GROUP GRADINGS ON LIE ALGEBRAS, WITH APPLICATIONS TO GEOMETRY. I

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ABSTRACT. In this article, which is the first part of a sequence of two, we discuss modern approaches to the classification of group gradings on simple and nilpotent Lie algebras. In the second article we discuss applications and related topics in differential geometry.

1. INTRODUCTION

This article is the first of the sequence of two articles in which we discuss several topics related to the first author’s presentations given at the workshops “Lie Groups, Lie Algebras and their Representations” carried out at the University of Southern California and several campuses of the University of California. The main topics covered were the theory and representations of simple locally finite Lie algebras, Lie superalgebras and group gradings on simple Lie algebras, with applications to differential geometry. The most recent talk, in 2010 at the University of Southern California, was devoted to a more abstract topic: the study of the distortion arising when one Lie algebra is embedded in another as a subalgebra.

At least in one of the areas, namely, in the theory of group gradings on simple Lie algebras, the development that followed, was truly spectacular. Thanks to the efforts of several researchers in several countries, now we have a complete classification of abelian group gradings on all classical simple Lie algebras over an algebraically closed fields of characteristic different from 2. The detailed exposition of this theory can now be found in a monograph [19], which has incorporated the results of [6], where the above classification, up to the isomorphism, has been completed, with the exception of $D_4$ and $psl_3$. In the same monograph [19], the authors present the classification of group grading on almost all exceptional algebras, which heavily depends on the study of various classes of nonassociative algebras, such as octonions, Albert algebra, etc. In the case char $F = 0$, a description of fine gradings on simple Lie algebras, with the same exception, as above, has appeared in [22] and a description of all group gradings was obtained in [9] and [11].

Lately, the study of the gradings on Lie algebras went beyond the boundaries of the area of classical or exceptional simple finite-dimensional algebras. In the paper [3] the authors classify up to isomorphism all abelian group gradings on finitary simple Lie algebras of linear transformations. In the paper [7] the authors classify all abelian group gradings on two types of restricted simple Lie algebras of Cartan type over algebraically closed field of characteristic $p > 3$.

These results became possible thanks to the extensive use of the methods involving the techniques of the theory of Hopf algebras. Probably the first papers where this approach was successfully implemented were [5] and [8]. This is a well-known fact that an algebra $A$ over a field $F$ graded by an abelian group $G$ bears a canonical structure of the (right) comodule algebra over the group algebra $H = FG$ (see [24]). Conversely, the $H$-comodule algebra
canonically becomes a $G$-graded algebra. If $A$ is an associative envelope of a Lie algebra $L$ then in some cases of interest to us, the $H$-comodule structure of $L$ can be extended to $H$-comodule structure of $A$. Then $A$ becomes $G$-graded and the grading of $L$ becomes the restriction of a $G$-grading of an associative algebra $A$. It is a usual pattern that the gradings of simple associative algebras are easier to determine.

The problem of extending the $H$-comodule structure from Lie to associative algebras in the case of finite-dimensional algebras is approached using the technique of affine group schemes [32]. These techniques, however, do not apply in the case of infinite-dimensional algebras. In the case where the algebras are infinite-dimensional (or just of sufficiently great dimension) quite another approach works. As it turns out, the question about the possibility of extending the $H$-comodule structure from a Lie to an associative algebra is analogous to the problem of extending automorphisms of Lie algebras to the automorphisms or negatives of antiautomorphisms of their associative envelopes. This latter problem, named after I. Herstein, was successfully solved in the 90’s using so called functional identities (see [18]). An appropriate adaptation of the methods of this book made it possible to classify in [3] all abelian group gradings on simple finitary Lie algebras over algebraically closed fields of characteristic different from 2 and 3. Note that these results are also related to the theory of locally finite simple Lie algebras, which was one of the topics of research of several people related to the West Coast Lie Theory workshop.

Since Cartan decompositions of semisimple Lie algebras are a special case of fine gradings, it is our strong belief that the classification of all gradings on simple Lie algebras should be important for the study of their structure, representations and other properties.

For examples, the study of graded identical relations of algebras lately became an important branch of the theory of algebras with polynomial identities (PI-algebras). Graded polynomial identities of algebras are often much easier to study, yet they define the ordinary ones.

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2. **Group gradings of Lie algebras**

In what follows $\mathbb{K}$ is an algebraically closed field of characteristic different from 2. For more details on group gradings of algebras see a recent monograph [19]. These definitions apply to other classes of algebras, with an appropriate change of notation.

**2.1. Definition.** Let $L$ be a Lie algebra over $\mathbb{K}$ and $S$ a set.

**Definition 2.1.** An $S$-grading $\Gamma$ of $L$ with support $S$ is a decomposition

$$\Gamma : L = \bigoplus_{s \in S} L_s$$

of $L$ as the sum on nonzero vector subspaces $L_s$ satisfying the following condition. For any $s_1, s_2 \in S$ such that $[L_{s_1}, L_{s_2}] \neq 0$ there is $\mu(s_1, s_2) \in S$ such that $[L_{s_1}, L_{s_2}] \subset L_{\mu(s_1, s_2)}$.

If $S$ is a subset of a group $G$ such that $\mu(s_1, s_2) = s_1s_2$ in $G$, we say that $\Gamma$ is a group grading by the group $G$ with support $S$. Such a group $G$ is not defined uniquely, but for any group grading there a universal grading group $U(\Gamma)$ such that any other grading group $G$ of $\Gamma$ is a factor group of $U(\Gamma)$. The universal group is given in terms of generators and defining relations if one chooses $S$ as the set of generators and all the above equations $s_1s_2 = \mu(s_1, s_2)$ as the set of defining relations.
2.2. Equivalent and isomorphic gradings. If $\Gamma' : L = \bigoplus_{s' \in S'} L'_{s'}$ is another grading of $L$, we say that $\Gamma$ is equivalent to $\Gamma'$ if there is an automorphism $\varphi \in \text{Aut} L$ and a bijection $\sigma : S \to S'$ such that $\varphi(L_s) = L'_{\sigma(s)}$. It follows that $\sigma(\mu(s_1, s_2)) = \mu(\sigma(s_1), \sigma(s_2))$.

Two group gradings $\Gamma$ and $\Gamma'$ of a Lie algebra $L$ by groups $G$ and $G'$ with supports $S$ and $S'$ are called weakly isomorphic if they are equivalent, as above, and the map $\sigma : S \to S'$ extends to the isomorphism of groups $G$ to $G'$. The strongest relation is the isomorphism of $G$-gradings. In this case both $\Gamma$ and $\Gamma'$ are gradings by the same group, they have the same support and the isomorphism of groups $\sigma$ is identity. As a result, we have $\varphi(L_g) = L'_{g'}$, for any $g \in G$. In the case of group $G$-gradings with support $S$ we will write $\Gamma : L = \bigoplus_{g \in G} L_g$, assuming that $L_g = \{0\}$ if $g \in G \setminus S$.

2.3. Refinements, coarsenings, fine gradings.

**Definition 2.2.** Let $\Gamma$, as above, and $\Gamma' : L = \bigoplus_{t \in T} L'_t$ be two gradings on $L$ with supports $S$ and $T$, respectively. We say that $\Gamma$ is a refinement of $\Gamma'$ or $\Gamma'$ is a coarsening of $\Gamma$ if for any $s \in S$ there exists $t \in T$ such that $L_s \subset L'_t$.

Coarsening and refinements often arise as follow. Let $\Gamma : L = \bigoplus_{g \in G} L_g$ and $\varepsilon : G \to K$ be a homomorphism of groups. We set $L'_k = \bigoplus_{\varepsilon(g) = k} L_g$, for any $k \in K$. Then we obtain the image-grading $\varepsilon(\Gamma) = \bigoplus_{k \in K} L'_k$. If $\varepsilon$ is an onto homomorphism, we say that $\varepsilon(\Gamma)$ is a factor-grading of the grading $\Gamma$. Clearly, $\varepsilon(\Gamma)$ a coarsening of $\Gamma$ and $\Gamma$ is a refinement of $\varepsilon(\Gamma)$.

A group grading $\Gamma$ of $L$ is called a fine grading if doesn’t admit proper group refinements.

**Remark 2.1.** If $\Gamma'$ is a refinement of $\Gamma$ then $\Gamma$ viewed as a $U(\Gamma)$-grading is a factor grading of $\Gamma'$ viewed as a $U(\Gamma')$-grading.

2.4. Abelian groups gradings. In the case of Lie algebras it is natural to assume that all groups involved in the group gradings are abelian. In fact, for many Lie algebras, like finite-dimensional simple ones, this is satisfied (see [11]) in the sense that the partial function $\mu : S \times S \to S$ appearing in the definition of the grading is symmetric or that the elements of the support in the group grading commute. So in what follows we always deal with gradings by abelian groups. In addition, when we study finite-dimensional algebras, the supports of the gradings are finite sets, so our groups are finitely generated.

Now given a finitely generated abelian group $G$, we denote by $\widehat{G}$ the group of (1-dimensional) characters of $G$, that is, the group of all homomorphisms $\chi : G \to F^\times$ where $F^\times$ is the multiplicative group of the field $F$. If $\Gamma : L = \bigoplus_{g \in G} L_g$ is a grading of a Lie algebra $L$ with a grading group $G$, there is an action of $\widehat{G}$ by semisimple automorphisms of $L$ given on the homogeneous elements of $L$ by $\chi \ast x = \chi(g)x$ where $\chi \in \widehat{G}$ and $x \in L_g$. If $G$ is generated by the support $S$ of $\Gamma$, different characters act differently. Indeed, assume $\chi_1, \chi_2 \in \widehat{G}$ are such that $\chi_1 \ast x = \chi_2 \ast x$, for any $x \in L$. Choose any $s \in S$ and $0 \neq x \in L_s$. In this case $\chi_1(s)x = \chi_1 \ast x = \chi_2 \ast x = \chi_2(s)x$. As a result, $\chi_1(s) = \chi_2(s)$, for any $s \in S$. Since $\chi_1$ and $\chi_2$
are homomorphisms and $G$ is generated by $S$, we have $\chi_1 = \chi_2$, as claimed. This allows us, in this important case, to view $\hat{G}$ as a subgroup of $\text{Aut } L$. We will view $\text{Aut } L$ as an algebraic group. When we study finite-dimensional algebras, then $G$ is finitely generated abelian and so $\hat{G}$ is the group of characters of a finitely generated abelian group. If $G \cong \mathbb{Z}^m \times A$, where $m$ is an integer, $m \geq 0$, and $A$ a finite abelian group, then $\hat{G} \cong (F^\times)^m \times \hat{A}$. Such subgroups of algebraic groups, consisting of semisimple elements, are called quasi-tori.

A quasi-torus is a generalization of the notion of a torus, which is an algebraic subgroup of $\text{Aut } L$ isomorphic to $\hat{G} \cong (F^\times)^m$, for some $m$. A torus, which is not contained in a larger torus is called maximal. The following result is classical.

**Theorem 2.1.** In any algebraic group any two maximal tori are conjugate by an inner automorphism.

Another well-known result, is often attributed to Platonov [27] (but see also [28])

**Theorem 2.2.** Any quasi-torus is isomorphic to a subgroup in the normalizer of a maximal torus.

Thus, if we find a maximal torus $D$ in $\text{Aut } L$ equal to its normalizer in $\text{Aut } L$, then for any grading of $L$ by a finitely generated abelian group $G$ there is $\varphi \in \text{Aut } L$ such that $\varphi\hat{G}\varphi^{-1} \subset D$.

Every time we have a quasitorus $Q$ in $\text{Aut } L$ there is root space decomposition of $L$ with roots from the group of (algebraic) characters $\mathcal{X}(Q)$ for the group $Q$, the root subspace for $\lambda \in \mathcal{X}(Q)$ given by

$$L_\lambda = \{ x \in L \mid \alpha(x) = \lambda(\alpha)x \text{ for any } \alpha \in Q \}.$$  

If $Q \subset D$ then, by duality, $\mathcal{X}(Q)$ is a factor group of $\mathcal{X}(D) \cong \mathbb{Z}^m$ where $m = \text{dim } D$. The root space decomposition by $D$ is the refinement of the root space decomposition by $Q$ and so the grading by $\mathcal{X}(Q)$ is a coarsening of a grading by $\mathcal{X}(D) \cong \mathbb{Z}^m$.

Now assume that we deal with a grading of $L$ by a finitely generated abelian group $G$, $\Gamma : L = \bigoplus_{g \in G} L_g$. Assume that $p = \text{char } F$ and write $G = G_p \times G_{p'}$, where $G_p$ is the Sylow $p$-subgroup and $G_{p'}$ its complement in $G$ that has no elements of order $p$. If char $F = 0$ then $G = G_{p'}$. Let us consider the quasitorus $\hat{G} \subset \text{Aut } L$. Then there is $\varphi \in \text{Aut } L$ such that $\varphi\hat{G}\varphi^{-1} \subset D$. Let us switch to another $G$-grading $\varphi(\Gamma) : L = \bigoplus_{g \in G} \varphi(L_g)$. The action of $\hat{G}$ on $L$ induced by $\varphi(\Gamma)$ gives rise to another copy of $\hat{G}$ in $\text{Aut } L$, namely, $\varphi\hat{G}\varphi^{-1}$ and now this subgroup is a subgroup in $D$. Replacing $D$ by another maximal torus $\varphi^{-1}D\varphi$ we may assume from the very beginning that $\hat{G} \subset D$. Then, the root decomposition by $D$ is a refinement of the root decomposition under the action of $\hat{G}$. Thus the grading $\Gamma' : L = \bigoplus_{\lambda \in \mathcal{X}(\hat{G})} L_\lambda$ induced by this root decomposition resulting from the action of $\hat{G}$ is a coarsening of the $\mathcal{X}(D) \cong \mathbb{Z}^m$ grading of $L$ induced by by the action of the torus $D$.

In the case where $G$ has no elements of order $p = \text{char } F$, we have the original grading being a coarsening of the grading induced from the action of a maximal torus. Usually, this is some “standard” torus, and the grading induced by its action is also called “standard”.

We summarize the above discussion as follows.
Theorem 2.3. Let $\Gamma : L = \bigoplus_{g \in G} L_g$ be a grading of a finite-dimensional algebra $L$ over an algebraically closed field $K$ by a finitely generated abelian group $G$. If $\operatorname{char} K = p > 0$, let $G_p$ denote the Sylow $p$-subgroup of $G$. Consider the automorphism group $A = \operatorname{Aut} L$ of $L$ and assume $D$ is a maximal torus of $A$, of dimension $m$, equal to its normalizer in $A$. Then the factor-grading $\Gamma/G_p$ is isomorphic to a factor-grading of the standard $\mathbb{Z}^m$-grading of $L$ induced by the action of $D$ on $L$.

An important particular case is the following.

Theorem 2.4. Let $\Gamma : L = \bigoplus_{g \in G} L_g$ be a grading of a finite-dimensional Lie algebra $L$ over a algebraically closed field $K$ by a finitely generated abelian group $G$. If $\operatorname{char} K = p > 0$, assume $G$ has no elements of order $p$. Consider the automorphism group $A = \operatorname{Aut} L$ of $L$ and assume $D$ is a maximal torus of $A$, of dimension $m$, equal to its normalizer in $A$. Then $\Gamma$ is isomorphic to a factor-grading of the standard $\mathbb{Z}^m$-grading of $L$ induced by the action of $D$ on $L$.

Even more special is the following.

Theorem 2.5. Let $\Gamma : L = \bigoplus_{g \in G} L_g$ be a grading of a finite-dimensional algebra $L$ over a algebraically closed field of characteristic zero $K$ by a finitely generated abelian group $G$. Consider the automorphism group $A = \operatorname{Aut} L$ of $L$ and assume $D$ is a maximal torus of $A$, of dimension $m$, equal to its normalizer in $A$. Then $\Gamma$ is isomorphic to a factor-grading of the standard $\mathbb{Z}^m$-grading of $L$ induced by the action of $D$ on $L$.

2.5. Automorphism group of a grading. Since we completely classify gradings up to equivalence for certain classes of algebras, we quote some more results from [19]. Given a grading $\Gamma : L = \bigoplus_{s \in S} L_s$ of an algebra $L$, the subgroup of the group $\operatorname{Aut} L$ permuting the components of $\Gamma$ is called the automorphism group of the grading $\Gamma$ and denoted by $\operatorname{Aut} \Gamma$. Each $\varphi \in \operatorname{Aut} \Gamma$ defines a bijection on the support $S$ of the grading: if $\varphi(L_s) = L_{s'}$ then $s \mapsto s'$ is the desired permutation $\sigma(\varphi)$, an element of the symmetric group $\operatorname{Sym} S$. The kernel of the homomorphism $\varphi \mapsto \sigma(\varphi)$ is called the stabilizer of the grading $\Gamma$, denoted by $\operatorname{Stab} \Gamma$. Finally a subgroup of $\operatorname{Stab} \Gamma$, whose elements are scalar maps on each graded component of $\Gamma$ is called the diagonal group of $\Gamma$ and denoted by $\operatorname{Diag} \Gamma$.

Definition 2.3. Let $Q \subset \operatorname{Aut} \Gamma$ be a quasitorus. Let $\Gamma$ be the eigenspace decomposition of $L$ with respect to $Q$. Then the quasitorus $\operatorname{Diag} \Gamma$ in $\operatorname{Aut} \Gamma$ is called the saturation of $Q$. We always have $Q \subset \operatorname{Diag} \Gamma$. If $Q = \operatorname{Diag} \Gamma$ then we say that $Q$ is a saturated quasitorus.

A quasitorus $Q$ is saturated if and only if the group $\mathcal{X}(Q)$ of algebraic characters of $Q$ is $U(\Gamma)$, the universal group of $\Gamma$.

Proposition 2.1. The equivalence classes of gradings on $L$ are in one-to-one correspondence with the conjugacy classes of saturated quasitori in $\operatorname{Aut} L$.

Notice that if we already know that every quasitorus is conjugate to a subgroup of a fixed maximal torus and that two subgroups of the maximal torus are conjugate if and only if they are equal, we can say that the equivalence classes of gradings are in one-to-one correspondence with the saturated subquasitori of a fixed maximal torus.
2.6. Group gradings and actions of Hopf algebras. In the case where the action of $\tilde{\Gamma}$ by automorphisms does not completely reflect the $G$-grading, one can still find the transformations which are responsible for the gradings. For this one need to consider the group algebra $H = FG$ of the group $G$. The Hopf algebra structure on $H$, that is, the coproduct $\Delta$, the counit $\varepsilon$ and the antipode $S$ are given as follows: $\Delta(g) = g \otimes g$, $\varepsilon(g) = 1$ and $S(g) = g^{-1}$, for any $g \in G$. Now let us consider the finite dual $K = H^*$ of $H$, which is just the ordinary dual $H^*$ in the case where $|G| < \infty$, and the set of linear functions $f : H \to \mathbb{K}$ such that $\text{Ker} f$ contains a two-sided ideal of finite codimension in $H$. All operations on $K$ are defined by duality. The action of $f \in K$ on $A = \bigoplus_{g \in G} A_g$ is defined by setting $f^*a = f(g)a$ if $a \in A_g$. If $G$ is a finitely generated abelian group (or, more generally, a residually linear group), then $K$ separates points of $G$ and the $G$-grading can be recovered from the $K$-action. One has to set $A_g = \{ x \in A | f^*x = f(g)x \text{ for all } f \in K \}$.

If $G$ is a finite group of order coprime to the characteristic of the base field, the basis of $K$ is formed by the characters of $G$, they act as automorphisms and so we are bounced back to the situation described earlier. In the case where $\text{char} F = p > 0$ and $G$ is an elementary abelian group of order $p^n$, it is known that $K$ is a restricted enveloping algebra for an abelian $p$-Lie algebra spanned by $n$ semisimple commuting derivations. In this case the study of gradings on an algebra is reduced to the study of action by derivations. In all other cases, although $K$ has a very simple structure as an algebra, just the direct sum of the copies of the ground field, the action of the elements of $K$ on the products of element of $A$ can be extremely complex. However, this approach was successfully applied to the study of gradings on algebras of type $A$ in characteristic $p > 0$ [5].

3. Transfer

If $A$ is an associative algebra over a field $\mathbb{K}$ then we denote by $A^{(-)}$ the Lie algebra on the same vector space $A$ under the bracket operation $[a, b] = ab - ba$, for all $a, b \in A$. By Poincaré–Birkhoff–Witt Theorem, for any Lie algebra $L$ there is an associative algebra $A$ such that $L$ is (isomorphic to) a subalgebra of $A^{(-)}$. If $A$ is generated by (the image of) $L$, we say that $A$ is an associative envelope of $L$. For any $L$ one can choose the universal enveloping algebra $U(L)$ as an associative envelope, the drawback being that $U(L)$ is an infinite-dimensional algebra, for any nonzero $L$. In many cases, however, there are more manageable associative envelopes. For example, by Ado–Iwasawa Theorem, every finite-dimensional Lie algebra has a finite-dimensional associative envelope. A Lie algebra consisting of linear transformations of a vector space $V$ is a subalgebra of the associative algebra $\text{End}(V)$, etc.

It was noted a while ago [26] that the study of the group gradings on simple Lie algebras is closely related to the study of gradings on their associative envelopes. This is especially true in the case of the classical simple Lie algebras over algebraically closed fields of characteristic zero by two reasons. First, because, as we saw, the study of gradings is equivalent to the study of quasitori in the automorphism groups of these algebras. Second, because all these algebras have matrix realizations and, with the exception of $D_4$, their automorphisms extend
to the automorphisms or negatives of the automorphisms of the respective matrix algebras. The situation is especially benign in the case of algebras of the types $B,C,D$, except $D_4$. In this case, every automorphism of a Lie algebra of one of these types is given by a matrix conjugation. As a result, any $G$-grading of a Lie algebra, of the type $so(n)$ or $sp(n)$, $n$ even, can be obtained by restriction from a $G$-grading of the matrix algebra $M_n$.

3.1. **Affine group schemes.** An object that most fully reflects the structure of gradings on a finite-dimensional algebra $A$ by abelian groups is called the automorphism group scheme of $A$ over a field $K$, denoted by $\text{Aut}_A$. This is a representable functor $\mathcal{F}$ from the category $\text{Comm}$ of commutative associative unital algebras over $K$ to the category $\text{Ab}$ of abelian groups, which associates with each commutative associative unital algebra $R$ the group $\text{Aut}_R(A \otimes R)$. The value $\mathcal{F}(K)$ is just the ordinary $\text{Aut}_A$. Being representable for a functor $\mathcal{G}$ means that there is a finitely generated Hopf algebra $H$ such that $\mathcal{G}(R) = \text{Alg}_K(H, R)$, the group of algebra homomorphisms from $H$ to $R$ under the convolution product $(f * g)(h) = \sum f(h_1)g(h_2)$, where $\Delta(h) = \sum h_1 \otimes h_2$, $h$ being an arbitrary element of $H$. Given a finitely generated abelian group $G$, the affine group scheme represented by the group algebra $H = K G$ is denoted by $G^D$.

We quote the following results from [19, Section 1.4].

