. This fact can be interpreted as follows: let

$$\mu_t = \mu + \sum_{i \geq 1} t^i \varphi_i$$

a deformation of $\mu$ with coefficient in the valued ring of formal series $\mathbb{C}[[t]]$. As $\mu_t$ is a multiplication of Lie algebra, this implies in particular that $\varphi_1 \in H^2(\mathfrak{g},\mathfrak{g})$. If $\mu$ is rigid, then, from the first previous result, $\mu_t$ is isomorphic to $\mu$, the isomorphism belonging to $Gl(n,\mathbb{C}[[t]])$. If $H^2(\mathfrak{g},\mathfrak{g}) = 0$, then every deformation of $\mu$ is isomorphic to $\mu$ and $\mu$ is rigid. If $H^2(\mathfrak{g},\mathfrak{g}) \neq 0$, then $\mu$ is not rigid or $\mu$ is rigid and there is $\varphi_1 \neq 0 \in H^2(\mathfrak{g},\mathfrak{g})$ which never is the first term of a deformation of $\mu$.

### 7.3.2 Finite dimensional rigid Poisson algebras

We recall in this section a result of [29] which precises the structure of a finite dimensional complex Poisson algebras whose the underlying Lie bracket is rigid. Let $\mathcal{P}$ a finite dimensional complex Lie algebra. We denote by $[X,Y]$ and $X.Y$ the corresponding Lie bracket and associative multiplication. If the bracket corresponds to a simple complex (and thus rigid) Lie algebra, then the associative product is trivial : $X.Y = 0$. Let us assume now that the Lie bracket corresponds to a rigid solvable Lie algebra. We recall the following result:

**Proposition 45** *Let $\mathfrak{g}$ be a $n$-dimensional complex solvable rigid Lie algebra. Then $\mathfrak{g}$ is written:*

$$\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{n}$$

*where $\mathfrak{n}$ is the nilradical of $\mathfrak{g}$ and $\mathfrak{t}$ a maximal abelian subalgebra such that the adjoint operators $adX$ are diagonalizable for every $X \in \mathfrak{t}$.*

This subalgebra $\mathfrak{t}$ is usually called the Malcev torus. All these maximal torus are conjugated and their common dimension is called the rank of $\mathfrak{g}$. Then we have [29]

**Proposition 46** *Let $\mathfrak{g}$ be a rigid solvable Lie algebra of rank 1 with non-zero roots. Then there is only one Poisson algebra $\mathcal{P}$ such that $\mathfrak{g}_{\mathcal{P}} = \mathfrak{g}$. It is defined by*

$$X_i \cdot X_j = \{X_i, X_j\}.$$

### 7.3.3 Cohomology of $\mathcal{A}_{n+1} = \mathbb{C}[X_0, X_1, \cdots, X_n]$ associated to a rigid Lie bracket

In this section we consider a linear Poisson bracket on $\mathbb{C}[X_0, \cdots, X_n]$ such that the brackets $\{X_i, X_j\} = \mathcal{P}(X_i, X_j)$ corresponds to a solvable rigid Lie algebra $\mathfrak{g}$ of rank 1. We assume that the roots [?] of this rigid Lie algebras are $1, \cdots, n$. In this case we have

$$\left\{ \begin{array}{l} \{X_0, X_i\} = iX_i, \quad i = 1, \cdots, n \\ \{X_1, X_i\} = X_{i+1}, \quad i = 2, \cdots, n-1 \\ \{X_2, X_i\} = X_{i+2}, \quad i = 3, \cdots, n-2. \end{array} \right.$$

We denote this $(n+1)$-dimensional Poisson algebra by $\mathcal{P}(\mathfrak{g})$. This algebra is a deformation of the Poisson algebra studied in section 1.2. The corresponding $(n-1)-$exterior form is

$$\begin{array}{ll} \Omega & = \sum_{i=1}^{n} (-1)^{i-1} X_i d_1 \wedge ... \wedge \hat{d_i} \wedge ... \wedge d_n + \sum_{i=2}^{n-1} (-1)^i X_{i+1} d_0 \wedge d_2 \wedge ... \wedge \hat{d_i} \wedge ... \wedge d_n \\ & + \sum_{i=3}^{n-2} (-1)^{i+1} X_{i+2} d_0 \wedge d_1 \wedge d_3 \wedge ... \wedge \hat{d_i} \wedge ... \wedge d_n \end{array}$$