**Proposition 3.1.** The $G$-gradings on an algebra $A$ are in one-to-one correspondence with the morphisms of affine group schemes $G^D \to \text{Aut}_A$. Two $G$-gradings are isomorphic if and only if the corresponding morphisms are conjugate by an element of $\text{Aut}_A$.

**Theorem 3.1.** Let $A$ and $B$ be finite-dimensional algebras. Assume that we have a morphism $\theta : \text{Aut}_A \to \text{Aut}_B$. Then, for any abelian group $G$, we have a mapping $\Gamma \to \theta(\Gamma)$ from $G$-gradings of $A$ to $G$-gradings of $B$. If $\Gamma \cong \Gamma'$ then $\theta(\Gamma) \cong \theta(\Gamma')$. If $\theta$ is an isomorphism and $G$ is the universal grading group of a fine grading $\Gamma$ then $\theta(\Gamma)$ is a fine grading with universal group $G$.

Using this theorem and the results about the gradings of nonassociative algebras, such as octonions or Albert algebra, made it possible to describe abelian group gradings on simple Lie algebras of the types $G_2$ and $F_4$ in an amazing generality of arbitrary algebraically closed fields of characteristic not 2 or 3 in the case of $G_2$ and not 2, in the case of $F_4$. The group scheme $\text{Aut}_L \cong \text{Aut}_O$, where $O$ is the octonion algebra in the case where $L$ is of the type $G_2$ and $\text{Aut}_L \cong \text{Aut}_A$ where $A$ is the Albert algebra in the case where $L$ is of the type $F_4$. For the details see [19, Chapters 4,5]. In both cases, the Lie algebra $L$ is the derivation algebra of the respective nonassociative algebra $C$ and the isomorphism of the group schemes is given by the adjoint map (see below).

The approach via group schemes paved the way to the classification of abelian group gradings on simple Cartan Lie algebras. These algebras arise as subalgebras of the derivation algebras of so called divided power algebras (Witt algebras, type W; Special algebras, type S; Hamiltonian algebras, type H; Contact algebras, type K). For the classification of Cartan...
Type Lie algebras see [29, 30]. A fairly recent paper [7] is devoted to the classification of all abelian group gradings on the restricted Lie algebras of the types $W$ and $S$, with important information in the case $S$. In the case of restricted algebras, we only need to consider the derivation algebras of so the truncated polynomial algebras, that is, the algebras of the form $O_p(n) = \mathbb{K}[x_1, \ldots, x_n]/(x_1^p, \ldots, x_n^p)$, where $n \geq 1$, $p = \text{char} \mathbb{K}$. Using the isomorphism of the group schemes $\text{Ad} : \text{Aut} O_p(n) \rightarrow \text{Aut} W_p(n)$, where $\varphi : (D \rightarrow (\varphi^{-1} \circ D \circ \varphi))$ where $\varphi$ is an automorphism of $O_p(n)$ and $D$ is a derivation of $O_p(n)$ one easily transfers the gradings from $O_p(n)$ to $W_p(n)$. For more details see [19, Chapter 7].

3.2. Group gradings, comodules and functional identities. Suppose a Lie algebra $L$ over a field $F$ is graded by an abelian group $G$. This is well known [24] to be equivalent to $L$ being a (right) $H$-comodule Lie algebra over the group algebra $H = FG$, that is, to the existence of a Lie homomorphism $\rho : L \rightarrow L \otimes H$ such that

\[(\rho \otimes \text{id}_H) \rho = (\text{id}_L \otimes \Delta) \rho\]

and

\[(\text{id}_L \otimes \varepsilon) \rho = \text{id}_L.\]

In the case of a graded algebra, $\rho$ is determined by $\rho(a_g) = a_g \otimes g$ where $a_g$ is a homogeneous element of degree $g$. If $L$ is a Lie subalgebra generating an associative algebra $A$ and $\rho$ extends to an associative homomorphism $\rho : A \rightarrow A \otimes H$, then $A$ also becomes $G$-graded. One easily checks that (1)/(2) will still be satisfied because $L$ generates $A$ as an associative algebra. Since both $L_g$ and $A_g$ are defined as the sets of elements $x$ in $L$ and $A$ satisfying $\rho(x) = x \otimes g$ we have $L_g = L \cap A_g$. In what follows, we will see that this extension of $\rho$ from $L$ to $A$ indeed happens in the case where $L$ is the Lie algebra of skew symmetric elements in a central simple associative algebra over any field of characteristic not 2, provided that $\dim L \geq 21$. The case of $L$ being infinite-dimensional is not excluded but is rather welcome! On the other hand, it is clear that some gradings of simple Lie algebras cannot be induced from the associative gradings. This is true for the grading of $\mathfrak{sl}(n)$ by $\mathbb{Z}_2$, which represents every matrix as the sum its of skew-symmetric and symmetric components. To correctly handling the situation arising, let us extend the Lie homomorphism $\rho : L \rightarrow L \otimes H$ to $\tilde{\rho} : L \otimes H \rightarrow L \otimes H$ by setting $\tilde{\rho}(x \otimes h) = \rho(x)(1 \otimes h)$. Clearly, this is a surjective Lie homomorphism. We want to extend $\tilde{\rho}$ to $A \otimes H$.

Recall that a map $\sigma$ from an associative algebra $\mathcal{A}'$ to a unital associative algebra $\mathcal{A}''$ is a direct sum of a homomorphism and the negative of an antihomomorphism if there exist central idempotent $e_1$ and $e_2$ in $\mathcal{A}''$ with $e_1 + e_2 = 1$ such that $x \mapsto e_1 \sigma(x)$ is a homomorphism and $x \mapsto e_2 \sigma(x)$ is the negative of an antihomomorphism. Maps with this property are clearly Lie homomorphisms; the nontrivial question is whether all Lie homomorphisms can be described in terms of such maps.

The answer to these questions is partially given in the following theorems from [2]. The technique used, so called Functional Identities, was developed in course of solution of the famous Herstein problems about Lie homomorphisms of associative algebras (see [15, 16,
We call a unital associative algebra over a field $\mathbb{K}$ *central* if its center equals $\mathbb{K} \cdot 1$.

**Theorem 3.2.** Let $A$ be a central simple algebra such that $\dim_{\mathbb{F}} A \geq 64$. Let $H$ be a unital commutative algebra. If $A$ is not unital, then assume that $H$ is finite dimensional. Then every surjective Lie homomorphism $\rho : [A, A] \otimes H \to [A, A] \otimes H$ can be extended to a direct sum of a homomorphism and the negative of an antihomomorphism $\sigma : A \otimes H \to A \otimes H$.

**Theorem 3.3.** Let $\text{char} \mathbb{F} \neq 2$ and let $A$ be a central simple algebra such that $\dim_{\mathbb{F}} A \geq 441$. Suppose that $A$ has an involution and set $K = K_A$. Let $H$ be a unital commutative algebra. If $A$ is not unital, assume that $H$ is finite dimensional. Then every surjective Lie homomorphism $\rho : [K, K] \otimes H \to [K, K] \otimes H$ can be extended to a homomorphism $\sigma : A \otimes H \to A \otimes H$.

These transfer theorems and the techniques of affine group schemes have been successfully applied and the final classification of abelian group gradings on classical simple Lie algebras of types except $D_4$ over algebraically closed fields of characteristic different from 2 (in one case, different from 3) was obtained in [4]. The case of $D_4$ was accomplished by A. Elduque (see the details in [19]).

### 3.3. Case of finitary simple algebras

Theorems 3.2 and 3.3 work well for infinite-dimensional algebras. For example, one can apply them to the case of Lie algebras $L$, which are direct limits of algebras $\{L_i = \mathfrak{sl}(n_i, \mathbb{K}) \mid k \in \mathbb{N}\}$, where each $L_i$ is embedded in $L_{i+1}$ by a diagonal embedding $X \to \text{diag}(X, \ldots, X)$. In this case the unital algebra $A$ is the direct limit of $\{A_i = M_n(\mathbb{K}) \mid k \in \mathbb{N}\}$, with the same embeddings. One could also consider the orthogonal or symplectic Lie algebras of skew-symmetric elements in $A$ with respect to appropriate involutions. One drawback is that these theorems give no answer in the case of gradings of non-unital simple algebras by infinite groups, which is exactly the case when one deals with simple Lie algebras of finitary linear transformations. These algebras are direct limit of algebras of the same kind as before but the embedding is given by setting $X \to \text{diag}(X, 0)$. Let us remind few definitions.

An infinite-dimensional simple Lie algebra $L$ of linear operators on an infinite-dimensional space over a field $\mathbb{K}$ is called *finitary* if $L$ consists of linear operators of finite rank. These algebras were classified in [13] over any $\mathbb{K}$ with $\text{char} \mathbb{K} = 0$ and in [14] over an algebraically closed $\mathbb{K}$ with $\text{char} \mathbb{K} \neq 2, 3$. Under the latter assumption, finitary simple Lie algebras over $\mathbb{K}$ can be described in the following way.

Let $U$ be an infinite-dimensional vector space over $\mathbb{K}$. Let $\Pi \subset U^*$ be a total subspace, i.e., for any $v \neq 0$ in $U$ there is $f \in \Pi$ such that $f(v) \neq 0$. Let $\mathfrak{F}_\Pi(U)$ be the space spanned by the linear operators of the form $v \otimes f$, $v \in U$ and $f \in \Pi$, defined by $(v \otimes f)(u) = f(u)v$ for all $u \in U$. It is known from [23] that $R = \mathfrak{F}_\Pi(U)$ is a (non-unital) simple associative algebra.

The commutator $[R, R]$ is a simple Lie algebra, which is denoted by $\mathfrak{fsl}(U, \Pi)$. The algebra $R$ admits an ($\mathbb{K}$-linear) involution if and only if there is a nondegenerate bilinear form
Φ: \(U \times U \to \mathbb{K}\) that identifies \(U\) with \(\Pi\) and that is either symmetric or skew-symmetric. The set of skew-symmetric elements with respect to this involution, i.e., the set of \(r \in R\) satisfying \(\Phi(ru, v) + \Phi(u, rv) = 0\) for all \(u, v \in U\), is a simple Lie algebra, which is denoted by \(\mathfrak{fso}(U, \Phi)\) if \(\Phi\) is symmetric and \(\mathfrak{fsp}(U, \Phi)\) if \(\Phi\) is skew-symmetric.

In [14] it is shown that if \(\mathbb{K}\) is algebraically closed and \(\text{char } \mathbb{K} \neq 2, 3\), then any finitary simple Lie algebra over \(\mathbb{K}\) is isomorphic to one of \(\mathfrak{fsl}(U, \Pi)\), \(\mathfrak{fso}(U, \Phi)\) or \(\mathfrak{fsp}(U, \Phi)\). The most important special case is that of countable (infinite) dimension. Then \(U\) has countable dimension and the isomorphism class of the Lie algebra does not depend on \(\Pi\) or \(\Phi\). Hence, there are exactly three finitary simple Lie algebras of countable dimension: \(\mathfrak{sl}(\infty)\), \(\mathfrak{so}(\infty)\) and \(\mathfrak{sp}(\infty)\).

In the paper [3] we have given a complete classification of the abelian group gradings on finitary simple Lie algebras. This classification was obtained using the transfer to the case of associative algebras. Since the authors did not want to restrict themselves to the case of finite grading groups, as suggested by Theorems 3.2 or 3.3, they proved the following result.

To state the results, we set \(A^\# = A + \mathbb{K}1\); thus, \(A^\# = A\) if \(1 \in A\). If \(X\) is a subset of an associative algebra, then by \(\langle X \rangle\) we denote the subalgebra generated by \(X\).

**Theorem 3.4.** Let \(A\) be any algebra satisfying \(\mathfrak{fsl}(U) \subset A \subset \text{End}(U)\), where \(U\) is infinite-dimensional, and let \(L\) be a noncentral Lie ideal of \(A\). If \(H\) is a unital commutative associative algebra, \(M\) is a Lie ideal of some associative algebra, and \(\rho: M \to L \otimes H\) is a surjective Lie homomorphism, then there exist a direct sum of a homomorphism and the negative of an antihomomorphism \(\sigma: \langle M \rangle \to A^\# \otimes H\) and a linear map \(\tau: M \to 1 \otimes H\) such that \(\tau([M, M]) = 0\) and \(\rho(x) = \sigma(x) + \tau(x)\) for all \(x \in M\).

If \(A\) is any associative algebra with an involution \(\varphi\), then by \(\mathcal{K}(A, \varphi)\) we denote the Lie algebra of skew-symmetric elements in \(A\):

\[
\mathcal{K}(A, \varphi) = \{a \in A \mid \varphi(a) = -a\}.
\]

**Theorem 3.5.** Let \(A\) be any algebra satisfying \(\mathfrak{fsl}(U) \subset A \subset \text{End}(U)\), where \(U\) is infinite-dimensional and the characteristic of the ground field is different from 2. Assume that \(A\) has an involution \(\varphi\) and let \(L\) be a noncentral Lie ideal of \(\mathcal{K}(A, \varphi)\). If \(H\) is a unital commutative associative algebra, \(U\) is an associative algebra with involution \(\ast\), \(M\) is a Lie ideal of \(\mathcal{K}(U, \ast)\), and \(\rho: M \to L \otimes H\) is a surjective Lie homomorphism, then there exist a homomorphism \(\sigma: \langle M \rangle \to A^\# \otimes H\) and a linear map \(\tau: M \to 1 \otimes H\) such that \(\tau([M, M]) = 0\) and \(\rho(x) = \sigma(x) + \tau(x)\) for all \(x \in M\).

Note that in the case of simple algebras the map \(\tau\) is always zero.
4. Group gradings on algebras of finitary linear transformations

For the transfer approach to work, one needs to classify the gradings by groups on associative simple finitary algebras.

Now the basic idea about the gradings in the case of finite-dimensional associative algebras (even graded Artinian algebras) is as follows. Let us call an algebra graded simple if it has no nontrivial proper ideals which are graded subspaces. Let us call an $G$-graded algebra a graded division algebra if any nonzero homogeneous element is invertible. An algebra is called graded left Artinian if it satisfies the descending chain condition for its left ideals. A well-known adaptation of the classical Wedderburn-Artin Theorem (see [25]) is the following.

**Theorem 4.1.** Every $G$-graded left Artinian graded simple algebra $A$ is isomorphic to an algebra of endomorphisms $\text{End}_D(V)$ of a $G$-graded right vector space $V$ over a graded division algebra $D$.

Basically the same result holds in the case of simple associative algebras or finitary linear transformations, which belong to a wider class of primitive algebras with minimal ideals.

As shown in [23, IV.9], $R$ is a primitive algebra with minimal one-sided ideals if and only if it is isomorphic to a subalgebra of $\mathfrak{L}_\Pi(U)$ containing the ideal $\mathfrak{F}_\Pi(U)$ where $U$ is a right vector space over a division algebra $\Delta$, $\Pi$ is a total subspace of the left vector space $U^*$ over $\Delta$, $\mathfrak{L}_\Pi(U)$ is the algebra of all continuous $\Delta$-linear operators on $U$, and $\mathfrak{F}_\Pi(U)$ is the set of all operators in $\mathfrak{L}_\Pi(U)$ whose image has finite dimension over $\Delta$. The term continuous refers here to the topology on $U$ with a neighbourhood basis at 0 consisting of the sets of the form $\ker f_1 \cap \ldots \cap \ker f_k$ where $f_1, \ldots, f_k \in \Pi$ and $k \in \mathbb{N}$. A linear operator $A: U \to U$ is continuous with respect to this topology if and only if the adjoint operator $A^*: U^* \to U^*$ leaves the subspace $\Pi$ invariant. (In particular, if $\Delta$ is $\mathbb{R}$ or $\mathbb{C}$, $U$ is a Banach space and $\Pi$ consists of all bounded linear functionals on $U$, then a linear operator on $U$ is continuous in our sense if and only if it is bounded.)

In our paper [3], we show that if an algebra $R$ as above is given a grading by a group $G$, then it becomes a graded primitive algebra with minimal one-sided graded ideals. To state the graded analogue of the quoted result from [23, IV.9], let us remind that a linear transformation $f: M \to N$ of $G$-graded vector spaces is said to be homogeneous of degree $h$ if $f(M_g) \subset N_{hg}$, for all $g \in G$. Thus, the set $\text{Hom}^e(M, N)$ of finite sums of homogeneous maps from $M$ to $N$ is a $G$-graded vector space. A homomorphism of graded vector spaces is a homogeneous map of degree $e$.

4.1. **Graded division algebras.** In the case where $\mathbb{K}$ is algebraically closed and we consider a $G$-grading on $R = \mathfrak{F}_\Pi(U)$, which is a locally finite simple algebra with minimal ideals, we have that in the presentation of $R \cong \mathfrak{F}_W^e(V)$ in Theorem 4.4, the graded division algebra is isomorphic to a matrix algebra $M_\ell(\mathbb{K})$, as an ungraded algebra. If we restrict ourselves to the case where $G$ is abelian (this is all we need when we deal with simple Lie algebras!) such graded division algebras have been described in [10] and classified up to isomorphism [4]. The following statement can be found in [19].
Theorem 4.2. Let $T$ be a finite abelian group and let $\mathbb{K}$ be an algebraically closed field. There exists a grading on the matrix algebra $M_n(\mathbb{K})$ with support $T$ making $M_n(\mathbb{K})$ a graded division algebra if and only if $\text{char } \mathbb{K}$ does not divide $n$ and $T \cong \mathbb{Z}_{\ell_1}^2 \times \cdots \times \mathbb{Z}_{\ell_r}^2$ where $\ell_1 \cdots \ell_r = n$. The isomorphism classes of such gradings are in one-to-one correspondence with nondegenerate alternating bicharacters $\beta : T \times T \rightarrow \mathbb{K}^\times$. All such gradings belong to one equivalence class. \hfill \Box

A standard realization can be obtained as follows.

If $R$ is finite-dimensional or a finitary algebra then $D$ is finite-dimensional and $T = \text{Supp } D$ is a finite subgroup of $G$. Every homogeneous component of $D$ is one-dimensional and, one can easily see that $D$ is isomorphic to a group algebra of $T$ twisted by a 2-cocycle $\sigma : T \times T \rightarrow \mathbb{K}^\times$, usually denoted by $\mathbb{K}\sigma T$. This means that we can choose a basis $X_i$ in each $D_t$, $t \in T$ and then we will have $X_iX_u = \sigma(t,u)X_{tu}$. Let us introduce an alternating bicharacter $\beta = \beta_\ell$ by setting $\beta(t,u) = \sigma(t,u)\sigma(u,t)^{-1}$. This depends only on the cohomology class of $\sigma$, which defines the isomorphism class of $\mathbb{K}\sigma T$. The simplicity of $\mathbb{K}\sigma T$ is then equivalent to the nondegeneracy of $\beta$.

Suppose there exists a nondegenerate alternating bicharacter $\beta$ on $T$. One easily shows that $T$ admits a “symplectic basis”, i.e., there exists a decomposition of $T$ as the direct product of cyclic subgroups:

\begin{equation}
T = H_1' \times H_1'' \times \cdots \times H_r' \times H_r''
\end{equation}

such that $H_1' \times H_1''$ and $H_j' \times H_j''$ are $\beta$-orthogonal for $i \neq j$, and $H_i'$ and $H_i''$ are in duality by $\beta$.

4.2. Pauli gradings on matrix algebras. Given a matrix algebra $D = M_n(\mathbb{K})$ over a field $\mathbb{K}$ possessing a primitive $n$th root of 1, one can define a division grading by the group $T = \langle a \rangle_n \times \langle b \rangle_n \cong \mathbb{Z}_n^2$ on $D$, as follows. Let

\begin{equation}
X = X(n, \varepsilon) = \begin{bmatrix}
\varepsilon^{n-1} & 0 & 0 & \cdots & 0 & 0 \\
0 & \varepsilon^{n-2} & 0 & \cdots & 0 & 0 \\
\varepsilon & 0 & \cdots & \varepsilon & 0 \\
0 & 0 & 0 & \cdots & \varepsilon & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 \\
0 & 0 & 0 & \cdots & 0 & 0
\end{bmatrix}
\end{equation}

and

\begin{equation}
Y = Y(n) = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
\varepsilon & 0 & \cdots & \varepsilon & 0 \\
0 & 0 & 0 & \cdots & \varepsilon & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 \\
1 & 0 & 0 & \cdots & 0 & 0
\end{bmatrix}
\end{equation}

be so called generalized Pauli matrices. Clearly, $X$ and $Y$ are periodic matrices of order $n$ each. Moreover, $YX = \varepsilon^{-1}XY$. For any $t = a^ib^j \in T$, we set $X_t = X^tY^{-t}$. There are $n^2$ of such different matrices, they form a basis of $R$ and setting $D_t = \mathbb{K}X_t$, for each $t \in T$ turns $D$ into a graded algebra. We call this grading a Pauli grading of $D = M_n(\mathbb{K})$ and denote by $\Pi(n, \varepsilon)$. Since for any $t \in T$ every nonzero element of $D_t$ is invertible, $D$ is a graded division algebras.

Using Pauli gradings allows us to describe the classes $[\sigma]$ such that $\beta_\sigma$ is nondegenerate and the isomorphisms from $\mathbb{K}\sigma T$ onto a matrix algebra, as follows. Denote by $\ell_i$ the order of $H_i'$ and $H_i''$. (We may assume without loss of generality that $\ell_i$ are prime powers.) If we
pick generators $a_i$ and $b_i$ for $H'_i$ and $H''_i$, respectively, then $\varepsilon_i = \beta(a_i, b_i)$ is a primitive $\ell_i$-th root of unity, and all other values of $\varepsilon$ on the elements $a_1, b_1, \ldots, a_r, b_r$ are 1. We can scale the elements $X_{a_i}$ and $X_{b_i}$ so that $X^{f}_{a_i} = X^{f}_{b_i} = 1$. Then we consider the Kronecker product $M_{\ell_i}(K) \otimes \cdots \otimes M_{\ell_r}(K)$ of matrix algebras $M_{\ell_i}(K)$, each with a Pauli grading $\Pi(\ell_i, \varepsilon_i)$. The degree of the product $X_1 \otimes \cdots \otimes X_r$ is equal to the product of the degrees of the factors. Then we map the generators $X_{t_i}$ of $K^\sigma T$, $t_i \in T_i$, $i = 1, \ldots, r$ as follows:

$$X_{t_i} \mapsto I \otimes \cdots \otimes I \otimes X_{t_i} \otimes I \otimes \cdots I,$$

where the only nonzero factor, on the $i$-th position, is the generalized Pauli matrix from the definition of the Pauli grading $\Pi(\ell_i, \varepsilon_i)$. Comparing defining relations and the dimensions shows that this is an isomorphism of graded algebras.

If we scale $X_{a_i}$ and $X_{b_i}$, as above, to have $X^{f}_{a_i} = X^{f}_{b_i} = 1$, and set $X^{a_1^{i_1} b_1^{j_1} \cdots a_r^{i_r} b_r^{j_r}} = X^{a_1^{i_1} X_{b_1}^{j_1} \cdots X_{a_r}^{i_r} X_{b_r}^{j_r}}$, then

$$X^{a_1^{i_1} b_1^{j_1} \cdots a_r^{i_r} b_r^{j_r}} X^{a_1^{i_1'} b_1^{j_1'} \cdots a_r^{i_r'} b_r^{j_r'}} = \varepsilon_1^{-ji_1} \cdots \varepsilon_r^{-ji_r} X^{a_1^{i_1+i_1'} b_1^{j_1+j_1'} \cdots a_r^{i_r+i_r'} b_r^{j_r+j_r'}}.$$

Hence, with this choice of $X_t$, we obtain a representative of the cohomology class $[\sigma]$ that is multiplicative in each variable, i.e., it is a bicharacter (not alternating unless $T$ is trivial).

Summarizing, we derive the following.

**Theorem 4.3.** If a matrix algebra $R = M_n(K)$, $K$ an algebraically closed field, is turned into a $G$-graded division algebra then there are $l_1, \ldots, l_r$, $n = l_1 \cdots l_r$, a subgroup $T \cong \mathbb{Z}_{l_1}^2 \oplus \cdots \oplus \mathbb{Z}_{l_r}^2$ and $\varepsilon_1, \ldots, \varepsilon_r$, where $\varepsilon_i$ is an $l_i$-th root of 1, for each $i = 1, \ldots, r$, such that $R$ is isomorphic as a graded algebra to the graded Kronecker product $M_{\ell_i}(K) \otimes \cdots \otimes M_{\ell_r}(K)$ of matrix algebras, on each of which the grading is Pauli.

In many arguments, however, it is convenient to simply use that any finite-dimensional $G$-graded division algebra $D$ is a twisted group algebra of a group $T$, with basis $X_t$, $t \in T$, $D_t = KX_t$ and $X_t X_{t'} = \sigma(t, t') X_{t'}$ for some 2-cocycle $\sigma: T \times T \to K^\times$. The isomorphism classes of such gradings are in bijection with the pairs $(T, \beta)$ where $T \subset G$ is a subgroup of order $\ell^2$ and $\beta: T \times T \to K^\times$ is a nondegenerate alternating bicharacter. Here $T$ is the support of the grading and $\beta$ is given by $\beta(t, t') = \sigma(t, t')/\sigma(t', t)$, so we get

$$X_t X_{t'} = \beta(t, t') X_{t'} X_t \quad \text{for all} \quad t, t' \in T.$$

Note that char $K$ cannot divide the order of $T$.

### 4.3. Graded primitive algebras with minimal graded left ideals

As mentioned in the Introduction, we want to study gradings on the associative algebras of the form $\mathfrak{H}(U)$ where $U$ is a vector space over $K$ and $\Pi \subset U^*$ is a total subspace. A natural class to which these algebras belong is the primitive algebras with minimal left ideals. In fact, the simple algebras in this class are precisely the algebras $\mathfrak{H}(U)$ where $U$ is a right vector space over a division algebra and $\Pi \subset U^*$ is a total subspace. We are now going to develop a graded version of the theory in [23, Chapter IV], which will apply to the setting we are interested
It is easy to see that $V_{\text{ideal}}$ of $R$ is homogeneous idempotent (hence of degree $e$). Indeed, if $I^2 \neq 0$ then there is a homogeneous $x \in I$ such that $Ix \neq 0$ and hence $Ix = I$. The left annihilator $L$ of $x$ in $R$ is a graded left ideal of $R$ such that $I \nsubseteq L$ and hence $L \cap I = 0$. Let $\varepsilon$ be an element of $I$ such that $\varepsilon x = x$. Since $x$ is homogeneous and every homogeneous component of $\varepsilon$ is in $I$, we may assume that $\varepsilon$ is homogeneous of degree $e$. Now $\varepsilon^2 x = \varepsilon x = x$ and hence $\varepsilon^2 - \varepsilon \in L$. Since $I \cap L = 0$, we conclude that $\varepsilon^2 = \varepsilon$. Since $Ix \neq 0$, it follows that $I\varepsilon \neq 0$ and hence $I\varepsilon = I$. Since we are assuming $R$ graded primitive and hence graded prime, the case $I^2 = 0$ is not possible. Hence $I = R\varepsilon$ is a graded simple $R$-module.

The following is a graded version of a result in [23, III.5].
Lemma 4.2. Let $R$ be a graded primitive algebra (or ring) with a minimal graded left ideal $I$. Let $V$ be a faithful graded simple left $R$-module. Then there exists $g \in G$ such that $V$ is isomorphic to $I[g]$ as a graded $R$-module. Hence, all faithful graded simple left $R$-modules are isomorphic up to a right shift of grading.

Proof. Since $IV$ is a graded submodule of $V$, we have either $IV = 0$ or $IV = V$. But $V$ is faithful, so $IV = V$. Pick a homogeneous $v \in V$ such that $Iv \neq 0$ and let $g = \deg v$. Then the map $I \to V$ given by $r \mapsto rv$ is a homomorphism of $R$-modules and sends $I_h$ to $V_{hg}$, $h \in G$. By graded simplicity of $I$ and $V$, this map is an isomorphism of $R$-modules. Hence $I[g]$ is isomorphic to $V$ as a graded $R$-module.

Now we will obtain a graded version of the structure theorem in [23, IV.9].

Theorem 4.4. Let $R$ be a $G$-graded algebra (or ring). Then $R$ is graded primitive with minimal graded left ideals if and only if there exists a $G$-graded division algebra $D$, a graded right vector space $V$ over $D$ and a total graded subspace $W \subset V^{gr*}$ such that $R$ is isomorphic to a graded subalgebra (subring) of $D_W(V)$ containing $S_W(V)$. Moreover, $S_W(V)$ is the only such subalgebra (subring) that is graded simple.

4.5. An isomorphism theorem. We are going to investigate under what conditions two graded simple algebras described by Theorem 4.4 are isomorphic. By Lemma 4.2, $V$ is determined by $R$ up to isomorphism and shift of grading. If $\varphi \in \text{End}_{D}^{gr}V$ is homogeneous of degree $t$, then $\varphi$ regarded as a map $V[g] \to V[g]$ will be homogeneous of degree $g^{-1}tg$. Indeed, $(V_h)\varphi \subset V_{ht}$, for all $h \in G$, can be rewritten as $(V_h^{[g]})\varphi \subset V_{ht^g}^{[g]}$. Hence if $D = \text{End}_{R}^{gr}(V)$, then $\text{End}_{D}^{gr}(V[g]) = [s^{-1}]D[g]$. It remains to include the total graded subspace $W \subset V^{gr*}$ in the picture. It is convenient to introduce the following terminology. Fix a group $G$.

Definition 4.1. Let $D$ and $D'$ be $G$-graded division algebras (or rings) and let $V$ and $V'$ be graded right vector spaces over $D$ and $D'$, respectively. Let $\psi_0: D \to D'$ be an isomorphism of $G$-graded algebras (or rings). A homomorphism $\psi_1: V \to V'$ of $G$-graded vector spaces over $\mathbb{K}$ (or $G$-graded abelian groups) is said to be $\psi_0$-semilinear if $\psi_1(vd) = \psi_1(v)\psi_0(d)$ for all $v \in V$ and $d \in D$. The adjoint to $\psi_1$ is the mapping $\psi_1^*: (V')^{gr*} \to V^{gr*}$ defined by $(\psi_1^*(f))(v) = \psi_0^{-1}(f(\psi_1(v)))$ for all $f \in (V')^{gr*}$ and $v \in V$.

One easily checks that $\psi_1^*$ is $\psi_0^{-1}$-semilinear.

Definition 4.2. Let $D$ and $D'$ be $G$-graded division algebras (or rings), let $V$ and $V'$ be graded right vector spaces over $D$ and $D'$, respectively, and let $W$ and $W'$ be total graded subspaces of $V^{gr*}$ and $(V')^{gr*}$, respectively. An isomorphism of triples from $(D,V,W)$ to $(D',V',W')$ is a triple $(\psi_0, \psi_1, \psi_2)$ where $\psi_0: D \to D'$ is an isomorphism of graded algebras (or rings) while $\psi_1: V \to V'$ and $\psi_2: W \to W'$ are isomorphisms of graded vector spaces over $\mathbb{K}$ (or graded abelian groups) such that $(\psi_2(w), \psi_1(v)) = \psi_0((w,v))$ for all $v \in V$ and $w \in W$. 

It follows that \( \psi_1 \) and \( \psi_2 \) are \( \psi_0 \)-seminilinear. Also, for given isomorphisms \( \psi_0 \) and \( \psi_1 \) there can exist at most one \( \psi_2 \) such that \( (\psi_0, \psi_1, \psi_2) \) is an isomorphism of triples. Such \( \psi_2 \) exists if and only if \( \psi_1 \) is \( \psi_0 \)-seminilinear and \( \psi_1^1(W') = W \). Indeed, we can take \( \psi_2 \) to be the restriction of \( (\psi_1^1)^{-1} \) to \( W \). The condition \( \psi_1^1(W') = W \) means that \( \psi_1: V \to V' \) is a homeomorphism with respect to the topologies induced by \( W \) and \( W' \).

The following is a graded version of the isomorphism theorem in [23, IV.11].

**Theorem 4.5.** Let \( G \) be a group. Let \( D \) and \( D' \) be \( G \)-graded division algebras (or rings), let \( V \) and \( V' \) be graded right vector spaces over \( D \) and \( D' \), respectively, and let \( W \) and \( W' \) be total graded subspaces of \( V^{gr} \) and \( (V')^{gr} \), respectively. Let \( R \) and \( R' \) be \( G \)-graded algebras (or rings) such that

\[
\mathfrak{S}^{gr}_W(V) \subset R \subset \mathfrak{L}^{gr}_W(V) \quad \text{and} \quad \mathfrak{S}^{gr}_{W'}(V') \subset R' \subset \mathfrak{L}^{gr}_{W'}(V').
\]

If \( \psi: R \to R' \) is an isomorphism of graded algebras, then there exist \( g \in G \) and an isomorphism \( (\psi_0, \psi_1, \psi_2) \) from \((\{g\}^{-1}D[g], V[g], \{g\}^{-1}W)\) to \((D', V', W')\) such that

\[
(6) \quad \psi(r) = \psi_1 \circ r \circ \psi_1^{-1} \quad \text{for all } r \in R.
\]

If other \( g' \in G \) and isomorphism \( (\psi_0', \psi_1', \psi_2') \) from \((\{g'\}^{-1}D[g'], V[g'], \{g'\}^{-1}W)\) to \((D', V', W')\) define \( \psi \) as above, then there exists a nonzero homogeneous \( d \in D \) such that \( g' = g d \), \( \psi_0'(x) = d^{-1} \psi_0(x') d \) for all \( x \in D \), \( \psi_1'(v) = \psi_1(v) d \) for all \( v \in V \), and \( \psi_2'(w) = d^{-1} \psi_2(w) \) for all \( w \in W \).

As a partial converse, if \( (\psi_0, \psi_1, \psi_2) \) is an isomorphism of triples as above, then setting \( \psi(r) = \psi_1 \circ r \circ \psi_1^{-1} \) defines an isomorphism of \( G \)-graded algebras (or rings) \( \psi: \mathfrak{L}^{gr}_W(V) \to \mathfrak{L}^{gr}_{W'}(V') \) such that \( \psi(\mathfrak{S}^{gr}_W(V)) = \mathfrak{S}^{gr}_{W'}(V') \).

Note that it follows that any isomorphism of graded algebras \( \psi: R \to R' \) extends to an isomorphism \( \mathfrak{L}^{gr}_W(V) \to \mathfrak{L}^{gr}_{W'}(V') \) and restricts to an isomorphism \( \mathfrak{S}^{gr}_W(V) \to \mathfrak{S}^{gr}_{W'}(V') \).

4.6. **Graded simple algebras with minimal graded left ideals.** Fix a grading group \( G \). In view of Theorems 4.4 and 4.5, graded simple algebras (or rings) with minimal graded left ideals are classified by the triples \((D, V, W)\) where \( D \) is a graded division algebra (or ring), \( V \) is a right graded vector space over \( D \) and \( W \) is a total graded subspace of \( V^{gr} \). For a fixed \( D \), the triples \((D, V, W)\) can be classified up to isomorphism as follows.

Let \( T \) be the support of \( D \) and let \( \Delta = D_e \). Clearly, \( T \) is a subgroup of \( G \) and \( \Delta \) is a division algebra (or ring). Consider the set \( G/T \) of left cosets and the set \( T \setminus G \) of right cosets of \( T \) in \( G \). The map \( A \to A^{-1} \) is a bijection between \( G/T \) and \( T \setminus G \). Clearly, the graded right \( D \)-modules \([g]\) and \([h]D \) are isomorphic if and only if \( gT = hT \) (similarly for graded left \( D \)-modules). Any graded right \( D \)-module \( V \) is a direct sum of modules of this form, which can be grouped into isotypic components. Namely, \( V_A = \bigoplus_{a \in A} V_a \) is the isotypic component of \( V \) corresponding to \([g]\) \( D \) where \( A = gT \). Note that \( V_g \) is a right \( \Delta \)-module and \( V_A \simeq V_g \otimes \Delta [g]D \) as graded right \( D \)-modules. Select a left transversal \( S \) for \( T \), i.e., a set of representatives for the left cosets of \( T \), and let \( \tilde{V}_A = V_g \) where \( g \) is the unique element of
A \cap S. Let \( \widetilde{V} = \bigoplus_{A \in G/T} \widetilde{V}_A \). Then \( \widetilde{V} \) is a right \( \Delta \)-module and \( V \cong \widetilde{V} \otimes_{\Delta} D \) as ungraded right \( D \)-modules. We can recover the \( G \)-grading on \( V \) if we consider \( \widetilde{V} \) as graded by the set \( G/T \). Similarly, any graded left \( D \)-module \( W \) can be encoded by a left \( \Delta \)-module \( \widetilde{W} \) with a grading by the set \( T \setminus G \).

Now observe that since \( T \) is the support of \( D \), we have \( (W_B, V_A) = 0 \) for all \( A \in G/T \) and \( B \in T \setminus G \) with \( BA \neq T \). It follows that \( W_{A^{-1}} \) is a total graded subspace of \( (V_A)^{gr} \).

Selecting a left transversal \( S \) and using \( S^{-1} \) as a right transversal, we obtain \( \widetilde{V} \) and \( W \) such that \( (\widetilde{W}_{A^{-1}}, \widetilde{V}_A) \subset \Delta \) and

\[
(7) \quad (\widetilde{W}_B, \widetilde{V}_A) = 0 \quad \text{for all} \quad A \in G/T \quad \text{and} \quad B \in T \setminus G \quad \text{with} \quad B \neq A^{-1}.
\]

Hence, for each \( A \in G/T \), the \( D \)-bilinear form \( W \times V \to D \) restricts to a nondegenerate \( \Delta \)-bilinear form \( \widetilde{W}_{A^{-1}} \times \widetilde{V}_A \to \Delta \), which identifies \( \widetilde{W}_{A^{-1}} \) with a total subspace of the \( \Delta \)-dual \((\widetilde{V}_A)^\ast\).

Conversely, let \( \widetilde{V} \) be a right \( \Delta \)-module that is given a grading by \( G/T \) and let \( \widetilde{W} \) be a left \( \Delta \)-module that is given a grading by \( T \setminus G \). Suppose \( \widetilde{W}_{A^{-1}} \) is identified with a total subspace of \((\widetilde{V}_A)^\ast\) for each \( A \in G/T \) or, equivalently, we have a nondegenerate \( \Delta \)-bilinear form \( \widetilde{W} \times \widetilde{V} \to \Delta \) that satisfies (7). For each \( A \in G/T \), choose \( g \in A \) and let \( V_A = \widetilde{V}_A \otimes \Delta \widetilde{W}_{A^{-1}} \). Then \( V_A \) is a graded right \( D \)-module whose isomorphism class does not depend on the choice of \( g \). Also, using the same \( g \), let \( W_{A^{-1}} = D_{[g]} \otimes_{\Delta} \widetilde{W}_{A^{-1}} \). Set \( V = \bigoplus_{A \in G/T} V_A \) and \( W = \bigoplus_{B \in T \setminus G} W_B \).

Extending the \( \Delta \)-bilinear form \( \widetilde{W} \times \widetilde{V} \to \Delta \) to a \( D \)-bilinear form \( W \times V \to D \), we can identify \( W \) with a total graded subspace of \((V^{gr})^\ast\). We will denote the corresponding \( G \)-graded algebra \( \mathfrak{F}^{gr}_W(V) \) by \( \mathfrak{F}(G, D, \widetilde{V}, \widetilde{W}) \).

**Definition 4.3.** We will write \((D, \widetilde{V}, \widetilde{W}) \sim (D', \widetilde{V}', \widetilde{W}')\) if there is an element \( g \in G \) and an isomorphism \( \psi_0 : [g^{-1}]D_{[g]} \rightarrow D' \) of graded algebras such that, for any \( A \in G/T \), there exists an isomorphism of triples from \((\Delta, \widetilde{V}_A, \widetilde{W}_{A^{-1}})\) to \((\Delta', \widetilde{V}'_{Ag}, \widetilde{W}'_{g^{-1}A^{-1}})\) whose component \( \Delta \rightarrow \Delta' \) is the restriction of \( \psi_0 \). (Note that \( Ag \) is a left coset for \( T' = g^{-1}Tg \).)

**Corollary 4.1.** Let \( G \) be a group and let \( R \) be a \( G \)-graded algebra (or ring). If \( R \) is graded simple with minimal graded left ideals, then \( R \) is isomorphic to some \( \mathfrak{F}(G, D, \widetilde{V}, \widetilde{W}) \). Two graded algebras \( \mathfrak{F}(G, D, \widetilde{V}, \widetilde{W}) \) and \( \mathfrak{F}(G, D', \widetilde{V}', \widetilde{W}') \) are isomorphic if and only if we have \((D, \widetilde{V}, \widetilde{W}) \sim (D', \widetilde{V}', \widetilde{W}')\).

**Proof.** The first claim is clear by Theorem 4.4 and the above discussion. Definition 4.4 is set up in such a way that \((D, \widetilde{V}, \widetilde{W}) \sim (D', \widetilde{V}', \widetilde{W}')\) if and only if the triples \(([g^{-1}]D_{[g]}, V_{[g]}, [g^{-1}]W)\) and \((D', V', W')\) are isomorphic for some \( g \in G \), so the second claim follows from Theorem 4.5.

An important special case of Corollary 4.1 is where the graded simple algebra \( R \) satisfies the descending chain condition on graded left ideals. Then \( V \) is finite-dimensional over \( D \), so \( W = V^{gr} \ast = V^\ast \) and \( R \) is isomorphic to the matrix algebra \( M_k(D) \) where \( k = \dim_D V \). Moreover, \( V = V_{A_1} \oplus \cdots \oplus V_{A_s} \) for some distinct \( A_1, \ldots, A_s \in G/T \), which can be encoded by
two $s$-tuples: $\kappa = (k_1, \ldots, k_s)$ and $\gamma = (g_1, \ldots, g_s)$ where $k_i = \dim_D V_{A_i}$ are positive integers with $k_1 + \cdots + k_s = k$ and $g_i$ are representatives for the cosets $A_i$. Therefore, the said algebras $R$ are classified by the triples $(D, \kappa, \gamma)$, up to an appropriate equivalence relation. Explicitly, the grading on $R$ can be written as follows. If $\{v_1, \ldots, v_k\}$ is a homogeneous basis of $V$ over $D$, with $\deg v_i = h_i$, and $\{v^1, \ldots, v^k\}$ is the dual basis of $V^*$, then $\deg v^i = h_i^{-1}$ and, for any homogeneous $d \in D$, the degree of the operator $v_i d \otimes v^j = v_i \otimes d v^j$ (which is represented by the matrix with $d$ in position $(i, j)$ and zeros elsewhere) equals $h_i(\deg d)h_j^{-1}$. This classification (under the assumption that $R$ is finite-dimensional over $\mathbb{K}$) already appeared in the literature — see [4] and references therein. The $G$-gradings on $M_k(D)$ defined in this way by $k$-tuples $(h_1, \ldots, h_k)$ of elements in $G$ will be called elementary.

In general, $\mathfrak{F}(G, D, \tilde{V}, \tilde{W})$ can be written as $\tilde{V} \otimes_D D \otimes_D \tilde{W}$ where, for any $\tilde{v} \in \tilde{V}$, $d \in D$, $\tilde{w} \in \tilde{W}$, the element $\tilde{v} \otimes d \otimes \tilde{w}$ acts on $V = \tilde{V} \otimes_D D$ as follows: $(\tilde{v} \otimes d \otimes \tilde{w})(u \otimes a) = \tilde{v} \otimes d(w, u)a$ for all $\tilde{u} \in \tilde{V}$ and $a \in D$. Clearly, the multiplication of $\mathfrak{F}(G, D, \tilde{V}, \tilde{W})$ is given by

\[(\tilde{v} \otimes d \otimes \tilde{w})(\tilde{v}' \otimes d' \otimes \tilde{w}') = \tilde{v} \otimes d(\tilde{w}, \tilde{v}')d' \otimes \tilde{w}' .\]

Fixing a left transversal $S$ for $T$, the $G$-grading on $\mathfrak{F}(G, D, \tilde{V}, \tilde{W})$ is given by

\[\deg(\tilde{v} \otimes d \otimes \tilde{w}) = \gamma(A)t\gamma(B)^{-1}\]

for all $\tilde{v} \in \tilde{V}_A$, $d \in D$, $\tilde{w} \in \tilde{W}_{B^{-1}}$,

where $t \in T$, $A, B \in G/T$, and $\gamma(A)$ denotes the unique element of $A \cap S$. The isomorphism class of the grading does not depend on the choice of the transversal.