where $d_i$ denotes $dX_i$ and $\hat{d_i}$ signifies that this term does not appear is the considered expression. Let $\varphi$ be a 2-cochain. We denote by $\varphi(i,j)$ the vector $\varphi(X_i, X_j)$. The $\varphi$ is a 2 cocycle if and only if

$$\begin{array}{rcl} \Phi_{n-1}(\varphi) & = & (-1)^{n-2} \varphi(1,i) d_0 \wedge d_2 \wedge ... \wedge \hat{d_i} \wedge ... \wedge d_n \\ & + & \sum_{i=3}^{n} (-1)^{i-1} \varphi(2,i) d_0 \wedge d_1 \wedge d_3 \wedge ... \wedge \hat{d_i} \wedge ... \wedge d_n \\ & + & \sum_{3 \leq i < j \leq n} (-1)^{j-i-1} \varphi(i,j) d_0 \wedge ... \wedge \hat{d_i} \wedge ... \wedge \hat{d_j} \wedge ... \wedge d_n \end{array}$$

satisfies

$$d(i(\partial_{\sigma(1)}, ..., \partial_{\sigma(n-2)})\Omega) \wedge \Phi_{n-1}(\varphi) + \Omega \wedge d[i(\partial_{\sigma(1)}, ..., \partial_{\sigma(n-2)}) \Phi_{n-2}(\varphi)] = 0 \tag{7.1}$$

for any $\sigma \in S_{3,n-2}$.

As $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{n}$, we have the decomposition $\mathcal{P}(\mathfrak{g}) = \mathcal{P}(\mathfrak{t}) \oplus \mathcal{P}(\mathfrak{n})$ where $\mathcal{P}(\mathfrak{t})$ (respect. $\mathcal{P}(\mathfrak{n})$) is the polynomial algebra generated by $X_0$ (respec. by $X_1, ..., X_n$]). From the Hochschild-Serre factorization theorem, we assume that the cocycles are $\mathfrak{t}$-invariant and with values in $\mathcal{P}(\mathfrak{n})$. We denote this space by $\chi^k(\mathcal{P}(\mathfrak{g}), \mathcal{P}(\mathfrak{g}))^{\mathfrak{t}}$. If $f \in \chi^1(\mathcal{P}(\mathfrak{g}), \mathcal{P}(\mathfrak{g}))^{\mathfrak{t}}$ then

$$\{X_0, f(X_i)\} = if(X_i)$$

and we obtain

$$f(X_1) = a_1^1 X_1, f(X_2) = a_1^{11} X_1^2 + a_2^2 X_2, ..., f(X_i) = \sum_{l_1 + ... + l_k = i} a_i^{l_1 ... l_k} X_1^{l_1} ... X_k^{l_k}.$$

Thus $\delta f(X_1, X_i) = a_1^1 \{X_1, X_i\} + \{X_1, f(X_i)\} - f(X_{i+1})$ and we can reduce any element $\varphi \in Z^2(\mathcal{P}(\mathfrak{g}, \mathcal{P}(\mathfrak{g})^{\mathfrak{t}}$ to a 2-cocycle satisfying

$$\varphi(X_1, X_i) = 0 \text{ for } i = 2, ..., n-1.$$

We denote by $Z_k^*(\mathcal{P}(\mathfrak{g}, \mathcal{P}(\mathfrak{g}))^{\mathfrak{t}}$ (resp $B_k^*, H_k^*$) the subspace homogeneous cocycle of degree $k$.

Let us look the system on the $\varphi(i,j)$ which is deduced from equation(1).