It is known that, for any finite-dimensional subspaces $\tilde{V}_1 \subset \tilde{V}$ and $\tilde{W}_1 \subset \tilde{W}$, there exist finite-dimensional subspaces $\tilde{V}_0 \subset \tilde{V}$ and $\tilde{W}_0 \subset \tilde{W}$ such that $\tilde{V}_1 \subset \tilde{V}_0$, $\tilde{W}_1 \subset \tilde{W}_0$, and the restriction of the bilinear form $\tilde{W} \times \tilde{V} \to \Delta$ to $\tilde{W}_0 \times \tilde{V}_0$ is nondegenerate (see e.g. [13, Lemma 5.7]). Selecting dual bases in $\tilde{V}_0$ and $\tilde{W}_0$, we see that $\tilde{V}_0 \otimes_D D \otimes_D \tilde{W}_0$ is a subalgebra of $\mathfrak{F}(G, T, \tilde{V}, \tilde{W})$ isomorphic to $M_k(D)$ where $k = \dim_D \tilde{V}_0 = \dim_D \tilde{W}_0$. Without loss of generality, we may assume that $\tilde{V}_0$ is a graded subspace of $\tilde{V}$ with respect to the grading by $G/T$ and $\tilde{W}_0$ is a graded subspace of $\tilde{W}$ with respect to the grading by $T \setminus G$. Then our subalgebra $\tilde{V}_0 \otimes_D D \otimes_D \tilde{W}_0$ is graded. Moreover, in terms of the matrix algebra $M_k(D)$, this grading is elementary. Thus we obtain the following graded version of Litoff’s Theorem [23, IV.15] (cf. Theorem 4 in [12]):

**Corollary 4.2.** Let $G$ be a group and let $R$ be a $G$-graded algebra (or ring). If $R$ is graded simple with minimal graded left ideals, then there exists a graded division algebra $D$ such that $R$ is a direct limit of matrix algebras over $D$ with elementary gradings. □

4.7. **Classification of $G$-gradings on the algebras $\mathfrak{F}_\Pi(U)$**. In this work, we are primarily interested in the case $R = \mathfrak{F}_\Pi(U)$ where $U$ is a vector space over $\mathbb{K}$ and $\Pi$ is a total subspace of $U^*$. We will assume that $\mathbb{K}$ is algebraically closed. Then the algebras of the form $\mathfrak{F}_\Pi(U)$ have the following abstract characterization: they are precisely the locally finite simple algebras with minimal left ideals. Indeed, $\mathfrak{F}_\Pi(U) = U \otimes \Pi$ is a direct limit of matrix algebras over $\mathbb{K}$ and hence is simple and locally finite. Conversely, if $R$ is a locally finite simple algebra with minimal left ideals, then $R$ is isomorphic to $\mathfrak{F}_\Pi(U)$ where $U$ is a right vector space over a
division algebra $\Delta$ and $\Pi$ is a total subspace of $U^*$. But $\Delta$ is isomorphic to a subalgebra of $R$, hence algebraic over $K$. Since $K$ is algebraically closed, this implies $\Delta = K$.

If $R$ is given a $G$-grading, then $R$ is graded simple with minimal graded left ideals (Lemma 4.1), so we can apply Corollary 4.1. Hence $R$ is isomorphic to some $\mathfrak{g}(G, D, \tilde{V}, \tilde{W})$ as a graded algebra. We claim that, disregarding the grading, $D$ is isomorphic to $M_\ell(K)$ for some $\ell$.

Recall from the proof of Theorem 4.4 that we can represent $R$ as $\mathfrak{g}(I)$ where $I = R\varepsilon$ is a minimal graded left ideal, $\varepsilon$ is a homogeneous idempotent, and $J = \varepsilon R$. Recall also that $D = \text{End}_R^g(I)$ coincides with $\text{End}_R(I)$ and is isomorphic to $\varepsilon R\varepsilon$. It is known that $R$ is semisimple as a left or right $R$-module (see e.g. [23, IV.9]). In fact, it is easy to see that, if $R$ is represented as $\mathfrak{g}(U)$, then the mapping $U_0 \mapsto U_0 \otimes \Pi$ is a one-to-one correspondence between the subspaces of $U$ and the right ideals of $R$ whereas the mapping $\Pi_0 \mapsto U \otimes \Pi_0$ is a one-to-one correspondence between the subspaces of $\Pi$ and the left ideals of $R$. Hence we can write $I = R\varepsilon_1 \oplus \cdots \oplus R\varepsilon_\ell$ where $\varepsilon_i$ are orthogonal idempotents with $\varepsilon_1 + \cdots + \varepsilon_\ell = \varepsilon$ and $R\varepsilon_i$ are minimal (ungraded) left ideals. Each of the $R\varepsilon_i$ is isomorphic to $U$ as a left $R$-module. Since $\text{End}_R(U) = K$, it follows that the algebra $\text{End}_R(I)$ is isomorphic to $M_\ell(K)$, completing the proof of the claim.

If $R$ is represented as $\mathfrak{g}(U)$, we can construct this isomorphism explicitly. Namely, write $I = U \otimes \Pi_0$ where $\Pi_0 \subset \Pi$ is an $\ell$-dimensional subspace and select $U_0 \subset U$ such that the restriction of the bilinear form $\Pi \times U \to K$ to $\Pi_0 \times U_0$ is nondegenerate. Let $\{e_1, \ldots, e_\ell\}$ be a basis of $U_0$ and let $\{e^1, \ldots, e^\ell\}$ be the dual basis of $\Pi_0$. Then we can take $\varepsilon_i = e_i \otimes e^i$, so $R\varepsilon_i = U \otimes \mathbb{K}e^i$, and the elements $e_i \otimes e^i$ constitute a basis of matrix units for $\varepsilon R\varepsilon$.

As for the graded division algebra $D$, this is a matrix algebra which is given a $G$-grading that makes it a graded division algebra. We have described them completely in Section 4.2. In what follows we will use notation introduced therein.

We want to understand the relation between, on the one hand, $\tilde{V}$ and $\tilde{W}$ and, on the other hand, $U$ and $\Pi$. We will use $I$ as $V$ and $J$ as $W$. The mapping $u \mapsto u \otimes e^i$ is an isomorphism of left $R$-modules $U \to R\varepsilon_i$. Also, the mapping $f \mapsto e_i \otimes f$ is an isomorphism of right $R$-modules $\Pi \to \varepsilon_i R$. This allows us to identify $I$ with $U^\ell$ and $J$ with $\Pi^\ell$. Recall that the $D$-bilinear form $J \times I \to D$ is just the multiplication of $R$. Hence, under the above identifications, this $D$-bilinear form maps $(f_1, \ldots, f_\ell) \in \Pi^\ell$ and $(u_1, \ldots, u_\ell) \in U^\ell$ to the matrix $[(f_i, u_j)]_{i,j}$ in $D$. Let $M$ be the unique simple right $D$-module and let $N$ be the unique simple left $D$-module, i.e., $M = \mathbb{K}^\ell$ written as rows and $N = \mathbb{K}^\ell$ written as columns. Then, disregarding the $G$-gradings on $I$ and $J$, we can identify $I$ with $U \otimes M$ as an $(R, D)$-bimodule and also identify $J$ with $N \otimes \Pi$ as a $(D, R)$-bimodule. Under these identifications, the $D$-bilinear form $J \times I \to D$ coincides with the extension of the $K$-linear form $\Pi \times U \to K$.

Now, we have $U \cong I \otimes_D N$ and $\Pi \cong M \otimes_D J$ as $R$-modules. If we identify $U$ with $I \otimes_D N$ and $\Pi$ with $M \otimes_D J$, then the $K$-bilinear form $\Pi \otimes U \to K$ is related to the $D$-bilinear form $J \times I \to D$ by the following formula:

\[(m \otimes y, x \otimes n) = m(y, x)n \quad \text{for all} \quad m \in M, n \in N, x \in I, y \in J.\]
where the right-hand side is the scalar in $\mathbb{K}$ obtained by multiplying a row, a matrix and a column. Recall that $\tilde{V}$ and $\tilde{W}$ associated to $V$ and $W$ are defined in such a way that $V = \tilde{V} \otimes D$ and $W = D \otimes \tilde{W}$ as ungraded $D$-modules, and the $D$-bilinear form $W \times V \to D$ is the extension of the $\mathbb{K}$-bilinear form $\tilde{W} \times \tilde{V} \to \mathbb{K}$ (recall that $D_\mathbb{K} = \mathbb{K}$). Hence $U = V \otimes_D N = (\tilde{V} \otimes D) \otimes_D N \cong \tilde{V} \otimes N$ and $\Pi = M \otimes_D W = M \otimes_D (D \otimes \tilde{W}) \cong M \otimes \tilde{W}$ as vector spaces over $\mathbb{K}$, where the isomorphism $(\tilde{V} \otimes D) \otimes_D N \cong \tilde{V} \otimes N$ is given by $\tilde{v} \otimes d \otimes n \mapsto \tilde{v} \otimes dn$ and the isomorphism $M \otimes_D (D \otimes \tilde{W}) \to M \otimes \tilde{W}$ is given by $m \otimes d \otimes \tilde{w} \mapsto md \otimes \tilde{w}$. Substituting $x = \tilde{v} \otimes a$ and $y = b \otimes \tilde{w}$, for any $\tilde{v} \in \tilde{V}$, $\tilde{w} \in \tilde{W}$, $a, b \in D$, into (10), we obtain

$$(m \otimes b \otimes \tilde{w}, \tilde{v} \otimes a \otimes n) = m(b \otimes \tilde{w}, \tilde{v} \otimes a)n = mb(\tilde{w}, \tilde{v})an.$$  

Hence, if we identify $U$ with $\tilde{V} \otimes N$ and $\Pi$ with $M \otimes \tilde{W}$, then the $\mathbb{K}$-bilinear forms $\Pi \times U \to \mathbb{K}$ and $\tilde{W} \times \tilde{V} \to \mathbb{K}$ are related by the following formula:

$$(m \otimes \tilde{w}, \tilde{v} \otimes n) = (\tilde{w}, \tilde{v})mn \quad \text{for all} \quad m \in M, n \in N, \tilde{v} \in \tilde{V}, \tilde{w} \in \tilde{W}. $$

In other words, we can identify $U$ with $\tilde{V}^\ell$ and $\Pi$ with $\tilde{W}^\ell$ so that the above $\mathbb{K}$-bilinear forms are related as follows:

$$(\tilde{w}_1, \ldots, \tilde{w}_\ell), (\tilde{v}_1, \ldots, \tilde{v}_\ell) = \sum_{i=1}^\ell (\tilde{w}_i, \tilde{v}_i) \quad \text{for all} \quad \tilde{v}_1 \in \tilde{V} \quad \text{and} \quad \tilde{w}_1 \in \tilde{W}. $$

Finally, we observe that Definition 4.3 simplifies because $D_e = D'_e = \mathbb{K}$. For brevity, an isomorphism of triples from $(\mathbb{K}, U, \Pi)$ to $(\mathbb{K}, U', \Pi')$ will be called an isomorphism of pairs from $(U, \Pi)$ to $(U', \Pi')$. In other words, pairs $(U, \Pi)$ and $(U', \Pi')$ are isomorphic if and only if there exists an isomorphism $U \to U'$ of vector spaces (over $\mathbb{K}$) whose adjoint $(U')^* \to U^*$ maps $\Pi'$ onto $\Pi$.

**Definition 4.4.** We will write $(D, \tilde{V}, \tilde{W}) \sim (D', \tilde{V}', \tilde{W}')$ if there is an element $g \in G$ such that $[g^{-1}]D[g] \cong D'$ as graded algebras and, for any $A \in G/T$, we have $(\tilde{V}_A, \tilde{W}_{A^{-1}}) \cong (\tilde{V}_{Ag}, \tilde{W}_{g^{-1}A^{-1}})$.

To summarize:

**Theorem 4.6.** Let $R$ be a locally finite simple algebra with minimal left ideals over an algebraically closed field $\mathbb{K}$. If $R$ is given a grading by a group $G$, then $R$ is isomorphic to some $\mathfrak{F}(G, D, \tilde{V}, \tilde{W})$ where $D$ is a matrix algebra over $\mathbb{K}$ equipped with a division grading. Conversely, if $D = M_t(\mathbb{K})$ with a division grading, then $\mathfrak{F}(G, D, \tilde{V}, \tilde{W})$ is a locally finite simple algebra with minimal left ideals, which can be represented as $\mathfrak{F}(G, U)$ where $U = \tilde{V}^\ell$, $\Pi = \tilde{W}^\ell$, and the nondegenerate bilinear form $\Pi \times U \to \mathbb{K}$ is given by (12). Moreover, two such graded algebras $\mathfrak{F}(G, D, \tilde{V}, \tilde{W})$ and $\mathfrak{F}(G, D', \tilde{V}', \tilde{W}')$ are isomorphic if and only if $(D, \tilde{V}, \tilde{W}) \sim (D', \tilde{V}', \tilde{W}')$ in the sense of Definition 4.4. \hfill $\square$

If $G$ is abelian, then $[g^{-1}]D[g] = D$ and the isomorphism class of $D$ is determined by $(T, \beta)$. Hence we may write $\mathfrak{F}(G, T, \beta, \tilde{V}, \tilde{W})$ for $\mathfrak{F}(G, D, \tilde{V}, \tilde{W})$.

**Definition 4.5.** We will write $(\tilde{V}, \tilde{W}) \sim (\tilde{V}', \tilde{W}')$ if there is an element $g \in G$ such that, for any $A \in G/T$, we have $(\tilde{V}_A, \tilde{W}_{A^{-1}}) \cong (\tilde{V}_{Ag}, \tilde{W}_{g^{-1}A^{-1}})$.
Corollary 4.3. If in Theorem 4.6 the group $G$ is abelian, then $R$ is isomorphic to some $\mathfrak{F}(G, T, \beta, \tilde{V}, \tilde{W})$. Two such graded algebras $\mathfrak{F}(G, T, \beta, \tilde{V}, \tilde{W})$ and $\mathfrak{F}(G, T', \beta', \tilde{V}', \tilde{W}')$ are isomorphic if and only if $T = T'$, $\beta = \beta'$ and $(\tilde{V}, \tilde{W}) \sim (\tilde{V}', \tilde{W}')$ in the sense of Definition 4.5. \hfill $\square$

In the case where $R$ has countable dimension, we can classify $G$-gradings on $R$ in combinatorial terms. Clearly, $\mathfrak{F}(\Pi)(U)$ has countable dimension if and only if both $\Pi$ and $U$ have countable dimension. It is known that then there exist dual bases in $U$ and $\Pi$, hence all such pairs $(U, \Pi)$ are isomorphic and there is only one such algebra $R$, which is denoted by $M_\infty(\mathbb{K})$. We will state the classification of $G$-gradings in a form that also applies to $M_n(\mathbb{K})$, for which this result is known under the assumption that $G$ is abelian [4]. Up to isomorphism, the pairs $(\tilde{V}_A, \tilde{W}_{A^{-1}})$ can be encoded by the function $\kappa: G/T \rightarrow \{0, 1, 2, \ldots, \infty\}$ that sends $A$ to $\dim \tilde{V}_A$. Note that the support of the function $\kappa$ is finite or countable, so $|\kappa| = \sum_{A \in G/T} \kappa(A)$ is defined as an element of $\{0, 1, 2, \ldots, \infty\}$. We will denote the associated graded algebra $\mathfrak{F}(G, D, \tilde{V}, \tilde{W})$ by $\mathfrak{F}(G, D, \kappa)$. Finally, for any $g \in G$, define $\kappa^g: G/(g^{-1}Tg) \rightarrow \{0, 1, 2, \ldots, \infty\}$ by setting $\kappa^g(Ag) = \kappa(A)$ for all $A \in G/T$.

Corollary 4.4. Let $\mathbb{K}$ be an algebraically closed field and let $R = M_n(\mathbb{K})$ where $n \in \mathbb{N} \cup \{\infty\}$. If $R$ is given a grading by a group $G$, then $R$ is isomorphic to some $\mathfrak{F}(G, D, \kappa)$ where $D = M_\ell(\mathbb{K})$, with $\ell \in \mathbb{N}$ and $n = |\kappa|\ell$, is equipped with a division grading. Moreover, two such graded algebras $\mathfrak{F}(G, D, \kappa)$ and $\mathfrak{F}(G, D', \kappa')$ are isomorphic if and only if there exists $g \in G$ such that $[g^{-1}D]^g \cong D'$ as graded algebras and $\kappa^g = \kappa'$.

The graded algebra $\mathfrak{F}(G, D, \kappa)$ can be constructed explicitly as follows. Select a left transversal $S$ for $T$ and let $\gamma(A)$ be the unique element of $A \cap S$. For each $A \in G/T$, select dual bases $\{v_i(A)\}$ and $\{v^j(A)\}$ for $\tilde{V}_A$ and $\tilde{W}_A$, respectively, consisting each of $\kappa(A)$ vectors. Then the algebra $\mathfrak{F}(G, D, \kappa)$ has a basis

$$\{E_{i,j}^{A,B}(t) \mid A, B \in G/T, i, j \in \mathbb{N}, i \leq \kappa(A), j \leq \kappa(B), t \in T\},$$

where $E_{i,j}^{A,B}(t) = v_i(A)X_t \otimes v^j(B) = v_i(A) \otimes X_tv^j(B)$. Equations (8) and (9) imply that the multiplication is given by the formula (using Kronecker delta):

$$E_{i,j}^{A,B}(t)E_{i',j'}^{A',B'}(t') = \delta_{B,A}\delta_{j,i'}\sigma(t, t')E_{i,j'}^{A',B'}(tt')$$

and the $G$-grading is given by

$$\deg E_{i,j}^{A,B}(t) = \gamma(A)t\gamma(B)^{-1}.$$

Thus we recover Theorem 5 in [12], which asserts the existence of such a basis under the assumption that $G$ is a finite abelian group and $\mathbb{K}$ is algebraically closed of characteristic zero. Theorem 6 in the same paper gives our condition for isomorphism of two $G$-gradings in the special case $D = \mathbb{K}$.
4.8. Antiautomorphisms and sesquilinear forms. We want to investigate under what conditions a graded algebra described by Theorem 4.4 admits an antiautomorphism. So, we temporarily return to the general setting: \( R \) is a \( G \)-graded primitive algebra (or ring) with minimal graded left ideals.

We may assume \( \mathfrak{F}^r_W(V) \subset R \subset \mathfrak{L}^r_W(V) \) where \( V \) is a right vector space over a graded division algebra \( D \), \( W \) is a left vector space over \( D \), and \( W \) is identified with a total graded subspace of \( V^{gr} \) by virtue of a \( D \)-bilinear form \( (\cdot, \cdot) \). Thus, we have

\[
(dw, v) = d(w, v) \quad \text{and} \quad (w, vd) = (w, v)d \quad \text{for all} \ v \in V, w \in W, d \in D.
\]

Note that, since the adjoint of any operator \( r \in R \) leaves \( W \) invariant, \( W \) becomes a graded right \( R \)-module such that

\[
(wr, v) = (w, rv) \quad \text{for all} \ v \in V, w \in W, r \in R.
\]

It follows that \( \mathfrak{F}^r_W(W) \subset R^{op} \subset \mathfrak{L}^r_W(W) \) where \( W \) is regarded as a right vector space over \( D^{op} \), \( V \) as a left vector space over \( D^{op} \), and \( V \) is identified with a total graded subspace of \( W^{gr} \) by virtue of \( (v, w)^{op} = (w, v) \). (The gradings on all these objects are by the group \( G^{op} \).)

Now suppose that we have an antiautomorphism \( \varphi \) of the graded algebra \( R \). Since \( S = \mathfrak{F}^r_W(V) \) is the unique minimal graded two-sided ideal of \( R \), we have \( \varphi(S) = S \). Thus \( \varphi \) restricts to an antiautomorphism of the graded simple algebra \( S \). It is known (see [9]) that if a graded simple algebra admits an antiautomorphism, then the support of the grading generates an abelian group. Now observe that any element of the support of \( \text{End}^r_D(V) \) has the form \( gh^{-1} \) where \( g \) and \( h \) are in the support of \( V \), and all elements of this form already occur in the support of \( \mathfrak{F}^r_W(V) \). Hence, the support of \( R \) equals the support of \( S \) and generates an abelian group. For this reason, we will assume from now on that \( G \) is abelian.

Applying Theorem 4.5 to the isomorphism \( \varphi: R \to R^{op} \) and taking into account that \( G \) is abelian, we see that there exists \( g_0 \in G \) and an isomorphism \( \varphi_0, \varphi_1, \varphi_2 \) from \((D, V^{[g_0]}, W^{[g_0^{-1}]}))\) to \((D^{op}, W, V)\) such that \( \varphi(r) = \varphi_1 \circ r \circ \varphi_1^{-1} \). In particular, \( \varphi_0 \) is an antiautomorphism of the graded algebra \( D \) and \( \varphi_1 \) is \( \varphi_0 \)-sesquilinear:

\[
\varphi_1(vd) = \varphi_0(d)\varphi(v) \quad \text{for all} \ v \in V \text{ and} \ d \in D.
\]

Now define a nondegenerate \( \mathbb{K} \)-bilinear form \( B: V \times V \to D \) as follows:

\[
B(u, v) = (\varphi_1(u), v) \quad \text{for all} \ u, v \in V.
\]

Then \( B \) has degree \( g_0 \) when regarded as a map \( V \otimes V \to D \). Combining (13) and (15), we see that, over \( D \), the form \( B \) is linear in the second argument and \( \varphi_0 \)-sesquilinear in the first argument, i.e.,

\[
B(ud, v) = \varphi_0(d)B(u, v) \quad \text{and} \quad B(u, vd) = B(u, v)d \quad \text{for all} \ u, v \in V, d \in D.
\]

For brevity, we will say that \( B \) is \( \varphi_0 \)-sesquilinear.
Applying (14), we obtain for all \( u, v \in V \) and \( r \in R \):

\[
B(ru, v) = (\varphi_1(ru), v) = (\varphi_1(u)\varphi(r), v)
= (\varphi_1(u), \varphi(r)v) = B(u, \varphi(r)v),
\]

which means that \( \varphi(r) \) is \textit{adjoint to} \( r \) \textit{with respect to} \( B \). In particular, \( \varphi \) can be recovered from \( B \).

We will need one further property of \( B \). Consider

\[
\bar{B}(u, v) = \varphi_1^{-1}(B(v, u)).
\]

Then \( \bar{B} \) is a nondegenerate \( \varphi_0^{-1} \)-sesquilinear form of the same degree as \( B \). Clearly, we have \( \bar{B}(ru, v) = \bar{B}(u, \varphi^{-1}(r)v) \), so \( \bar{B} \) is related to \( \varphi^{-1} \) in the same way as \( B \) is related to \( \varphi \). We claim that there exists a \( \varphi_0^{-2} \)-semilinear isomorphism of graded vector spaces \( Q: V \to V \) such that

\[
\bar{B}(u, v) = B(Qu, v) \quad \text{for all} \quad u, v \in V. \tag{17}
\]

Indeed, \( Q = \varphi_1^{-1} \circ \varphi_2^{-1} \) satisfies the requirements: it is clearly an invertible \( \varphi_0^{-2} \)-semilinear map, homogeneous of degree \( e \), and we have

\[
B(Qu, v) = (\varphi_2^{-1}(u), v) = (v, \varphi_2^{-1}(u))^* = \varphi_0^{-1}((\varphi_1(v), u)) = \bar{B}(u, v).
\]

It is important to note that the adjoint \( Q^* = (\varphi_2^{-1})^* \circ (\varphi_1^{-1})^* = \varphi_1 \circ \varphi_2 \) maps \( W \) onto \( W \), i.e., \( Q \) is a homeomorphism.

**Definition 4.6.** We will say that a nondegenerate homogeneous \( \varphi_0 \)-sesquilinear form \( B: V \times V \to D \) is \textit{weakly Hermitian} if there exists a \( \varphi_0^{-2} \)-semilinear isomorphism \( Q: V \to V \) of graded vector spaces such that (17) holds. (Note that, since \( B \) is nondegenerate, (17) uniquely determines \( Q \).)

The following is a graded version of the main result in [23, IV.12].

**Theorem 4.7.** Let \( G \) be an abelian group. Let \( D \) be a \( G \)-graded division algebra (or ring), let \( V \) be a graded right vector space over \( D \), and let \( W \) be a total graded subspace of \( V^{\text{gr}*} \).

Let \( R \) be a \( G \)-graded algebra (or ring) such that

\[
\mathfrak{F}^*_W(V) \subset R \subset \mathfrak{L}^*_W(V).
\]

If \( \varphi \) is an antiautomorphism of the graded algebra \( R \), then there exist an antiautomorphism \( \varphi_0 \) of the graded algebra \( D \) and a weakly Hermitian nondegenerate homogeneous \( \varphi_0 \)-sesquilinear form \( B: V \times V \to D \), such that the following conditions hold:

(a) the mapping \( V \to V^{\text{gr}*} \): \( u \mapsto f_u \), where \( f_u(v) = B(u, v) \) for all \( v \in V \), sends \( V \) onto \( W \);

(b) for any \( r \in R \), \( \varphi(r) \) is the adjoint to \( r \) with respect to \( B \), i.e., \( B(ru, v) = B(u, \varphi(r)v) \), for all \( u, v \in V \).
If \( \varphi' \) is an antiautomorphism of \( D \) and \( B' \) is a \( \varphi'_0 \)-sesquilinear form \( V \times V \to D \) that define \( W \) and \( \varphi \) as in (a) and (b), then there exists a nonzero homogeneous \( d \in D \) such that \( B' = dB \) and \( \varphi'_0(x) = d\varphi_0(x)d^{-1} \) for all \( x \in D \).

As a partial converse, if \( \varphi_0 \) is an antiautomorphism of the graded algebra \( D \) and \( B : V \times V \to D \) is a weakly Hermitian nondegenerate homogeneous \( \varphi_0 \)-sesquilinear form, then the adjoint with respect to \( B \) defines an antiautomorphism \( \varphi \) of the \( G \)-graded algebra \( \mathfrak{L}^\text{gr}_W(V) \), with \( W = \{ f_u \mid u \in V \} \), such that \( \varphi(\mathfrak{L}^\text{gr}_W(V)) = \mathfrak{L}^\text{gr}_W(V) \).

**Proof.** Given an antiautomorphism \( \varphi \), the existence of the pair \((\varphi_0, B)\) is already proved. If \((\varphi'_0, B')\) is another such pair, then the corresponding mapping \( u \mapsto f'_u \) is an isomorphism of graded \( R \)-modules \( V^{[g]} \to W^{[g']} \). Hence \( \varphi'_1 \circ \varphi_1^{-1} \) is a nonzero homogeneous element of \( \text{End}^{\text{gr}}_R(W) \), so there exists a nonzero homogeneous \( d \in D \) such that \( \varphi'_1(v) = d\varphi_1(v) \) for all \( v \in V \), which implies \( B' = dB \). Now the equation \( \varphi'_0(x) = d\varphi_0(x)d^{-1} \) follows easily from (16).

Conversely, for a given antiautomorphism \( \varphi_0 \) and a form \( B \) of degree \( g_0 \), define \( \varphi_1 : V \to W \) by setting \( \varphi_1(u) = f_u \). This is a homogeneous \( \varphi_0 \)-semilinear isomorphism of degree \( g_0 \). Take \( \varphi_2 = Q^{-1} \circ \varphi_1^{-1} \). Then one checks that \((\varphi_0, \varphi_1, \varphi_2)\) is an isomorphism from \((D, V^{[g_0]}, W^{[g_0]})\) to \((D^{op}, W, V)\), so Theorem 4.5 tells us that \( \varphi(r) = \varphi_1 \circ r \circ \varphi_1^{-1} \) defines an isomorphism of graded algebras \( \mathfrak{L}^\text{gr}_W(V) \to \mathfrak{L}^\text{gr}_V(W) = \mathfrak{L}^\text{gr}_W(V)^{op} \) that restricts to an isomorphism \( \mathfrak{L}^\text{gr}_W(V) \to \mathfrak{L}^\text{gr}_V(W) = \mathfrak{L}^\text{gr}_W(V)^{op} \). It remains to observe that the definition of \( \varphi_1 \) implies that \( \varphi(r) \) is the adjoint to \( r \) with respect to \( B \), for any \( r \in \mathfrak{L}^\text{gr}_W(V) \). \( \square \)

Note that it follows that any antiautomorphism \( \varphi \) of the graded algebra \( R \) extends to an antiautomorphism of \( \mathfrak{L}^\text{gr}_W(V) \) and restricts to an antiautomorphism of \( \mathfrak{L}^\text{gr}_W(V) \).

**Remark 4.1.** It is easy to compute \( \varphi \) on \( \mathfrak{L}^\text{gr}_W(V) \) explicitly. We have \( \varphi(v \otimes f_u) = \varphi_2(f_u) \otimes \varphi_1(v) \), so, taking into account \( \varphi_1(v) = f_v \) and \( \varphi_2 = Q^{-1} \circ \varphi_1^{-1} \), we obtain:

\[
\varphi(v \otimes f_u) = Q^{-1}u \otimes f_v \quad \text{for all} \quad u, v \in V.
\]

4.9. **Antiautomorphisms that are involutive on the identity component.** We restrict ourselves to the case where \( R \) is a locally finite simple algebra with minimal left ideals over an algebraically closed field \( K \). If \( R \) is given a grading by an abelian group \( G \), then, by Theorem 4.6, \( R \) is isomorphic to some \( \mathfrak{L}(G, D, \tilde{V}, \tilde{W}) \) where \( D \) is a matrix algebra with a division grading. Suppose that the graded algebra \( R \) admits an antiautomorphism \( \varphi \). Then, by Theorem 4.7, we obtain an antiautomorphism \( \varphi_0 \) for \( D \). It is known [11] that this forces the support \( T \) of \( D \) to be an elementary 2-group and hence char \( K \neq 2 \) or \( T = \{ e \} \). From now on, we assume char \( K \neq 2 \).

Since \( G \) is abelian, any division grading on a matrix algebra can be realized using generalized Pauli matrices. If the support \( T \) is an elementary 2-group, then the matrix transpose preserves this grading. Choose a nonzero element \( X_t \) in each component \( D_t \). Then the transpose of \( X_t \) equals \( \beta(t)X_t \) where \( \beta(t) \in \{ \pm 1 \} \). It is easy to check (see [4]) that \( \beta : T \to \{ \pm 1 \} \) is a quadratic form on \( T \) if we regard it as a vector space over the field of two elements,
with the nondegenerate alternating bicharacter $\beta: T \times T \to \mathbb{K}^\times$ being the associated bilinear form: $\beta(t,t') = \beta(t)\beta(t')$ for all $t,t' \in T$. It is easy to see that any automorphism of the graded algebra $D$ is a conjugation by some $X_t$. Hence $\varphi_0$ is given by $\varphi_0(X_r) = \beta(t')X_t^{-1}X_rX_t$ for some $t \in T$. In particular, $\varphi_0$ is an involution. Hence, the isomorphism $Q$ associated to the $\varphi_0$-sesquilinear form $B$ in Theorem 4.7 is linear over $D$ and thus $Q$ is an invertible element of the identity component of $\mathfrak{L}_W^\text{gr}(V)$. Adjusting $B$, we may assume without loss of generality that

$$\varphi_0(X_t) = \beta(t)X_t \quad \text{for all } t \in T.$$ 

This convention makes the choice of $B$ unique up to a scalar in $\mathbb{K}^\times$.

Assume that $\varphi$ restricts to an involution on $R_e$. Then $B$ has certain symmetry properties, which we are going to investigate now. In particular, $B$ is balanced, i.e., for any pair of homogeneous $u, v \in V$, we have

$$B(u, v) = 0 \iff B(v, u) = 0.$$ 

Recall that $\mathfrak{g}(G, D, \tilde{V}, \tilde{W}) = \mathfrak{g}_W^\text{gr}(V)$ where $V$ and $W$ are constructed from $\tilde{V}$ and $\tilde{W}$ as follows. Select a transversal $S$ for $T$ and, for each $A \in G/T$, set $V_A = \tilde{V}_A \otimes D$ and $W_A = D \otimes \tilde{W}_A$, with the degree of the elements of $\tilde{V}_A \otimes 1$ and $1 \otimes \tilde{W}_A$ set to be the unique element of $A \cap S$, which we denote by $\gamma(A)$. It will be convenient to identify $\tilde{V}_A$ with $\tilde{V}_A \otimes 1$ and $\tilde{W}_A$ with $1 \otimes \tilde{W}_A$.

Using the definition of $\tilde{B}$ and equation (17), we compute, for all $u, v \in V$ and $r \in \mathfrak{L}_W^\text{gr}(V)$:

$$B(u, \varphi^2(r)v) = B(\varphi(r)u, v) = \varphi_0(\tilde{B}(v, \varphi(r)u)) = \varphi_0(B(Qv, \varphi(r)u)) = \varphi_0(B(rQv, u)) = \tilde{B}(u, rQv) = B(u, Q^{-1}rQv).$$

Substituting $r = 1$, we obtain $B(u, v) = B(Qu, Qv)$ for all $u, v \in V$ and hence $B(Qu, v) = B(u, Q^{-1}v)$. Continuing with (19) to obtain, for all $u, v \in V$, $B(u, \varphi^2(r)v) = B(u, Q^{-1}rQv)$. Therefore,

$$\varphi^2(r) = Q^{-1}rQ \quad \text{for all } r \in R.$$ 

Observe that the identity component $R_e$ is the direct sum of subalgebras $R^A$, $A \in G/T$, where $R^A$ consists of all operators in $R_e$ that map the isotypic component $V_A$ into itself and other isotypic components to zero. Clearly, $R^A$ is spanned by the operators of the form $w \otimes v$ where $v \in \tilde{V}_A$ and $w \in \tilde{W}_A$. Being homogeneous of degree $e$, $Q$ maps $V_A$ onto $V_A$. The restriction of $Q$ to $V_A$ is linear over $D$ and, by (20), commutes with all elements of $R^A$. It follows that $Q$ acts on $V_A$ as a scalar $\lambda_A \in \mathbb{K}^\times$. Now (17) implies that $B$ is balanced, as claimed.

The fact that $B$ is balanced allows us to define the concept of orthogonality for homogeneous elements and for graded subspaces of $V$. Since $B$ is homogeneous of degree, say, $g_0$, we have, for all $u \in V_{g_1}$ and $v \in V_{g_2}$, that $B(u, v) = 0$ unless $g_0g_1g_2 \in T$. It follows that $V_A$ is orthogonal to all isotypic components except $V_{g^{-1}_0A^{-1}}$, and hence the restriction of $B$ to $V_A \times V_{g^{-1}_0A^{-1}}$ is nondegenerate. It will be important to distinguish whether or not $A$ equals $g_0^{-1}A^{-1}$. 
If $g_0 A^2 = T$, then the element $g_0 \gamma(A)^2 \in T$ does not depend on the choice of the transversal and will be denoted by $\tau(A)$. The restriction of $B$ to $V_A \times V_A$ is a nondegenerate $\varphi_0$-sesquilinear form over $D$. It is uniquely determined by its restriction to $\tilde{V}_A \times \tilde{V}_A$, which is a bilinear form over $\mathbb{K}$ with values in $D_{\varphi_0(A)}$. Set

\begin{equation}
B(u, v) = \tilde{B}_A(u, v)X_\tau(A) \quad \text{for all } u, v \in \tilde{V}_A \text{ where } g_0 A^2 = T.
\end{equation}

Then $\tilde{B}_A$ is a nondegenerate bilinear form on $\tilde{V}_A$ with values in $\mathbb{K}$. Setting $t = \tau(A)$ for brevity, we compute:

\begin{align*}
\tilde{B}_A(v, u)X_t &= B(v, u) = \varphi_0(B(Qu, v)) = \varphi_0(B(\Lambda_A u, v)) \\
&= \varphi_0(\Lambda\tilde{B}_A(u, v)X_t) = \Lambda\tilde{B}_A(u, v)\varphi_0(X_t) = \Lambda\beta(t)\tilde{B}_A(u, v)X_t,
\end{align*}

so that (22) holds for any $A$. Setting $g = A_A^T$ for brevity, we compute:

\begin{equation}
\tilde{B}(v, u) = \tilde{B}_A(u, v)1 \quad \text{for all } u, v \in \tilde{V}_A \text{ where } g_0 A^2 \neq T.
\end{equation}

It is easy to see how $\tilde{B}_A$ and $\tilde{B}_A^0A^{-1}$ are related: $\tilde{B}_A^0A^{-1}(v, u) = \Lambda\tilde{B}_A(u, v)$. Putting all pieces together, we set $\tilde{V}_A = V_A \oplus \tilde{V}_A^0A^{-1}$ and define a nondegenerate bilinear form $\tilde{B}: \tilde{V} \times \tilde{V} \to \mathbb{K}$ so that all summands in (24) are orthogonal to each other, the restriction of $\tilde{B}$ to $\tilde{V}_A \times \tilde{V}_A$ is $\tilde{B}_A$ if $g_0 A^2 = T$ while the restriction of $\tilde{B}$ to $\tilde{V}_A$ is zero and the restriction to $\tilde{V}_A \times \tilde{V}_A^0A^{-1}$ is $\tilde{B}_A$ if $g_0 A^2 \neq T$.

Conversely, let $\tilde{V}$ be a vector space over $\mathbb{K}$ that is given a grading by $G/T$ and let $\tilde{B}$ be a nondegenerate bilinear form on $\tilde{V}$ that is compatible with the grading in the sense that $\tilde{B}(\tilde{V}_{g_1T}, \tilde{V}_{g_2T}) = 0$ for all $g_1, g_2 \in G$ with $g_0g_1g_2T \neq T$ and, for all $A \in G/T$, satisfies the following symmetry condition:

\begin{equation}
\tilde{B}(v, u) = \mu_A\tilde{B}(u, v) \quad \text{for all } u, v \in \tilde{V}_A \text{ where } \mu_A \in \mathbb{K}^\times.
\end{equation}

It follows that $\mu_A(g_0 A^{-1}) = 1$. Hence, if $g_0 A^2 = T$, then $\tilde{V}$ restricts to a symmetric or a skew-symmetric form on $\tilde{V}_A$. For any $A \in G/T$, let $\tilde{B}_A$ be the restriction of $\tilde{B}$ to $\tilde{V}_A \times \tilde{V}_A^0A^{-1}$. It follows that $\tilde{B}_A$ is nondegenerate. Choose a transversal $S$ for $T$ so that (22) holds for all $A$ with $g_0 A^2 \neq T$. Set $V_A = \tilde{V}_A \otimes [\gamma(A)]D$. Then $V_A$ is a graded $D$-module whose isomorphism class does not depend on the choice of $S$. Set $V = \bigoplus_{A \in G/T} V_A$. Define $B: V \times V \to D$ using (21) and (23), setting $B$ equal to zero in all other cases and then extending
by \( \varphi_0 \)-sesquilinearity. Clearly, \( B \) is nondegenerate. Set \( W = \{ f_u \mid u \in V \} \) where \( f_u(v) = B(u, v) \). We will denote the corresponding \( G \)-graded algebra \( \mathfrak{F}_{W}^g(V) \) by \( \mathfrak{F}(G, D, \tilde{V}, \tilde{B}, g_0) \) or \( \mathfrak{F}(G, T, \beta, \tilde{V}, \tilde{B}, g_0) \), since \( D \) is determined by the support \( T \) and bicharacter \( \beta \). The graded algebra \( \mathfrak{F}_{W}^g(V) \) has an antiautomorphism \( \varphi \) defined by the adjoint with respect to \( B \). Indeed, let \( Q : V \to V \) act on \( V_A \) as the scalar \( \lambda_A \) where
\[
\lambda_A = \begin{cases} 
\mu_A \beta(\tau(A)) & \text{if } g_0 A^2 = T; \\
\mu_A & \text{if } g_0 A^2 \neq T.
\end{cases}
\]
(26)

Then \( Q \) satisfies \( B(u, v) = B(Qu, v) \) for all \( u, v \in V \) and hence \( B \) is weakly Hermitian. Since \( Q \) commutes with the elements of \( R_e \), equation (20) tells us that \( \varphi^2 \) is the identity on \( R_e \).

**Definition 4.7.** With fixed \( D \) and \( \varphi_0 \), we will write \( (\tilde{V}, \tilde{B}, g_0) \sim (\tilde{V}', \tilde{B}', g_0') \) if there is an element \( g \in G \) such that \( g_0' = g_0 g^{-2} \) and, for any \( A \in G/T \) with \( g_0 A^2 = T \), we have \( \tilde{V}_A \cong \tilde{V}'_{g_A} \) as inner product spaces while, for any \( A \in G/T \) with \( g_0 A^2 \neq T \), we have \( (\tilde{V}_A, \tilde{V}_{g_0^{-1} A^{-1}}) \cong (\tilde{V}'_{g_A}, \tilde{V}'_{(g_0')^{-1} g_0^{-1} A^{-1}}) \) and \( \mu_A = \mu'_{g A} \) (where \( \mu_A \) is defined by (25) and \( \mu' \) by the same equation with \( B \) replaced by \( \tilde{B}' \)).

Recall that, disregarding the grading, \( R \) can be represented as \( \mathfrak{F}_N(U) \) with \( U = \tilde{V} \otimes N \) and \( \Pi = M \otimes \tilde{W} \), where \( M \) and \( N \) are the natural right and left modules for \( D = M_\ell(\mathbb{K}) \), respectively (see the analysis preceding Theorem 4.6). In our case, \( W = \{ f_u \mid u \in V \} \) and \( \tilde{W} = \{ f_{\tilde{u}} \mid \tilde{u} \in \tilde{V} \} \). Note that \( f_{\tilde{u}}(\tilde{v}) = B(\tilde{u}, \tilde{v}) \) does not necessarily belong to \( D_e = \mathbb{K} \) for all \( \tilde{u}, \tilde{v} \in \tilde{V} \), so equation (11) for the \( \mathbb{K} \)-bilinear form \( \Pi \) should be modified as follows:
\[
(27) \quad (m \otimes \tilde{u}, \tilde{v} \otimes n) = m B(\tilde{u}, \tilde{v}) n \quad \text{for all} \quad m \in M, n \in N, \tilde{u}, \tilde{v} \in \tilde{V}.
\]
If we identify \( U \) and \( \Pi \) with \( \tilde{V}_\ell \), then the above \( \mathbb{K} \)-bilinear form is given by
\[
(28) \quad ((\tilde{u}_1, \ldots, \tilde{u}_\ell), (\tilde{v}_1, \ldots, \tilde{v}_\ell)) = \sum_{i,j=1}^{\ell} x_{ij} B(\tilde{u}_i, \tilde{v}_j) \quad \text{for all} \quad \tilde{u}_i \in \tilde{V}_{A_i} \text{ and } \tilde{v}_j \in \tilde{V}_{A'_j},
\]
where \( x_{ij} \) is the \((i, j)\)-entry of the matrix \( X_{\tau(A)} \) if \( A_i = A'_j = A \) with \( g_0 A^2 = T \) and \( x_{ij} = \delta_{i,j} \) otherwise.

Now we are ready to state the result:

**Theorem 4.8.** Let \( R \) be a locally finite simple algebra with minimal left ideals over an algebraically closed field \( \mathbb{K} \), \( \text{char} \mathbb{K} \neq 2 \). If \( R \) is given a grading by an abelian group \( G \) and an antiautomorphism \( \varphi \) that preserves the grading and restricts to an involution on \( R_e \), then \( (R, \varphi) \) is isomorphic to some \( \mathfrak{F}(G, D, \tilde{V}, \tilde{B}, g_0) \) where \( D \) is a matrix algebra over \( \mathbb{K} \) equipped with a division grading and an involution \( \varphi_0 \) given by matrix transpose. Conversely, if \( D = M_\ell(\mathbb{K}) \) with a division grading and \( \varphi_0 \) is the matrix transpose, then \( \mathfrak{F}(G, D, \tilde{V}, \tilde{B}) \) is a locally finite simple algebra with minimal left ideals, which can be represented as \( \mathfrak{F}_U(U) \) where \( U = \tilde{V}_\ell \) and the nondegenerate bilinear form \( U \times U \to \mathbb{K} \) is given by (28). Two such graded algebras \( \mathfrak{F}(G, D, \tilde{V}, \tilde{B}, g_0) \) and \( \mathfrak{F}(G, D', \tilde{V}', \tilde{B}', g_0) \) are not isomorphic unless \( D \cong D' \) as graded algebras, whereas, for fixed \( D \) and \( \varphi_0 \), \( \mathfrak{F}(G, D, \tilde{V}, \tilde{B}, g_0) \) and \( \mathfrak{F}(G, D, \tilde{V}', \tilde{B}', g_0) \) are isomorphic as graded algebras with antiautomorphism if and only if \( (\tilde{V}, \tilde{B}, g_0) \sim (\tilde{V}', \tilde{B}', g_0) \) in the sense of Definition 4.7.
In the case where $R$ has finite or countable dimension, i.e., $R = M_n(\mathbb{K})$ with $n \in \mathbb{N} \cup \{\infty\}$, we can express the classification in combinatorial terms. Since $\widetilde{A}$ is algebraically closed, two vector spaces with symmetric inner products are isomorphic if they have the same finite or countable dimension. The same is true for vector spaces with skew-symmetric inner product. Therefore, for $A \in G/T$ with $g_0 A^2 = T$, the isomorphism class of $\widetilde{V}_A$ is encoded by $\mu_A$ and $\dim \widetilde{V}_A$. For $A \in G/T$ with $g_0 A^2 \neq T$, the isomorphism class of $(\widetilde{V}_A, \widetilde{V}_{g_0^{-1} A^{-1}})$ is encoded by $\dim \widetilde{V}_A = \dim \widetilde{V}_{g_0^{-1} A^{-1}}$. We introduce functions $\mu : G/T \to \mathbb{K}^\times$ sending $A$ to $\mu_A$ (where we set $\mu_A = 1$ if $\widetilde{V}_A = 0$) and, as before, $\kappa : G/T \to \{0, 1, 2, \ldots, \infty\}$ sending $A$ to $\dim \widetilde{V}_A$. Recall that $\mu$ satisfies $\mu_A \mu_{g_0^{-1} A^{-1}} = 1$ for all $A \in G/T$ and $\kappa$ has a finite or countable support.

We will denote the associated graded algebra with antiautomorphism $\bar{F}$ by $\mathfrak{F}(G, D, \bar{V}, \bar{B}, g_0)$ by $\mathfrak{F}(G, D, \kappa, \mu, g_0)$ or by $\mathfrak{F}(G, T, \beta, \kappa, \mu, g_0)$. For a given elementary 2-subgroup $T \subset G$ and a bicharacter $\beta$, we fix a realization of $D$ using Pauli matrices and thus fix an involution $\varphi_0$ on $D$. Finally, for any $g \in G$, define $\mu^g$ and $\kappa^g$ by setting $\mu^g(gA) = \mu(A)$ and $\kappa^g(gA) = \kappa(A)$ for all $A \in G/T$.