- If $(\sigma(1),...,\sigma(n-2))=(3,4,...,n)$ then (1) is trivial.
- If $(\sigma(1),...,\sigma(n-2))=(2,3,...,\hat{l},...,n)$ then (1) is trivial as soon as $l\neq n$. If $l=n$ then (1) is reduced to

$$n\varphi(X_1,X_n)+(-1)^{n-1}\sum iX_i\partial_i\varphi(X_1,X_n)=0$$

and $\varphi(X_1,X_n)$ is of weight $n+1$.
- If $(\sigma(1),...,\sigma(n-2))=(1,2,...,\hat{\imath},...,\hat{\jmath}...,n)$ we obtain

$$(i+j)\varphi(X_i,X_j)=\sum kX_k\partial_k\varphi(X_i,X_j)$$

and $\varphi(X_i,X_j)$ is of weight $i+1$.
- If $(\sigma(1),...,\sigma(n-2))=(0,1,2,...,\hat{\imath},...,\hat{\jmath},...,\widehat{k},\dots,n)$ with $i\geq 3$, then we obtain

$$\Omega\wedge d(\varphi(X_i,X_j)dX_k+\varphi(X_i,X_k)dX_j+\varphi(X_j,X_k)dX_i)=0$$

- If $(\sigma(1),...,\sigma(n-2))=(0,3,...,\hat{l},...,n)$ we obtain relation between $\varphi(3,l)$ and $\varphi(2,l+1)$. We deduce that $\varphi(2,i)$ generates $Z_1^2(A_p,A_p)^t$ more precisely we have

$$\varphi(X_2,X_{i+1})+\varphi(X_3,X_i)=X_3\partial_2\varphi(2,i)+X_4\partial_3\varphi(2,i)+...+X_n\partial_{n-1}\varphi(2,i).$$

In case of $k=1$, $\varphi(2,i)=a_{2i}X_{2+i}$ if $i\leq n-2$.

**Case** $k=1$ As $\varphi(X_1,X_n)$ is of weight $n+1$, then $\varphi(X_1,X_n)=0$. We have also $\varphi(X_i,X_j)=a_{ij}^{i+j}X_{i+j}$ if $i+j\leq n$.

If $(\sigma(1),...,\sigma(n-2))=(1,2,...,i\hat{},...,j\hat{},...,n)$ we obtain $(i+j)\varphi(X_i,X_j)=\sum kX_k\partial_k\varphi(X_i,X_j)$ and $\varphi(X_i,X_j)$ is of weight $i+1$. Then $\varphi(X_i,X_j)=a_{ij}^{i+j}X_{i+j}$ if $i+j\leq n$.

If $(\sigma(1),...,\sigma(n-2))=(0,3,...,l\hat{},...,n)$ we obtain relation between $\varphi(3,l)$ and $\varphi(2,l+1)$. We deduce that $\varphi(2,i)$ generates $Z_1^2(A_p,A_p)^t$ more precisely we have

$$\varphi(X_2,X_{i+1})+\varphi(X_3,X_i)=X_3\partial_2\varphi(2,i)+X_4\partial_3\varphi(2,i)+...+X_n\partial_{n-1}\varphi(2,i).$$

In case of $k=1$, $\varphi(2,i)=a_{2i}X_{2+i}$ if $i\leq n-2$.

Then we have $a_{2(i+1)}+a_{3i}=a_{2i}$ if $3\leq i\leq n-2$. Their relation gives

$$\begin{cases} a_{i+1j}+a_{ij+1}=a_{ij}\\ a_{i+2j}+a_{ij+2}=a_{ij}\end{cases}.$$

We deduce that

**Lemma 8** *If $n\geq 7$, then $H_1^2(A_p,A_p)$ is of dimension 1 and generated by the cocycle given by*

$$\begin{cases} \varphi(X_2,X_i)=(i-1)X_{2+i} \quad i=4,...,n-2\\ \varphi(X_3,X_i)=X_{3+i} \quad i=4,...,n-3\\ \varphi(X_i,X_j)=0 \quad \textit{in other case}\end{cases}.$$