**Corollary 4.5.** Let $\mathbb{K}$ be an algebraically closed field, $\text{char } \mathbb{K} \neq 2$, and let $R = M_n(\mathbb{K})$ where $n \in \mathbb{N} \cup \{\infty\}$. If $R$ is given a grading by an abelian group $G$ and an antiautomorphism $\varphi$ that preserves the grading and restricts to an involution on $R_e$, then $(R, \varphi)$ is isomorphic to some $\mathfrak{F}(G, D, \kappa, \mu, g_0)$ where $D = M_\ell(\mathbb{K})$, with $\ell \in \mathbb{N}$ and $n = |\kappa|\ell$, is equipped with a division grading and an involution $\varphi_0$ given by matrix transpose. Moreover, $\mathfrak{F}(G, T, \beta, \kappa, \mu, g_0)$ and $\mathfrak{F}(G, T', \beta', \kappa', \mu', g_0')$ are isomorphic as graded algebras with antiautomorphism if and only if $T = T'$, $\beta' = \beta$, and there exists $g \in G$ such that $g_0 = g_0 g^{-2}$, $\kappa' = \kappa^g$ and $\mu' = \mu^g$. □

### 4.10. Graded involutions

We can specialize Theorem 4.8 to obtain a classification of involutions on the graded algebra $R$. By (20), $\varphi$ is an involution if and only if $Q : V \to V$ is a scalar operator, i.e., $\bar{B} = \lambda B$ for some $\lambda \in \mathbb{K}^\times$, which implies $\lambda \in \{\pm 1\}$. Disregarding the grading, $R = \mathfrak{F}_U(U)$ where $U$ is an inner product space. Since $\varphi_0$ is matrix transpose, equation (27) implies that $(v, u) = (u, v)$ for all $u, v \in U$. Hence, in the case $\bar{B} = B$, we obtain an orthogonal involution on $R$ and write $\text{sgn}(\varphi) = 1$, whereas in the case $\bar{B} = -B$, we obtain a symplectic involution and write $\text{sgn}(\varphi) = -1$. Since all $\lambda_A$ must be equal to $\lambda = \text{sgn}(\varphi)$, equation (26) yields

$$
\mu_A = \begin{cases} 
\text{sgn}(\varphi) \beta(\tau(A)) & \text{if } g_0 A^2 = T; \\
\text{sgn}(\varphi) & \text{if } g_0 A^2 \neq T.
\end{cases}
$$

(29)

We note that, for $g_0 A^2 \neq T$, although the space $\widetilde{V}_{\{A, g_0^{-1} A^{-1}\}}$ now has a symmetric or skew-symmetric inner product, the equivalence relation in Definition 4.7 requires more than just an isomorphism of inner product spaces: the isomorphism must respect the direct sum decomposition $\widetilde{V}_{\{A, g_0^{-1} A^{-1}\}} = \widetilde{V}_A \oplus \widetilde{V}_{g_0^{-1} A^{-1}}$. To summarize:

**Proposition 4.1.** Under the conditions of Theorem 4.8, $\varphi$ is an involution if and only if $\bar{B}$ satisfies the symmetry condition (25) where $\mu_A$ is given by (29). □
In the case of finite or countable dimension, we again can reduce everything to combinatorial terms (for the finite case, this result appeared in [4]). Once \( \delta = \text{sgn}(\varphi) \) is specified, the function \( \mu : G/T \to \mathbb{K}^\times \) is determined by (29), so we will denote the corresponding graded algebra with involution by \( \mathfrak{F}(G, D, \kappa, \delta, g_0) \) or \( \mathfrak{F}(G, T, \beta, \kappa, \delta, g_0) \).

**Corollary 4.6.** Let \( R = \mathfrak{F}(G, T, \beta, \kappa) \) where the ground field \( \mathbb{K} \) is algebraically closed of characteristic different from 2, and \( G \) is an abelian group. The graded algebra \( R \) admits an involution \( \varphi \) with \( \text{sgn}(\varphi) = \delta \) if and only if \( T \) is an elementary 2-group and, for some \( g_0 \in G \), we have \( \kappa(A) = \kappa(g_0^{-1}A^{-1}) \) for all \( A \in G/T \) and we also have \( \beta(g_0a^2) = \delta \) for all \( A = aT \in G/T \) such that \( g_0A^2 = T \) and \( \kappa(A) \) is finite and odd. If \( \varphi \) is an involution on \( R \) with \( \text{sgn}(\varphi) = \delta \), then the pair \((R, \varphi)\) is isomorphic to some \( \mathfrak{F}(G, T, \beta, \kappa, \delta, g_0) \). Moreover, \( \mathfrak{F}(G, T, \beta, \kappa, \delta, g_0) \) and \( \mathfrak{F}(G, T', \beta', \kappa', \delta', g_0) \) are isomorphic as graded algebras with involution if and only if \( T' = T, \beta' = \beta, \delta' = \delta \), and there exists \( g \in G \) such that \( g_0' = g_0g^{-2} \) and \( \kappa' = \kappa^3 \).

\[ \square \]

5. **Gradings on Lie algebras of finitary linear transformations**

Throughout this section, \( \mathbb{K} \) is an algebraically closed field of characteristic different from 2, and \( G \) is an abelian group. Our goal is to classify \( G \)-gradings on the infinite-dimensional simple Lie algebras \( \mathfrak{fsl}(U, \Pi), \mathfrak{fso}(U, \Phi) \) and \( \mathfrak{fsp}(U, \Phi) \) of finitary linear transformation over \( \mathbb{K} \). We are going to transfer the classification results for the associative algebras \( \mathfrak{F}(U) \) in Section 4.3 to the above Lie algebras using the transfer Theorems 3.4 and 3.5.

We denote by \( \mathcal{H} \) the group algebra \( \mathbb{K}G \), which is a commutative and cocommutative Hopf algebra. A \( G \)-grading on an algebra \( \mathcal{U} \) is equivalent to a (right) \( \mathcal{H} \)-comodule structure \( \rho : \mathcal{U} \to \mathcal{U} \otimes \mathcal{H} \) that is also a homomorphism of algebras. Indeed, given a \( G \)-grading on \( \mathcal{U} \), we set \( \rho(x) = x \otimes g \) for all \( x \in L_g \) and extend by linearity. Conversely, given \( \rho \), we obtain a \( G \)-grading on \( \mathcal{U} \) by setting \( U_g = \{ x \in U \mid \rho(x) = x \otimes g \} \). We can extend \( \rho \) to a homomorphism \( \mathcal{U} \otimes \mathcal{H} \to \mathcal{U} \otimes \mathcal{H} \) by setting \( \rho(x \otimes h) = \rho(x)(1 \otimes h) \). It is easy to see that this extended homomorphism is surjective.

5.1. **Special linear Lie algebras.** Let \( \mathcal{L} = \mathfrak{fsl}(U, \Pi) \) where \( U \) is infinite-dimensional and let \( R = \mathfrak{F}(U) \). Suppose \( \mathcal{L} \) is given a \( G \)-grading and let \( \rho : \mathcal{L} \otimes \mathcal{H} \to \mathcal{L} \otimes \mathcal{H} \) be the Lie homomorphism obtained by extending the comodule structure map. Then \( \mathcal{L} = [R, R] \) is a noncentral Lie ideal of \( R \) and \( \mathcal{M} = \mathcal{L} \otimes \mathcal{H} \) is a Lie ideal of \( R \otimes \mathcal{H} \). Moreover, \( \langle \mathcal{L} \rangle = R \) implies \( \langle \mathcal{M} \rangle = R \otimes \mathcal{H} \), and \( \langle \mathcal{L}, \mathcal{L} \rangle = \mathcal{L} \) implies \( [\mathcal{M}, \mathcal{M}] = \mathcal{M} \). Applying Theorem 3.4 and observing that \( \tau = 0 \), we conclude that \( \rho \) extends to a map \( \rho' : R \otimes \mathcal{H} \to R \otimes \mathcal{H} \) which is a sum of a homomorphism and the negative of an antihomomorphism. Thus, there are central idempotents \( e_1 \) and \( e_2 \) in \( \mathcal{H} \) (which can be identified with the center of \( \mathbb{R}^e \otimes \mathcal{H} \)) with \( e_1 + e_2 = 1 \) such that the composition of \( \rho' \) and the projection \( R \otimes \mathcal{H} \to R \otimes e_1 \mathcal{H} \) is a homomorphism, while the composition of \( \rho' \) and the projection \( R \otimes \mathcal{H} \to R \otimes e_2 \mathcal{H} \) is the negative of an antihomomorphism.

If \( \psi \) is any automorphism of \( \mathcal{L} \), it can be extended to a map \( \psi' : R \to R \) that is a homomorphism or the negative of an antihomomorphism (use [18, Theorem 6.19] or our Theorem
3.4 with $\mathcal{H} = \mathbb{K}$). Clearly, $\psi'$ is surjective. Since $R$ is simple, we conclude that $\psi'$ is an automorphism or the negative of an antimorphism. We claim that $\psi$ cannot admit extensions of both types. Indeed, assume that $\psi'$ is an automorphism of $R$ and $\psi''$ is the negative of an antimorphism of $R$ such that both restrict to $\psi$. Let $\sigma = (\psi')^{-1} \psi''$. Then $\sigma$ is the negative of an antimorphism of $R$ that restricts to identity on $\mathcal{L}$. Hence, for any $x \in \mathcal{L}$ and $r \in R$, we have $[x, r] = \sigma([x, r]) = [\sigma(x), \sigma(r)] = [x, \sigma(r)]$. It follows that $r - \sigma(r)$ belongs to the center of $R$, which is zero, so $\sigma$ is the identity map — a contradiction. We have shown that, for any automorphism $\psi$ of $\mathcal{L}$, there is a unique extension $\psi': R \rightarrow R$ that is either an antimorphism or the negative of an antimorphism.

Now any character $\chi \in \hat{G}$ acts as an automorphism $\psi$ of $\mathcal{L}$ defined by $\psi(x) = \chi(g)x$ for all $x \in \mathcal{L}$. Denote this $\psi$ by $\eta(\chi)$, i.e., $\eta(\chi) = (id_R \otimes \chi)\rho$. Clearly, $\eta$ is a homomorphism from $\hat{G}$ to $Aut(\mathcal{L})$. Define $\eta'(\chi) = \eta(\chi)^*$. It follows from the uniqueness of extension that $\eta'$ is an automorphism of $\hat{G}$ to the group $Aut(\hat{R})$ consisting of all automorphisms and the negatives of antimorphisms of $R$. We can regard $\chi$ as a homomorphism of algebras $\mathcal{H} \rightarrow \mathbb{K}$ and define $\eta''(\chi) = (id_R \otimes \chi)\rho'$. Then $\eta''(\chi)$ is a map $R \rightarrow R$ that restricts to $\eta(\chi)$ on $L$. Clearly, $\chi(e_1)$ is either 1 or 0, and then $\chi(e_2)$ is either 0 or 1, respectively. An easy calculation shows that if $\chi(e_1) = 1$, then $\eta''(\chi)$ is a homomorphism, and if $\chi(e_1) = 0$, then $\eta''(\chi)$ is the negative of an antimorphism. We conclude that $\eta''(\chi) = \eta'(\chi)$. Setting $h = e_1 - e_2$, we see that, for any $\chi \in \hat{G}$,

$$\chi(h) = \begin{cases} 1 & \text{if } \eta'(\chi) \in Aut(R); \\ -1 & \text{if } \eta'(\chi) \notin Aut(R). \end{cases}$$

(30)

It is well known that any idempotent of $\mathcal{H} = \mathbb{K}G$ is contained in $\mathbb{K}K$ for some finite subgroup $K \subset G$ such that char $\mathbb{K}$ does not divide the order of $K$. Pick $K$ such that $e_1 \in \mathbb{K}G_0$, then also $h \in \mathbb{K}K$. Since any character of $K$ extends to a character of $G$, equation (30) implies that $(\chi_1\chi_2)(h) = \chi_1(h)\chi_2(h)$ for all $\chi_1, \chi_2 \in \hat{K}$, so $h$ is a group-like element of $(\mathbb{K}\hat{K})^* = \mathbb{K}K$. It follows that $h \in K$. Clearly, the order of $h$ is at most 2. Since char $\mathbb{K} \neq 2$, the characterization given by (30) shows that the element $h$ is uniquely determined by the given $G$-grading on $\mathcal{L}$. It follows that $e_1, e_2$ and $\rho'$ are also uniquely determined.

If $h$ has order 1, then $e_1 = 1$ and $e_2 = 0$, so the map $\rho': R \rightarrow R \otimes \mathcal{H}$ is a homomorphism of associative algebras. Since $\mathcal{L}$ generates $R$, it immediately follows that $\rho'$ is a comodule structure. This makes $R$ a $G$-graded algebra such that the given grading on $\mathcal{L}$ is just the restriction of the grading on $R$, i.e., $\mathcal{L}_g = R_g \cap \mathcal{L}$ for all $g \in G$. Such gradings on $\mathcal{L}$ will be referred to as grading of Type I. We can use Corollary 4.3 to obtain all such gradings.

If $h$ has order 2, then both $e_1$ and $e_2$ are nontrivial, so $\rho'$ is not a homomorphism of associative algebras, which means that the algebra $R$ does not admit a $G$-grading that would restrict to the given grading on $\mathcal{L}$. Such gradings on $\mathcal{L}$ will be referred to as grading of Type II (with the distinguished element $h = e_1 - e_2$). Let $\overline{G} = G/\langle h \rangle$ and $\overline{\mathcal{H}} = \mathbb{K}\overline{G}$. Denote the quotient map $G \rightarrow \overline{G}$ by $\pi$ and extend it to a homomorphism of Hopf algebras $\mathcal{H} \rightarrow \overline{\mathcal{H}}$. Since $\pi(e_2) = 0$, the map $\overline{\rho} = (id_R \otimes \pi)\rho$ is a homomorphism of associative algebras, so $\overline{\rho}: R \rightarrow R \otimes \overline{\mathcal{H}}$ is a comodule structure, which makes $R$ a $\overline{\mathcal{H}}$-graded algebra. The restriction
of this grading to \( \mathcal{L} \) is the coarsening of the given \( G \)-grading induced by \( \pi: G \to \bar{G} \), i.e.,
\[
R_g \cap \mathcal{L} = L_g \oplus L_{gh}
\]
for all \( g \in G \), where \( \bar{g} = \pi(g) \). The original grading can be recovered as follows.

Fix a character \( \chi \in \hat{G} \) satisfying \( \chi(h) = -1 \) and let \( \psi = \eta(\chi) \). Then we get
\[
L_g = \{ x \in L_{\bar{g}} \mid \psi(x) = \chi(g)x \}.
\]
Indeed, by definition of \( \psi \), \( L_g \) is contained in the right-hand side. Conversely, suppose \( x \in L_{\bar{g}} \) satisfies \( \psi(x) = \chi(g)x \). Write \( x = y + z \) where \( y \in L_g \) and \( y \in L_{gh} \). Then we have \( \chi(g)(y + z) = \psi(x) = \chi(g)(y - z) \). It follows that \( z = 0 \) and so \( x \in L_g \).

By (30), the extension \( \psi' \) of \( \psi \) is not an automorphism, so \( \psi' = -\varphi \) where \( \varphi \) is an antiautomorphism of \( R \). Since \( \psi \) leaves the components \( L_{\bar{g}} \) invariant and \( \mathcal{L} \) generates \( R \), it follows that \( \varphi \) leaves the components \( R_{\bar{g}} \) invariant. Since \( \varphi^2 = (\psi')^2 = \eta'(\chi^2) \) and \( \chi^2 \) can be regarded as a character of \( \bar{G} \), we obtain \( \varphi^2(r) = \chi^2(\bar{g})r \) for all \( r \in R_{\bar{g}} \). Set
\[
R_{\bar{g}} = \{ r \in R_{\bar{g}} \mid \varphi(r) = -\chi(g)r \}.
\]
Then \( R_{\bar{g}} = R_g \oplus R_{gh} \) and hence \( R = \bigoplus_{g \in G} R_g \). Since \( \psi' \) is not an automorphism of \( R \), this is not a \( G \)-grading on the associative algebra \( R \). It is, however, a \( G \)-grading on the Lie algebra \( R^{(-)} \) (corresponding to the comodule structure \( \rho' \)). The \( \bar{G} \)-grading on \( R \) and the antiautomorphism \( \varphi \) completely determine the \( G \)-grading on \( \mathcal{L} \):
\[
\mathcal{L}_g = \{ x \in R_{\bar{g}} \cap \mathcal{L} \mid \varphi(x) = -\chi(g)x \}.
\]

Conversely, suppose we have a \( \bar{G} \)-grading on \( R \) and an antiautomorphism \( \varphi \) of the \( \bar{G} \)-graded algebra \( R \) satisfying the following compatibility condition:
\[
\varphi^2(r) = \chi^2(\bar{g})r \quad \text{for all} \quad r \in R_{\bar{g}}, \bar{g} \in \bar{G}.
\]
Since \( -\varphi \) is an automorphism of \( R^{(-)} \), equation (31) gives a \( G \)-grading on \( R^{(-)} \) that refines the given \( \bar{G} \)-grading. Since \( \mathcal{L} = [R, R] \) is a \( G \)-graded subalgebra of \( R^{(-)} \), we see that (32) defines a Type II grading on \( \mathcal{L} \) with distinguished element \( h \). Note that the compatibility condition implies that \( \varphi \) acts as involution on the identity component of the \( \bar{G} \)-grading of \( R \), so Theorem 4.8 tells us that \( (R, \varphi) \) is isomorphic to some \( \mathfrak{g}(\bar{G}, D, \bar{V}, \tilde{B}, \bar{y}_0) \).

**Proposition 5.1.** The graded algebra \( R = \mathfrak{g}(\bar{G}, D, \bar{V}, \tilde{B}, \bar{y}_0) \) with its antiautomorphism \( \varphi \) satisfies the compatibility condition (33) if and only if \( \pi: G \to \bar{G} \) splits over the support \( T \) of \( D \) and there exists \( \mu_0 \in \mathbb{K}^\times \) such that \( \tilde{B} \) satisfies the symmetry condition (25) where \( \mu_A \), for all \( A \in \bar{G}/\mathcal{T} \), is given by
\[
\mu_A = \begin{cases} 
\mu_0\chi^{-2}(A)\beta(\tau(A)) & \text{if } \bar{y}_0A^2 = \mathcal{T}; \\
\mu_0\chi^{-2}(A) & \text{if } \bar{y}_0A^2 \neq \mathcal{T};
\end{cases}
\]
where we regard \( \chi^2 \) as a character of \( \bar{G}/\mathcal{T} \) (since \( \chi^2 \) is trivial on \( \mathcal{T} \)).
We note that, since $\mu_A \mu_{\tau_0^{-1}A^{-1}} = 1$ for all $A$, the scalar $\mu_0$ satisfies
$$\mu_0^2 = \chi^{-2}(g_0)$$
and hence can take only two values.

To state the classification of $G$-gradings on the Lie algebras $\mathfrak{fsl}(U, \Pi)$, we introduce the model $G$-graded algebras $\mathfrak{A}^{(I)}(G, T, \beta, \tilde{V}, \tilde{W})$ and $\mathfrak{A}^{(II)}(G, H, h, \beta, \tilde{V}, \tilde{B}, g_0)$.

Let $T \subset G$ be a finite subgroup with a nondegenerate alternating bicharacter $\beta$. The Lie algebra $\mathfrak{A}^{(I)}(G, T, \beta, \tilde{V}, \tilde{W})$ is just the commutator subalgebra of the $G$-graded associative algebra $R = \mathfrak{I}(G, T, \beta, \tilde{V}, \tilde{W})$ (introduced before Corollary 4.3). By Theorem 4.6, it is isomorphic to $\mathfrak{fsl}(U, \Pi)$ where $U = \tilde{V}^\ell$, $\Pi = \tilde{W}^\ell$, $\ell^2 = |T|$, and the bilinear form $\Pi \times U \to \mathbb{K}$ is given by (12).

Let $H \subset G$ be a finite elementary 2-subgroup, $h \neq e$ an element of $H$, and $\beta$ a nondegenerate alternating bicharacter on $H/\langle h \rangle$. Fix a character $\chi \in \hat{G}$ with $\chi(h) = -1$. Let $\mathfrak{G} = G/\langle h \rangle$ and $T = H/\langle h \rangle$. The Lie algebra $\mathfrak{A}^{(II)}(G, H, h, \beta, \tilde{V}, \tilde{B}, g_0)$ is the commutator subalgebra of the Lie algebra $R^{(\ell)}$ with a $G$-grading defined by refining the $\mathfrak{G}$-grading as in (31) where $R$ is the $\mathfrak{G}$-graded associative algebra $\mathfrak{I}(G, T, \beta, \tilde{V}, \tilde{B}, g_0)$ with antiautomorphism $\varphi$ (introduced before Theorem 4.8) and the bilinear form $\tilde{B}$ on $\tilde{V}$ satisfies the symmetry condition (25) with $\mu_A$ given by (34) for some $\mu_0 \in \mathbb{K}^\times$. By Theorem 4.8, $\mathfrak{A}^{(II)}(G, H, h, \beta, \tilde{V}, \tilde{B}, g_0)$ is isomorphic to $\mathfrak{fsl}(U)$ where $U = \tilde{V}^\ell$, $\ell^2 = |H|/2$, and the bilinear form $U \times U \to \mathbb{K}$ is given by (28).

The definition of $\mathfrak{A}^{(II)}(G, H, h, \beta, \tilde{V}, \tilde{B}, g_0)$ depends on the choice of $\chi$. However, regardless of this choice, we obtain the same collection of graded algebras as $(\tilde{V}, \tilde{B})$ ranges over all possibilities allowed by the chosen $\chi$. We assume that a choice of $\chi$ is fixed for any element $h \in G$ of order 2.

Now we can state our main result about gradings on special Lie algebras of finitary linear operators on an infinite-dimensional vector space.

**Theorem 5.1.** Let $G$ be an abelian group and let $\mathbb{K}$ be an algebraically closed field, $\text{char } \mathbb{K} \neq 2$. If a special Lie algebra $\mathcal{L}$ of finitary linear operators on an infinite-dimensional vector space over $\mathbb{K}$ is given a $G$-grading, then $\mathcal{L}$ is isomorphic as a graded algebra to some $\mathfrak{A}^{(I)}(G, T, \beta, \tilde{V}, \tilde{W})$ or $\mathfrak{A}^{(II)}(G, H, h, \beta, \tilde{V}, \tilde{B}, g_0)$. No $G$-graded Lie algebra with superscript (I) is isomorphic to one with superscript (II). Moreover,

- $\mathfrak{A}^{(I)}(G, T, \beta, \tilde{V}, \tilde{W})$ and $\mathfrak{A}^{(I)}(G, T', \beta', \tilde{V}', \tilde{W}')$ are isomorphic if and only if $T' = T$ and either $\beta' = \beta$ and $(\tilde{V}', \tilde{W}') \sim (\tilde{V}, \tilde{W})$, or $\beta' = \beta^{-1}$ and $(\tilde{V}', \tilde{W}') \sim (\tilde{W}', \tilde{V})$ as in Definition 4.5.
- $\mathfrak{A}^{(II)}(G, H, h, \beta, \tilde{V}, \tilde{B}, g_0)$ and $\mathfrak{A}^{(II)}(G, H', h', \beta', \tilde{V}', \tilde{B}', g_0)$ are isomorphic if and only if $H' = H$, $h' = h$, $\beta' = \beta$, and $(\tilde{V}', \tilde{B}', g_0) \sim (\tilde{V}, \tilde{B}, g_0)$ as in Definition 4.7.