The nilradical is filiform. These algebras have been studied in [**?**]. We have

- For $n=2,3,4$, then $\mathfrak{g}_n$ is not rigid.
- For $n=5,6$, then $\mathfrak{g}_n$ is rigid with $H^2(\mathfrak{g}_n,\mathfrak{g}_n)=\{0\}$.
- For $7\leq n\leq 11$, $\mathfrak{g}_n$ is not rigid.
- For $n\geq 12$, then $\mathfrak{g}_n$ is rigid with $dimH^2(\mathfrak{g}_n,\mathfrak{g}_n)=1$. In this case a basis of $H^2(\mathfrak{g}_n,\mathfrak{g}_n)$ is given by the cohomology class of the 2-cocycle which satisfies

$$\begin{cases} \phi(X,X_i)=0, \quad i\geq 1\\ \phi(X_1,X_i)=0, \quad i\geq 2\\ \phi(X_2,X_3)=0, \quad , \ \phi(X_2,X_i)=(4-i)X_{i+2}, \quad i\geq 4\\ \phi(X_3,X_i)=i+3, \quad i\geq 4\\ \phi(X_i,X_j)=0, \quad 4\leq i<j<n\end{cases}$$

### 7.3.4 Deformations of Enveloping algebra of rigid Lie algebra

In $[G, A]$ the rigid solvable Lie algebras such that $n$ is a filiform Lie algebras (that is of maximal nilindex) are classified using the determination of the set of roots associated to the action of $t$ on $n$. In the next section, we consider $H^2(A_3, A_3)$ where $A_3$ is the Poisson algebra on $A$ where linear part is nothing that this rigid Lie algebra.

**The Poisson algebra $A_3$**

Let $\mathcal{G}$ be the Lie algebra defined in the Poisson $\{X_0, X_1, ..., X_n\}$ by

$$\left\{ \begin{array}{ll} [X_0, X_i] = iX_i & i = 1, ..., n \\ [X_1, X_i] = X_{i+1} & i = 2, ..., n-1 \\ [X_2, X_i] = X_{i+2} & i = 3, ..., n-2 \end{array} \right. .$$

The rigidity of the Lie algebra is proved in $[CA]$. Let $P$ be the Poisson bracket on $\mathbb{C}[X_0, ..., X_n]$ such that

$$\{X_i, X_j\} = P(X_i, X_j) = [X_i, X_j].$$

The corresponding $(n-1)-$exterior form is

$$\Omega = \sum_{i=1}^{n}(-1)^{i-1}X_i d_1 \wedge ... \wedge \hat{d_i} \wedge ... \wedge d_n + \sum_{i=2}^{n-1}(-1)^{i}X_{i+1}d_0 \wedge d_2 \wedge ... \wedge \hat{d_i} \wedge ... \wedge d_n + \sum_{i=3}^{n-2}(-1)^{i+1}X_{i+2}d_0 \wedge d_1 \wedge d_3 \wedge ... \wedge \hat{d_i} \wedge ... \wedge d_n$$

where $d_i$ denotes $dX_i$ and $\hat{d_i}$ signifies that this faction does not appear is the considered expression.

The $\varphi$ is a 2 cocycle if and only if

$$\begin{array}{rcl} \Phi_{n-1}(\varphi) & = & (-1)^{n-2}\varphi(1, i)d_0 \wedge d_2 \wedge ... \wedge \hat{d_i} \wedge ... \wedge d_n \\ & + & \sum_{i=3}^{n}(-1)^{i-1}\varphi(2, i)d_0 \wedge d_1 \wedge d_3 \wedge ... \wedge \hat{d_i} \wedge ... \wedge d_n \\ & + & \sum_{3 \leq i < j \leq n}(-1)^{j-i-1}\varphi(i, j)d_0 \wedge ... \wedge \hat{d_i} \wedge ... \wedge \hat{d_j} \wedge ... \wedge d_n \end{array}$$

satisfies

$$d(i(\partial_{\sigma(1)}, ..., \partial_{\sigma(n-2)})\Omega) \wedge \Phi_{n-1}(\varphi) + \Omega \wedge d[i(\partial_{\sigma(1)}, ..., \partial_{\sigma(n-2)})\Phi_{n-2}(\varphi)] = 0$$

for any $\sigma \in S_{3,n-2}$.

As $\mathcal{G} = t \oplus n$, we have the decomposition $A_3 = A_3(t) \oplus A_3(n)$ where $A_3(t)$ is the polynomial algebra generated by $X_0$ and $A(n) = \mathbb{C}[X_1, ..., X_n]$. From the Hochschild-Serre factorization theorem and with assume that the cocycles are $t$-invariant and with values in $A_3(n)$. We denote this space by $\chi^k(A_3, A_3(n))^t$. If $f \in \chi^1(A_3, A_3(n))^t$ then $f\{X_0, X_i\} = if(X_i)$ and we obtain $f(X_1) = a_1^1 X_1$, $f(X_2) = a_1^{11}X_1^2 + a_2^2 X_2, ..., f(X_i) = \sum_{l_1+...+l_k=i} a_i^{l_1...l_k} X_1^{l_1}...X_k^{l_k}$.

Thus $\delta f(X_1, X_2) = a_1^1\{X_1, X_2\} + \{X_1, f(X_1)\} - f'(X_{i+1})$ and we can reduce any element $\varphi \in Z^2(A_P, A'_P(n))^t$ to a 2-cocycle satisfying

$$\varphi(X_1, X_i) = 0 \text{ for } i = 2, ..., n-1.$$

We denote by $Z_k^*(A_p, A_p(n))^t$ (resp $B_k^*, H_k^*$) the subspace homogeneous cocycle of degree $k$.

**Case** $k = 1$ If $(\sigma(1), ..., \sigma(n-2)) = (3, 4, ..., n)$ then is trivial.

If $(\sigma(1), ..., \sigma(n-2)) = (2, 3, ..., \hat{k}, ..., n)$ then is reduced to $n\varphi(X_1, X_n) + (-1)^{n-1}\sum iX_i\partial_i\varphi(X_1, X_n) = 0$ and $\varphi(X_1, X_n)$ is of weight $n+1$. Then, as $k = 1$, $\varphi(X_1, X_n) = 0$.

If $(\sigma(1), ..., \sigma(n-2)) = (1, 2, ..., \hat{i}, ..., \hat{j}, ..., n)$ we obtain $(i+j)\varphi(X_i, X_j) = \sum kX_k\partial_k\varphi(X_i, X_j)$ and $\varphi(X_i, X_j)$ is of weight $i+1$. Then $\varphi(X_i, X_j) = a_{ij}^{i+j}X_{i+j}$ if $i + j \leq n$.

If $(\sigma(1),...,\sigma(n-2)) = (0,3,...,\hat{l},...,n)$ we obtain relation between $\varphi(3,l)$ and $\varphi(2,l+1)$. We deduce that $\varphi(2,i)$ generates $Z_1^2(A_p,A_p)^t$ more precisely we have

$$\varphi(X_2,X_{i+1}) + \varphi(X_3,X_i) = X_3\partial_2\varphi(2,i) + X_4\partial_3\varphi(2,i) + ... + X_n\partial_{n-1}\varphi(2,i).$$

In case of $k=1$, $\varphi(2,i) = a_{2i}X_{2+i}$ if $i \leq n-2$.

Then we have $a_{2(i+1)} + a_{3i} = a_{2i}$ if $3 \leq i \leq n-2$. Their relation gives

$$\begin{cases} a_{i+1j} + a_{ij+1} = a_{ij} \\ a_{i+2j} + a_{ij+2} = a_{ij} \end{cases}.$$

We deduce that

**Lemma 9** *If $n \geq 7$, then $H_1^2(A_p,A_p)$ is of dimension 1 and generated by the cocycle given by*

$$\begin{cases} \varphi(X_2,X_i) = (i-1)X_{2+i} \quad i=4,...,n-2 \\ \varphi(X_3,X_i) = X_{3+i} \quad i=4,...,n-3 \\ \varphi(X_i,X_j) = 0 \quad \textit{in other case} \end{cases}.$$

**Application.** Let us consider the multiplication given by

$$\begin{cases} \mu(X_0,X_i) = iX_i \quad i=1,...,n \\ \mu(X_1,X_i) = X_{i+1} \quad i=2,...,n-1 \\ \mu(X_2,X_3) = X_5 \\ \mu(X_2,X_i) = (5-i)X_{2+i} \quad i=4,...,n-2 \\ \mu(X_3,X_i) = X_{3+i} \quad i=4,...,n-3 \end{cases}.$$

This product is not a Lie algebra product because $\varphi$ is a not integrable cocycle. But $\mu$ as a "non Lie" deformation of the product of the rigid algebra $\mathcal{G}$.

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# Contents