**5.1.1. Graded bases.** The Lie algebra $\mathcal{L} = \mathfrak{A}^{(I)}(G, T, \beta, \tilde{V}, \tilde{W})$, is the commutator subalgebra of $R = \mathfrak{I}(G, T, \beta, \tilde{V}, \tilde{W})$. As a vector space over $\mathbb{K}$, this latter can be written as $\tilde{V} \otimes D \otimes \tilde{W}$ — see Subsection 4.6. Recall that, for $A, A' \in G/T$, we have $(\tilde{W}_A, \tilde{V}_{A'}) = 0$ unless $A' = A^{-1}$, and $\tilde{W}_{A^{-1}}$ is a total subspace of $\tilde{V}_A^\ast$. The action of the tensor $v \otimes d \otimes w \in \tilde{V}_A \otimes D \otimes \tilde{W}_{(A')^{-1}}$...
on \( u \otimes d' \in \bar{V}_{A'} \otimes D \) is given by \((v \otimes d \otimes w)(u \otimes d') = v \otimes d(w, u)d' = (w, u)(v \otimes (dd'))\), which is zero unless \( A' = A' \). Also, the degree of the element \( v \otimes d \otimes w \) in the \( G \)-grading equals \( \gamma(A)(\deg d)\gamma(A')^{-1} \). Computing the commutators of such elements and using our standard notation \( X_t \) for the basis elements of \( D \), we find that \( \mathcal{L} \) is spanned by the elements of the form \( v \otimes X_t \otimes w \) where \((w, v) = 0 \) or \( t = e \). Note that since \( \bar{W} \) is a total subspace of \( \bar{V}^* \), there exist \( A_0 \in \bar{G}/\bar{T}, v_0 \in \bar{V}_{A_0} \) and \( w_0 \in \bar{W}_{A_0} \) such that \((w_0, v_0) = 1 \). Then \( K(v_0 \otimes 1 \otimes w_0) \oplus \mathcal{L} = R \). Given bases for the vector spaces \( \bar{V}_A \) and \( \bar{W}_A \) for all \( A \in \bar{G}/\bar{T} \) such that the basis for \( \bar{V}_{A_0} \) includes \( v_0 \) and the basis for \( \bar{W}_{A_0} \) includes \( w_0 \), we have to start with the basis for \( \bar{V}_A \) and \( \bar{W}_A \) of the form \( v \otimes 1 \otimes w \) — and we have identified \( \phi \) the notation of Subsection 4.9 — in particular, the \( \bar{\phi} \)-sesquilinear form \( \bar{\phi} \) given by (21) and (23) — and we have identified \( w \in \bar{V} \) with \( f_w \in W \) given by \( f_w(u) = B(w, u) \) for all \( u \in \bar{V} \). The degree of the element \( v \otimes d \otimes w \) in the \( \bar{G} \)-grading equals \( \gamma(A)(\deg d)\bar{g}_0\gamma(\bar{g}^{-1}_0(A')^{-1}) = \gamma(A)(\deg d)\tau(A')\gamma(A')^{-1} \) where, as before, \( \tau(A') = \bar{g}_0 \gamma(A')^2 \) if \( \bar{g}_0 \gamma(A')^2 = \bar{T} \), and we have set \( \tau(A') = \tau \) if \( \bar{g}_0 \gamma(A')^2 \neq \bar{T} \). Also, (18) yields \( \varphi(v \otimes d \otimes w) = Q^{-1}w \otimes \varphi_0(d)v \). Taking \( d = X_t \) and recalling that \( \varphi_0(X_t) = \beta(t)X_t \), we obtain

\[
\varphi(v \otimes X_t \otimes w) = \beta(t)(Q^{-1}(w) \otimes X_t \otimes v).
\]

With our fixed character \( \chi : G \to \bar{K}^* \) satisfying \( \chi(h) = -1 \), the \( G \)-grading on the vector space \( R \) is given by (31). Taking into account that \( \varphi^2(r) = \chi(g)^2 r \) for any \( r \in R_\bar{\gamma} \), we can write

\[
r = \frac{1}{2} \left( r - \frac{1}{\chi(g)}\varphi(r) \right) + \frac{1}{2} \left( r + \frac{1}{\chi(g)}\varphi(r) \right)
\]

where

\[
\frac{1}{2} \left( r - \frac{1}{\chi(g)}\varphi(r) \right) \in R_g \quad \text{and} \quad \frac{1}{2} \left( r + \frac{1}{\chi(g)}\varphi(r) \right) \in R_{gh}.
\]
Since \( \chi(gh) = -\chi(g) \), the second expression is identical to the first one if \( g \) is replaced by \( gh \). Substituting \( r = v \otimes X_t \otimes w \) and the above expression for \( \varphi(v \otimes X_t \otimes w) \), we find that \( R_g \) is spanned by the elements

\[
v \otimes X_t \otimes w - \frac{\beta(t)}{\chi(g)}Q^{-1}w \otimes X_t \otimes v,
\]

where \( v \in \tilde{V}_A, w \in \tilde{V}_{\gamma^{-1}(A')}^{-1}, \) and \( \gamma(A)t\tau(A')\gamma(A')^{-1} = \gamma \). The eigenvalues of \( Q \) are given by (26), and \( \mu_{\gamma^{-1}(A')^{-1}} = \mu_A^{-1} \), so we can rewrite the above spanning elements as follows:

\[
v \otimes X_t \otimes w - \mu_A(\beta(t))\beta(\tau(A'))\chi^{-1}(g)(w \otimes X_t \otimes v).
\]

Finally, recalling that \( \mu_A' \) is determined in Proposition 5.1 and \( A' = \gamma^{-1}A \), we conclude that \( R_g \) is spanned by the elements

\[
E_{g,v,w}^{(I)} = v \otimes X_t \otimes w - \frac{\mu_0(\beta(t))\chi(g)}{\chi(A)}(w \otimes X_t \otimes v)
\]

where \( v \) ranges over a basis of \( \tilde{V}_A \), \( w \) ranges over a basis of \( \tilde{V}_{\gamma^{-1}A^{-1}} \), and \( A \) ranges over \( \overline{G/T} = G/H \), while \( t = \gamma(A)^{-1}\gamma(\gamma^{-1}A)\tau(\gamma^{-1}A) \). If we discard zeros among the elements \( E_{g,v,w}^{(I)} \), the remaining ones form a basis for \( R_g \). This is also a basis for \( L_g \) unless \( g = e \) or \( g = h \). For these two cases, bases can be obtained using the same idea as for Type I, namely, subtracting a suitable scalar multiple of \( E_{g,v,w}^{(I)} \), where \( v_0 \in \tilde{V}_{A_0} \) and \( w_0 \in \tilde{V}_{\gamma^{-1}A^{-1}} \) are such that \( \tilde{B}(w_0,v_0) = 1 \). Specifically, for \( g = e \) and \( g = h \), we replace the elements (35) by

\[
E_{g,v,w}^{(I)} = v \otimes X_t \otimes w - \beta(t_0)\beta(t)\tilde{B}(w,v)E_{g,v,w}^{(I)}
\]

where \( t = \tau(A) \) and \( t_0 = \tau(A_0) \).

5.1.2. Countable case. In the case \( L = \mathfrak{sl}(\infty) \), we can express the classification of \( G \)-gradings in combinatorial terms. Here \( R = M_{\infty}(\mathbb{K}) \), whose \( G \)-gradings are classified in Corollary 4.4. Namely, \( R \) is isomorphic to \( \mathfrak{g}(G,D,\kappa) \), which we can write as \( \mathfrak{g}(G,T,\beta,\kappa) \) because \( G \) is abelian. Recall that the function \( \kappa: G/T \rightarrow \{0,1,2,\ldots,\infty\} \) has a finite or countable support; here \( |\kappa| = \sum_{A \in G/T} \kappa(A) \) must be infinite. The \( G \)-grading on \( \mathfrak{g}(G,T,\beta,\kappa) \) restricts to \( L \), and we denote the resulting \( G \)-graded Lie algebra by \( \mathfrak{A}(I)(G,T,\beta,\kappa) \). In this way we obtain all Type I gradings on \( L \).

For Type II, we can use Corollary 4.5, with \( G \) replaced by \( \overline{G} = G/\langle h \rangle \) and with the function \( \mu: A \mapsto \mu_A (A \in \overline{G/T}) \) satisfying (34). With the other parameters fixed, there are at most two such functions. Indeed, if there is \( A_0 \in \overline{G/T} \) such that \( \tilde{g}_0A_0 = \overline{T} \) and \( \kappa(A_0) \) is finite and odd (forcing \( \mu_{A_0} = 1 \)), then there is at most one admissible function \( \mu \), defined by (34) with \( \mu_0 = \chi^2(A_0)(\beta(\tau(A_0))) \). Such function exists if and only if all \( A_0 \) of this kind produce the same value \( \mu_0 \). If there is no \( A_0 \) of this kind, then there are exactly two admissible functions \( \mu \), defined by (34) where \( \mu_0 \) satisfies \( \mu_0^2 = \chi^{-2}(\tilde{g}_0) \). Denote by \( \mathfrak{A}(II)(G,H,h,\beta,\kappa,\mu_0,\tilde{g}_0) \) the \( G \)-graded Lie algebra obtained by restricting to \( L \) the refinement (31) of the \( \overline{G} \)-grading on \( \mathfrak{g}(G,T,\beta,\kappa,\mu,\tilde{g}_0) \).

Recall that \( \kappa^g \) is defined by \( \kappa^g(gA) = \kappa(A) \) for all \( A \in G/T \). Set \( \kappa(A) = \kappa(A^{-1}) \).
Corollary 5.1. Let $\mathbb{K}$ be an algebraically closed field, $\text{char } \mathbb{K} \neq 2$, $G$ an abelian group. Suppose $\mathcal{L} = \mathfrak{s}(\infty)$ over $\mathbb{K}$ is given a $G$-grading. Then, as a graded algebra, $\mathcal{L}$ is isomorphic to one of $\mathfrak{A}^{(1)}(G, T, \beta, \kappa)$ or $\mathfrak{A}^{(II)}(G, H, h, \beta, \kappa, \mu_0, \bar{g}_0)$. No $G$-graded Lie algebra with superscript (I) is isomorphic to one with superscript (II). Moreover, 

- $\mathfrak{A}^{(1)}(G, T, \beta, \kappa) \cong \mathfrak{A}^{(1)}(G, T', \beta', \kappa')$ if and only if $T' = T$ and either $\beta' = \beta$ and $\kappa' = \kappa^g$ for some $g \in G$, or $\beta' = \beta^{-1}$ and $\kappa' = \kappa^g$ for some $g \in G$.

- $\mathfrak{A}^{(II)}(G, H, h, \beta, \kappa, \mu_0, \bar{g}_0) \cong \mathfrak{A}^{(II)}(G, H', h', \beta', \kappa', \mu'_0, \bar{g}'_0)$ if and only if $H' = H$, $h' = h$, $\beta' = \beta$, and there exists $g \in G$ such that $\kappa' = \kappa^g$, $\mu'_0 = \mu_0 \chi^2(g)$ and $\bar{g}'_0 = \bar{g}_0 \delta^{-2}$. □

5.2. Orthogonal and symplectic Lie algebras. In the case $\mathcal{L} = \mathfrak{fso}(U, \Phi)$ or $\mathcal{L} = \mathfrak{fsp}(U, \Phi)$, we deal with simple Lie algebras of skew-symmetric elements in the associative algebra $R = \mathfrak{fr}(U)$ with respect to the involution $\varphi$ determined by the nondegenerate form $\Phi$, which is either orthogonal or symplectic. Here $\Pi$ is identified with $U$ by virtue of $\Phi$. We continue to assume that $U$ is infinite-dimensional. Suppose that $\mathcal{L}$ is given a $G$-grading. Applying Theorem 3.5 to the corresponding Lie homomorphism $\rho: \mathcal{L} \otimes \mathcal{H} \to \mathcal{L} \otimes \mathcal{H}$ and observing that $\tau = 0$ (because $\mathcal{L} = [\mathcal{L}, \mathcal{L}]$), we conclude that $\rho$ extends to a homomorphism of associative algebras $\rho': R \otimes \mathcal{H} \to R \otimes \mathcal{H}$. Since $\mathcal{L}$ generates $R$, it follows that $\rho'$ is a comodule structure. This gives $R$ a $G$-grading that restricts to the given grading on $\mathcal{L}$, i.e., $\mathcal{L}_g = R_g \cap \mathcal{L}$ for all $g \in G$. Moreover, since $\varphi$ restricts to the negative identity on $\mathcal{L}$, the restriction of the map $\varphi \otimes \text{id}_\mathcal{H}$ to $\mathcal{L} \otimes \mathcal{H}$ commutes with $\rho$, which implies that $\varphi \otimes \text{id}_\mathcal{H}$ commutes with $\rho'$. This means that $\varphi$ is an involution of $R$ as a $G$-graded algebra. Theorem 4.8 and Proposition 4.1 classify all the pairs $(R, \varphi)$ up to isomorphism.

To state the classification of $G$-gradings on the Lie algebras $\mathfrak{fso}(U, \Phi)$ and $\mathfrak{fsp}(U, \Phi)$, we introduce the model algebras $\mathfrak{B}(G, T, \beta, \tilde{V}, \tilde{B}, g_0)$ and $\mathfrak{E}(G, T, \beta, \tilde{V}, \tilde{B}, g_0)$.

Let $T \subset G$ be a finite elementary 2-subgroup with a nondegenerate alternating bicharacter $\beta$. Let $\mathcal{L}$ be the Lie algebra of skew-symmetric elements in the $G$-graded associative algebra $R = \mathfrak{fr}(G, T, \beta, \tilde{V}, \tilde{B}, g_0)$ with involution, where the bilinear form $\tilde{B}$ on $\tilde{V}$ satisfies the symmetry condition (25) with $\mu_\Lambda$ given by (29). If $\text{sgn}(\varphi) = 1$, we will denote $\mathcal{L}$ by $\mathfrak{B}(G, T, \beta, \tilde{V}, \tilde{B}, g_0)$. By Theorem 4.8, it is isomorphic to $\mathfrak{fso}(U, \Phi)$ where $U = \tilde{V}'$, $\ell^2 = |T|$, and the bilinear form $\Phi: U \times U \to \mathbb{K}$ is given by (28). If $\text{sgn}(\varphi) = -1$, we will denote $\mathcal{L}$ by $\mathfrak{E}(G, T, \beta, \tilde{V}, \tilde{B}, g_0)$. By Theorem 4.8, it is isomorphic to $\mathfrak{fsp}(U, \Phi)$ where $U$ and $\Phi$ are as above.

Theorem 5.2. Let $G$ be an abelian group and let $\mathbb{K}$ be an algebraically closed field, $\text{char } \mathbb{K} \neq 2$. If an orthogonal or symplectic Lie algebra $\mathcal{L}$ of finitary linear operators on an infinite-dimensional vector space over $\mathbb{K}$ is given a $G$-grading, then $\mathcal{L}$ is isomorphic as a graded algebra to some $\mathfrak{B}(G, T, \beta, \tilde{V}, \tilde{B}, g_0)$ in the orthogonal case or $\mathfrak{E}(G, T, \beta, \tilde{V}, \tilde{B}, g_0)$ in the symplectic case. The $G$-graded algebras with parameters $(T, \beta, \tilde{V}, \tilde{B}, g_0)$ and $(T', \beta', \tilde{V}', \tilde{B}', g'_0)$ are isomorphic if an only if $T' = T$, $\beta' = \beta$, and $(\tilde{V}', \tilde{B}', g'_0) \sim (\tilde{V}, \tilde{B}, g_0)$ as in Definition 4.7.
5.2.1. Graded bases. The calculations which we used to find graded bases in the case of Type II gradings on special linear Lie algebras apply also for orthogonal and symplectic Lie algebras, with the following simplifications: we always deal with the group \( G \) (hence we omit bars and \( \chi \)) and \( Q \) is id in the orthogonal case and \( -\text{id} \) in the symplectic case. Thus, \( \mathfrak{B}(G,T,\beta,\tilde{V},\tilde{B},g_0) \) is the set of skew-symmetric elements in \( \mathfrak{F}(G,D,\tilde{V},\tilde{B},g_0) \), where \( \tilde{B} \) is an “orthosymplectic” form on \( \tilde{V} \) that determines a “Hermitian” form on \( V \), as described in Subsection 4.10. The skew-symmetric elements
\[
 v \otimes X_i \otimes w - \beta(t)(w \otimes X_i \otimes v)
\]
span \( \mathcal{L} = \mathfrak{B}(G,T,\beta,\tilde{V},\tilde{B},g_0) \), so we obtain a basis for \( \mathcal{L}_g \) by letting \( A \) range over \( G/T \), \( v \) over a basis of \( \tilde{V}_A \), and \( w \) over a basis of \( \tilde{V}_{g^{-1}A^{-1}} \), while \( t = g^{-1}A^{-1}g^{-1}A \), and taking the nonzero elements given by (36).

In a similar way, \( \mathfrak{C}(G,T,\beta,\tilde{V},\tilde{B},g_0) \) is the set of skew-symmetric elements in the algebra \( \mathfrak{F}(G,D,\tilde{V},\tilde{B},g_0) \), where \( \tilde{B} \) is an “orthosymplectic” form on \( \tilde{V} \) that determines a “skew-Hermitian” form on \( V \). The skew-symmetric elements
\[
 v \otimes X_i \otimes w + \beta(t)(w \otimes X_i \otimes v)
\]
span \( \mathcal{L} = \mathfrak{C}(G,T,\beta,\tilde{V},\tilde{B},g_0) \), and we obtain a basis for \( \mathcal{L}_g \) as above, but using (37) instead of (36).

5.2.2. Countable case. In the cases \( \mathcal{L} = \mathfrak{so}(\infty) \) and \( \mathcal{L} = \text{Span} \{ \{ \infty \} \} \), we can express the classification of \( G \)-gradings in combinatorial terms. Here we have to deal with the pairs \((R,\varphi)\) where \( R = M_\infty(\mathbb{K}) \) is endowed with a \( G \)-grading and an involution \( \varphi \) that respects this grading. Such pairs are classified in Corollary 4.6. Namely, \((R,\varphi)\) is isomorphic to some \( \mathfrak{F}(G,T,\beta,\kappa,\delta,g_0) \) where \( |\kappa| \) must be infinite and \( \delta = \text{sgn}(\varphi) \).

We denote by \( \mathfrak{B}(G,T,\beta,\kappa,g_0) \) and \( \mathfrak{C}(G,T,\beta,\kappa,g_0) \) the \( G \)-graded Lie algebras of skew-symmetric elements (with respect to \( \varphi \)) in \( \mathfrak{F}(G,T,\beta,\kappa,\delta,g_0) \) where \( \delta = 1 \) and \( \delta = -1 \), respectively.

**Corollary 5.2.** Let \( \mathbb{K} \) be an algebraically closed field, char \( \mathbb{K} \neq 2 \), \( G \) an abelian group. Let \( \mathcal{L} \) be \( \mathfrak{so}(\infty) \), respectively \( \text{Span} \{ \{ \infty \} \} \), over \( \mathbb{K} \). Suppose \( \mathcal{L} \) is given a \( G \)-grading. Then, as a graded algebra, \( \mathcal{L} \) is isomorphic to some \( \mathfrak{B}(G,T,\beta,\kappa,g_0) \), respectively \( \mathfrak{C}(G,T,\beta,\kappa,g_0) \). No \( G \)-graded Lie algebra \( \mathfrak{B}(G,T,\beta,\kappa,g_0) \) is isomorphic to \( \mathfrak{C}(G,T',\beta',\kappa',g'_0) \). Moreover, \( \mathfrak{B}(G,T,\beta,\kappa,g_0) \) and \( \mathfrak{B}(G,T',\beta',\kappa',g'_0) \) are isomorphic if and only if \( T' = T, \beta' = \beta \), and there exists \( g \in G \) such that \( g_0 = g_0g^{-2} \) and \( \kappa' = \kappa^g \). The same holds for \( \mathfrak{C}(G,T,\beta,\kappa,g_0) \) and \( \mathfrak{C}(G,T',\beta',\kappa',g'_0) \).

6. Group gradings on nilpotent Lie algebras

Historically one of the deepest result on the grading of Lie algebras was Cartan’s theorem describing the symmetric decompositions (that is, \( \mathbb{Z}_2 \)-graded) of simple Lie algebras. A notable consequence of this theorem is the classification of compact riemannian symmetric spaces. In the previous sections we summarized recent results on general gradings on simple
Lie algebras. This will permit us to define in the second part new homogeneous reductive non-symmetric spaces associated to these gradings. The gradings on nilpotent Lie algebras have not been so much explored. It is then interesting one the one hand to classify the symmetric decompositions of a nilpotent Lie algebra and on the other hand to consider the most general gradings by finite abelian groups. The geometrical interest is double: one the one hand to construct numerous examples of symmetric nilpotent riemannian and non riemannian spaces and one the other hand to obtain a new approach of the affine geometry on nilpotent Lie groups. In this paper we restrict ourselves to the study of gradings on filiform Lie algebras of positive rank, in the case where the grading group is finitely generated abelian without elements of order \( p \) in case \( \text{char} \, \mathbb{K} = p > 0 \).

6.1. An example of an ad hoc argument. In some cases all gradings, up to isomorphism, can be determined based on the properties of algebras themselves rather than on the general theory of gradings. As an example, let us consider free nilpotent algebras. By definition, each such algebra \( L = L(X) \) is generated by a set of elements \( X \) such that any map of \( X \) to \( L \) extends to a homomorphism of \( L \). Additionally, \( L \) has a finite \( \mathbb{Z} \)-grading \( L = \bigoplus_{k \in \mathbb{Z}} L_k \), where \( L_k \) is the linear span of all monomials of degree \( k \) in \( X \). Any set of generators with the same property as \( X \) is called a free base of \( L \). A well-known example of such an algebra is the 3-dimensional Heisenberg Lie algebra \( H = \langle x, y, z \mid [x, y] = z \rangle \). Here \( X = \{x, y\} \) but actually any set \( Y \) of two elements linearly independent modulo the derived subalgebra \([H, H] = \langle z \rangle \) is also a free base of \( H \).

Now let us assume the base field \( \mathbb{K} \) infinite, the grading group \( G \) abelian and the base \( X = \{x_1, \ldots, x_n\} \) finite (one says that \( L(X) \) has finite rank \( n \)). In this case, it is easy to show, by a Vandermonde’s type argument \([1]\), that the structural \( \mathbb{Z} \)-grading on \( L \) can be refined to a \( \mathbb{Z}^n \)-grading, where given \( \alpha = (i_1, \ldots, i_n) \in \mathbb{Z}^n \), the component \( L_\alpha \) is defined as the linear span of monomials in \( x_1, \ldots, x_n \) whose degree in \( x_k \) equals \( i_k \), for any \( k = 1, \ldots, n \).

Now consider \( \gamma = (g_1, \ldots, g_n) \in G^n \). Given \( \alpha \in \mathbb{Z}^n \), define \( \gamma^\alpha = g_1^{i_1} \cdots g_n^{i_n} \in G \) and set \( L^{(g)} = \bigoplus_{\gamma^\alpha=g} L_\alpha \). Then \( \Gamma : \bigoplus_{g \in G} L^{(g)} \) is a \( G \)-grading of \( L \).

**Proposition 6.1.** Let \( \Gamma : L = \bigoplus_{g \in G} L_g \) be a \( G \)-grading of a free nilpotent algebra \( L = L(X) \) of finite rank over an infinite field \( \mathbb{K} \). If \( G \) is an abelian group then there is \( \gamma \in G^n \) such that \( \Gamma \) is isomorphic to \( \Gamma_{\gamma} \). If also \( \Gamma \) is isomorphic to \( \Gamma_{\delta} \), for some \( \delta \in G^n \), then \( \delta \) can be obtained from \( \gamma \) by permuting its components.

**Proof.** Notice that the subalgebra \( L^2 = \bigoplus_{k \geq 2} L_k \) is \( G \)-graded. Hence we have graded vector space bases \( B \) and \( C \) in \( L \) and \( L^2 \). We have \( \#B = \#C + n \). This shows that \( L \) has a basis \( \{y_1, \ldots, y_n\} \cup C \), where \( \{y_1, \ldots, y_n\} \subset B \). The elements of \( Y = \{y_1, \ldots, y_n\} \) are graded of degrees \( g_1, \ldots, g_n \). By a Nakayama type argument (see, for example, \([1, 1.7.2]\)), since \( L \) is nilpotent, a set generating \( L \) modulo \( L^2 \) is a generating set of \( L \). Therefore, the map \( \varphi : X \to L \) given by \( x_i \mapsto y_i, \ i = 1, \ldots, n \), extends to a homomorphism of \( L \) onto itself, hence to an isomorphism of \( L \). If \( \gamma = (g_1, \ldots, g_n) \), this map is an isomorphism of \( \Gamma_{\gamma} \) and our original map \( \Gamma \).
Since any permutation of the elements of \( X \) leads to an isomorphism of \( L \), it is obvious that \( \Gamma \gamma \) is isomorphic to \( \Gamma \delta \) if \( \delta \) is a permutation of \( \gamma \). Conversely, such an isomorphism leads to the isomorphism of the graded vector space \( L_1 \) onto itself. Now, for each \( g \in G \), \( \dim L^{(g)} = \dim L_\delta^{(g)} \). Clearly then the number of entries of \( g \) to both \( \gamma \) and \( \delta \) is the same, hence \( \delta \) is a permutation of \( \delta \).

6.2. Filiform Lie algebras. Let \( \mathbb{K} \) be a field and \( L \) be a Lie algebra over \( \mathbb{K} \). We denote by \( \{L^k \mid k = 1, 2, \ldots\} \) the lower central series of \( L \) defined by \( L^1 = L \) and \( L^k = [L^{k-1}, L] \), for \( k = 2, 3, \ldots \) One calls \( L \) nilpotent if there is a natural \( n \) such that \( L^{n+1} = \{0\} \). If \( L^n \neq \{0\} \) then \( n \) is called the nilpotent index of \( L \). As just above, a set of elements \( \{x_1, \ldots, x_m\} \subset L \) generates \( L \) if and only if the \( \{x_1 + L^2, \ldots, x_m + L^2\} \) is the spanning set in the vector space \( L/L^2 \). As a result, a nilpotent Lie algebra with \( \dim L/L^2 \leq 1 \) is at most 1-dimensional. If \( n > 1 \) then the nilpotent index of an \( n \)-dimensional nilpotent Lie algebra never exceeds \( n - 1 \).

Definition 6.1. Given a natural number \( n \), an \( n \)-dimensional Lie algebra \( L \) is called \( n \)-dimensional filiform if the nilpotent index of \( L \) is maximal possible, that is, \( n - 1 \). In this case we must have \( \dim L/L^2 = 2 \), \( \dim L^k/L^{k-1} = 1 \) for \( k = 2, 3, \ldots, n - 1 \).

In the situation described in Definition 6.1, the lower central series of \( L \) is “thready”, whence the French name “filiform”. If we choose a basis \( \{e_1, e_2, \ldots, e_{n-2}, e_{n-1}, e_n\} \) of \( L \) so that \( \{e_n\} \) is a basis of \( L^{n-1} \), \( \{e_{n-1}, e_n\} \) is a basis of \( L^{n-2} \), \( \{e_{n-2}, e_{n-1}, e_n\} \) is a basis of \( L^{n-3} \), etc., it is easy to observe that the center of \( L \) is 1-dimensional and equals \( L^{n-1} \).

Thus, the lower central sequence of \( L \) takes the form

\[
L = L^1 \supset L^2 \supset \ldots \supset L^{n-1} = Z(L) \supset \{0\},
\]

where \( Z(L) \) is the center of \( L \) and all containments are proper. In any Lie algebra the lower central series is a filtration in the sense that \([L^i, L^j] \subset L^{i+j}\).

Theorem 6.1 ([31]). Any \( n \)-dimensional filiform \( \mathbb{K} \)-Lie algebra \( L \) admits an adapted basis \( \{X_1, \ldots, X_n\} \), that is, a basis satisfying:

\[
[X_i, X_i] = X_{i+1}, \quad i = 2, \ldots, n - 1,
\]

\[
[X_2, X_3] = \sum_{k \geq 5} \gamma_k X_k,
\]

\[
[X_i, X_{n-i+1}] = (-1)^i+1 \alpha X_n, \text{ where } \alpha = 0 \text{ when } n = 2m + 1,
\]

\[
L^i = \mathbb{K} \{X_{i+1}, \ldots, X_n\} \text{ for all } i \geq 2.
\]

Now let us consider a collection of vector spaces \( W_i = L^i/L^{i+1}, \quad i = 1, 2, \ldots n - 1 \). The vector space direct sum \( \text{gr } L = \bigoplus_{i=1}^{n-1} W_i \) becomes a Lie algebra if one defines the bracket of the elements by setting \([X + L^{i+1}, Y + L^{j+1}] = [X, Y] + L^{i+j+1}, \text{ for } X \in L^i, Y \in L^j, 1 \leq i, j \leq n - 1 \). It follows from the above theorem that all the associated graded algebras for filiform Lie algebras are again filiform. They belong to one of the two types as follows.
Note for the future, that when we define a Lie algebra by writing a list of commutators $[X_i, X_j]$ of the elements of the basis, we always mean $[X_k, X_l] = 0$ for $(k, l)$ not on the list, except, naturally, that $[X_j, X_i] = -[X_i, X_j]$.

$L_n$: Each of these algebras has an adapted basis $\{X_1, X_2, \ldots, X_n\}$ such that $[X_1, X_i] = X_{i+1}$, for $i = 2, 3, \ldots, n - 1$.

$Q_n$: Here $n = 2m$. Each of these algebras has an adapted basis $\{X_1, X_2, \ldots, X_n\}$ such that $[X_1, X_i] = X_{i+1}$, if $i = 2, 3, \ldots, n - 1$, and $[X_j, X_{2m-j+1}] = (-1)^{j+1}X_{2m}$, if $2 \leq j \leq m$.

If $\text{gr} \ L \cong L$ then we call $L$ naturally graded (by the group $\mathbb{Z}$). So there are only two types of naturally graded algebras: $L_n$ and $Q_n$.

**Corollary 6.1.** [31] Any naturally graded filiform Lie algebra is isomorphic to

- $L_n$ if $n$ is odd,
- $L_n$ or $Q_n$ if $n$ is even.

We deduce that $L_n$ and $Q_n$ admits a $\mathbb{Z}$-grading with support $\{1, 2, \ldots, n\}$. From what follows, we will see the existence of other non-isomorphic $\mathbb{Z}$-gradings and we will determine the filiform Lie algebras admitting such gradings.

### 6.3. The automorphisms group of a filiform Lie algebra.

A special feature of filiform Lie algebras is the following.

**Theorem 6.2.** Let $L$ be a $n$-dimensional filiform $\mathbb{K}$-Lie algebra with $n \geq 4$. Then the group $\text{Aut} \ L$ is a solvable algebraic group, of toral rank at most 2.

**Proof.** Assume $\sigma \in \text{Aut} \ L$ and let $\{X_1, \ldots, X_n\}$ be an adapted basis of $L$. We know that in any algebra $L$, $\sigma(L^i) = L^i$ and so for any $i \geq 3$, $\sigma(X_i) = \lambda_i X_i + U_i$, where $\lambda_i \neq 0$ and $U_i \in L^i$.

We also know that $[X_1, X_i] = X_{i+1}$ and $[X_2, X_3] = \nu_3 X_5 + V_i$ where $V_i \in L^5$. Assume that $\sigma(X_2) = \mu_1 X_1 + \mu_2 X_2 + Y$, where $Y \in L^2$. Then

$$\sigma(X_2), \sigma(X_3)] = [\mu_1 X_1 + \mu_2 X_2 + Y, \lambda_3 X_3 + U_3]$$

$$= \mu_1 \lambda_3 X_4 + \mu_1 [X_1, U_3] + \mu_2 \lambda_3(\nu_3 X_5 + V_i) + \mu_2 [X_1, U_3] + \lambda_3[Y, X_3] + [Y, U_3].$$

Because $[L^i, L^j] \subset L^{i+j}$, all $[X_1, U_3]$, $[X_1, U_3]$, $[Y, X_3]$, $[Y, U_3]$ are in $L^4$. So $[\sigma(X_2), \sigma(X_3)] = \mu_1 \lambda_3 X_4 + Z$, where $Z \in L^4$. On the other hand,

$$\sigma([X_2, X_3]) = \sigma(\nu_3 X_5 + V_i) \in L^4.$$ 

Thus $\mu_1 \lambda_3 = 0$. Since dim $L \geq 4$, we have $\lambda_3 \neq 0$, and then $\mu_1 = 0$. If we set, for each $i = 1, \ldots, n$, $F_i = \text{Span} \{X_i, \ldots, X_n\}$, then

$$\mathcal{F} : L = F_1 \supset F_2 \supset \cdots \supset F_n \supset \{0\}$$

is a flag of subspaces in $L$. It is well-known that the set of all automorphisms of a linear space respecting a flag is a solvable subgroup of $\text{GL}(L)$. Now we have just proved that $\text{Aut} \ L$ respects $\mathcal{F}$; hence $\text{Aut} \ L$ is a solvable group, as claimed.
Finally, the matrices of the elements of Aut $L$ with respect to an adapted basis are triangular. Let $T_n$ be the group of all triangular matrices, $U_n$ the subgroup of unitriangular matrices, $D_n$ is the subgroup of the diagonal matrices and $C_n = \{ \text{diag}(t, s, ts, \ldots, t^{n-2}s) \mid s, t \in \mathbb{K}\setminus\{0\} \}$. Then $G/G \cap U_n \cong GU_n/U_n < C_n U_n/U_n \cong C_n$. Since $C_n$ is a 2-dimensional torus, the (toral) rank of $G$ does not exceed 2.

**Remark** If $\dim L = 2$ or 3, then Aut $L$ contains a subgroup isomorphic to $GL(2, \mathbb{K})$, hence not solvable.

6.4. Filiform Lie algebras of rank 1 or 2. In this section we list filiform Lie algebras of nonzero rank. It is proven in [21] that over $\mathbb{K} = \mathbb{C}$ every filiform algebra of nonzero rank is isomorphic to one of the algebras on the list.

Let $L$ be a $n$-dimensional filiform $\mathbb{K}$-Lie algebra whose rank $r(L)$ is not 0. Thus $r(L) = 2$ or 1 and

1. If $r(L) = 2$, $L$ is isomorphic to
   - (a) $L_n$, $n \geq 3$
     \[ [X_1, X_i] = X_{i+1}, \quad 1 \leq i \leq n - 1 \]
   - (b) $Q_n$, $n = 2m$, $m \geq 3$
     \[ [Y_1, Y_i] = Y_{i+1}, \quad 1 \leq i \leq n - 2, \]
     \[ [Y_i, Y_{n-i+1}] = (-1)^{i+1}Y_n, \quad 2 \leq 1 \leq m \]

2. If $r(L) = 1$, $L$ is isomorphic to
   - (a) $A^p_n(\alpha_1, \ldots, \alpha_t)$, $n \geq 4$, $t = \lfloor \frac{n-p}{2} \rfloor$, $1 \leq p \leq n - 4$
     \[ [X_1, X_i] = X_{i+1}, \quad 1 \leq i \leq n - 1, \]
     \[ [X_1, X_{i+1}] = \alpha_{i-1}X_{2i+p}, \quad 2 \leq i \leq t, \]
     \[ [X_1, X_j] = a_{i-1,j-1}X_{i+j+p-1}, \quad 2 \leq i < j, \quad i + j \leq n - p + 1 \]
   - (b) $B^p_n(\alpha_1, \ldots, \alpha_t)$, $n = 2m$, $m \geq 3$, $1 \leq p \leq n - 5$, $t = \lfloor \frac{n-p-3}{2} \rfloor$
     \[ [Y_1, Y_i] = Y_{i+1}, \quad 1 \leq i \leq n - 2, \]
     \[ [Y_i, Y_{n-i+1}] = (-1)^{i+1}Y_n, \quad 2 \leq i \leq m, \]
     \[ [Y_i, Y_{i+1}] = \alpha_{i-1}Y_{2i+p}, \quad 2 \leq i \leq t + 1, \]
     \[ [Y_i, Y_j] = a_{i-1,j-1}Y_{i+j+p-1}, \quad 2 \leq i < j, \quad i + j \leq n - p \]

In both cases
\[
\begin{align*}
  a_{i,i} &= 0, \\
  a_{i,i+1} &= \alpha_i, \\
  a_{i,j} &= a_{i+1,j} + a_{i,j+1}.
\end{align*}
\]

where $\{X_1, \ldots, X_n\}$ is an adapted basis and $\{Y_1, \ldots, Y_n\}$ a quasi-adapted basis.

Note that if $L$ is of the type $A^p_n$ then $\text{gr } L$ is of the type $L_n$. If $L$ is of the type $B^p_n$ then $\text{gr } L$ is of the type $Q_n$.

**Remarks.**
• In this proposition, the basis used to define the brackets is not always an adapted basis. More precisely, it is adapted for Lie algebras $L_n$ and $A^k_n$, and it is not adapted for Lie algebras $Q_n$ and $B^k_n$. But if $\{Y_1, \ldots, Y_n\}$ is a quasi-adapted basis, that is a basis which diagonalizes the semi-simple derivations, then the basis $\{X_1 = Y_1 - Y_2, X_2 = Y_2, \ldots, X_n = Y_n\}$ is adapted.

• In [21], the proof of the main result is given for $\mathbb{K} = \mathbb{C}$. Without any further restrictions, we can extend this result to arbitrary fields $\mathbb{K}$ which are algebraically closed and of characteristic 0. Since the proof is based on a simultaneous reduction of a semi-simple endomorphism and a nilpotent adjoint operator, this result can be extended to any field $\mathbb{K}$ which contains the eigenvalues of the semi-simple endomorphism. Thus the result is true over any algebraically closed field.

• In the statement of the above result in [21], there is a third family denoted $C_n$. But all the Lie algebras of this family are isomorphic to $Q_n$. This error was noticed after the publication of the paper.

6.5. Standard gradings of filiform Lie algebras of nonzero rank. In each of the four types of filiform algebras introduced in the previous section there are standard gradings, as follows.

(1) If $L = L_n$ then $L = \bigoplus_{(a,b) \in \mathbb{Z}^2} L_{(a,b)}$ where $L_{(a,b)} = \{0\}$ except $L_{(1,0)} = \langle X_1 \rangle$, $L_{(s-2,1)} = \langle X_s \rangle$, for all $s = 2, \ldots, n$.

(2) If $L = Q_n$ then $L = \bigoplus_{(a,b) \in \mathbb{Z}^2} L_{(a,b)}$ where $L_{(a,b)} = \{0\}$ except $L_{(1,0)} = \langle (X_1 + X_2) \rangle$, $L_{(s-2,1)} = \langle X_s \rangle$, for $s = 2, \ldots, n-1$, $L_{(n-3,2)} = \langle X_n \rangle$.

(3) If $L = A^n_r$ then $L = \bigoplus_{a \in \mathbb{Z}} L_a$ where $L_a = \{0\}$ except $L_1 = \langle X_1 \rangle$, $L_{s+p-1} = \langle X_s \rangle$, for $s = 2, \ldots, n$.

(4) If $L = B^n_r$ then $L = \bigoplus_{a \in \mathbb{Z}} L_a$ where $L_a = \{0\}$ except $L_1 = \langle (X_1 + X_2) \rangle$, $L_{s+p-1} = \langle X_s \rangle$, for $s = 2, \ldots, n-1$, $L_{n+2p-1} = \langle X_n \rangle$

where $\{X_1, \ldots, X_n\}$ is an adapted basis. Our main result says the following.

Theorem 6.3. Let $L$ be a finite-dimensional filiform $\mathbb{K}$-algebra of nonzero rank $r$ over an algebraically closed field $\mathbb{K}$ of characteristic 0. If $G$ is an abelian finitely generated group, then any $G$-grading of $L$ is a coarsening of a standard grading by $\mathbb{Z}^r$. The same result is true for the filiform Lie algebras of the types $L, Q, A, B$ over an algebraically closed field of characteristic $p > 0$, provided $G$ has no elements of order $p$.

Proof. Each of the four cases above gives rise to a maximal torus $D$ in $\text{Aut} L$. To describe $D$ we only need to indicate the action of an element of $D$ on the generators of $L$, which will be $X_1$ and $X_2$ in the cases of $L_n$ and $A_n$ or $X_1 + X_2$ and $X_2$ in the cases of $Q_n$ and $B_n$.

(1) If $L = L_n$ then $D = \{\varphi_{u,t} \mid u, t \in \mathbb{K}^\times\}$ where $\varphi_{u,t}(X_1) = uX_1$, $\varphi_{u,t}(X_2) = tX_2$.

(2) If $L = Q_n$ then $D = \{\varphi_{u,t} \mid u, t \in \mathbb{K}^\times\}$ where $\varphi_{u,t}(X_1 + X_2) = u(X_1 + X_2)$, $\varphi_{u,t}(X_2) = tX_2$.

(3) If $L = A^n_r$ then $D = \{\varphi_u \mid u \in \mathbb{K}^\times\}$ where $\varphi_u(X_1) = uX_1$, $\varphi_u(X_2) = u^{p+1}X_2$.
(4) If $L = B_n^p$, then $D = \{ \varphi_u \mid u \in \mathbb{K}^n \}$ where $\varphi_u(X_1 + X_2) = u(X_1 + X_2)$, $\varphi_u(X_2) = u^{p+1}X_2$.

**Lemma 6.1.** In each of the four cases in Theorem 6.3, the centralizer of $D$ is equal to $D$.

**Proof.** If $L$ is of the type $L_n$, then we have to determine all $\varphi \in \text{Aut} \ L$ such that $\varphi_{u,t}\varphi = \varphi\varphi_{v,s}$, for all $u, t, v, s \in \mathbb{K}^n$. Notice that $\varphi_{u,t}(X_1) = uX_1$ and $\varphi_{u,t}(X_i) = u^{-2}tX_i$, for all $i \geq 2$. Now let $\varphi(X_i) = \sum_{j \geq 1} a_{ji} X_j$, for $i = 1, 2, \ldots, n$. Then $\varphi_{u,t}(X_i) = a_{11} uX_1 + \sum_{j \geq 2} a_{ji} u^{-2}tX_j$ whereas $\varphi_{v,s}(X_1) = \sum_{j \geq 1} a_{ij} vX_j$. Also, $\varphi_{u,t}(X_i) = \sum_{j \geq 1} a_{ij} u^{-2}tX_j$ whereas $\varphi_{v,s}(X_i) = \sum_{j \geq 2} a_{ij} v^{-2}sX_j$, if $i \geq 2$. Thus we have $a_{11} u = a_{11} v$ and since $\varphi$ is a triangular automorphism, $a_{11} \neq 0$, hence $v = u$. Similarly, comparing $\varphi_{u,t}\varphi(X_2)$ and $\varphi\varphi_{v,s}(X_2)$, we obtain $s = t$. Thus the normalizer of $D$ is equal to its centralizer. Now we have $a_{ ji} (u^{-2}t - u) = 0$, for all $j \neq 1$ and $a_{ ji} (u^{-2}t - u^{-2}t) = 0$, for all $j \neq i$. Here $u, t$ are arbitrary elements of an infinite set $\mathbb{K}^n$. It follows that all $a_{ ji}$ are zero as soon as $j \neq i$. Notice that in any case, it follows from $[X_1, X_i] = X_{i+1}$ for $i \geq 2$, that $a_{ii} = a_{i1}^{-1}a_{22}$. As a result, $\varphi = \varphi_{a_{11},a_{22}}$, proving that the normalizer of $D$ is indeed, $D$ itself.

Now assume $L$ is of the type $Q_n$. Then, instead of comparing the values of the sides of $\varphi_{u,t}\varphi = \varphi\varphi_{v,s}$ at $X_1, X_2, \ldots, X_n$ we can compare them on the quasi-adapted basis $X_1 + X_2, X_2, X_2, \ldots, X_n$. Then we obtain $a_{ji} = 0$, for $j \neq i$, $i \geq 2$. At the same time,

$$
\varphi(X_1 + X_2) = a_{11}X_1 + \sum_{j \geq 2} (a_{1j} + a_{j2})X_j = a_{11}(X_1 + X_2) + (a_{21} + a_{22} + a_{11})X_2 + \sum_{i \geq 3} (a_{ij} - a_{j2})X_j.
$$

Applying the same argument, as before, we obtain $a_{j1} = a_{j2} = 0$, for $j \geq 3$ and $a_{21} = -a_{11} - a_{22}$. Hence $\varphi(X_1 + X_2) = a_{11}(X_1 + X_2)$. Thus, $\varphi = \varphi_{a_{11},a_{22}}$, as previously.

Next, assume $L$ is of one of the types $A_n^p$, where $p \geq 1$. In this case we can repeat the argument of the case $L_n$, bearing in mind that $t = u^{p+1}$. Then we will have equations $a_{j1}(u^{-2}u^{p+1} - u) = 0$, or $a_{j1}(u^{p-2} - 1) = 0$, for all $j \neq 1$ and $a_{ji}(u^{-2}u^{p+1} - u^{-2}u^{p+1}) = 0$, or $a_{j1}(u^{i} - u^{i}) = 0$ for all $j \neq i$ and all $u \in \mathbb{K}^n$. Since $j + p - 2 \neq 0$, we again can make the same conclusion $a_{ji} = 0$, for all $j \neq i$.

The case $B_n^p$, where $k \geq 1$, is reduced to the case $Q_n$ in the same manner as $A_n^p$ to $L_n$.

Thus the proof of our lemma is complete. \qed

To complete the proof of Theorem 6.3, we need only to refer to Theorem 2.3 from the Introduction.

**REFERENCES**


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