Habilitation à Diriger les Recherches

Non-associative algebras and Operads

Deformations of algebras over non-Koszul operads

the case of the 3-associative algebras

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UHA-LMIA

1. Non-associative algebras

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Notation. Let (V,μ) be an algebra over a field $\mathbb{K}$. We denote by A_μ its associator

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Theorem. Any invariant subspace of $\mathbb{K}[\Sigma_3]$ under the action of Σ_3 is a Σ_3-module of rank 1.

Consequence. Every invariant subspace of $\mathbb{K}[\Sigma_3]$ determines a symmetric non-associative relation

$$A_\mu \circ \Phi_v = 0,$$

where

$$\Phi_\sigma(x_1 \otimes x_2 \otimes x_3) = x_{\sigma^{-1}(1)} \otimes x_{\sigma^{-1}(2)} \otimes x_{\sigma^{-1}(3)}$$

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   Let us note that the invariant space associated with v_{Σ_3} and w_{Σ_3} are the only irreducible invariant spaces of dimension 1.

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(b) If $G =< \tau_{23} >$, v_G gives Vinberg Algebras, w_G gives right alternative algebras.

(c) If $G =< (123) >$, $v_G = w_G$ gives flexible Lie-admissible algebras.

2. Poisson algebras as Non-Associative algebras

Poisson algebra $(\mathcal{P}, \bullet, \{,\})$

$\bullet$ commutative associative,

$\{,\}$ anti-commutative satisfying the Jacobi identity,

distributive law:

$$\{x, y \bullet z\} = \{x, y\} \bullet z + y \bullet \{x, z\}.$$

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Polarization. Depolarization

polarization : decompose any multiplication

$$\cdot : V \otimes V \to V$$

into the sum of a commutative multiplication $\bullet$ and an anti-commutative one $[-,-]$:

$$x \bullet y := \frac{1}{\sqrt{2}}(x \cdot y + y \cdot x) \quad \text{and} \quad [x, y] := \frac{1}{\sqrt{2}}(x \cdot y - y \cdot x) \text{ for } \quad x, y \in V.$$

Inverse process: *depolarization* assembles a multiplication $\bullet$ with a multiplication $[-,-]$ into

$$x \cdot y := \frac{1}{\sqrt{2}}(x \bullet y + [x,y]), \quad \text{for} \quad x, y \in V.$$

Theorem.

The Poisson identities are equivalent to the following Non-Associative identity

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z - \frac{1}{3}\{(x \cdot z) \cdot y + (y \cdot z) \cdot x - (y \cdot x) \cdot z - (z \cdot x) \cdot y)\}$$

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3. If a Poisson-admissible algebra $\mathcal{P}$ has a non-zero idempotent, it admits the Pierce decomposition

$$\mathcal{P} = \mathcal{P}_{0,0} \oplus \mathcal{P}_{1,1},$$

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4. Classification of 2 and 3-dimensional Poisson algebras.

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7. Study of the deformations of the symmetric algebra $S(\mathfrak{g})$ of a rigid Lie algebra.

3. Operads and deformations

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The notion of operads appeared in algebraic topology in the '70. It gives a sense to the notion of type of algebras. For example, there are an operad $\mathcal{A}ss$ for associative algebras or $\mathcal{L}ie$ for Lie algebras.

Operad. A collection $\{\mathcal{P}(n)\}_{n\geq 1}$ of Σ_n-modules,
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$\mathcal{P}$-algebra on V $\quad\longleftrightarrow\quad$ $f(n) : \mathcal{P}(n) \otimes V^{\otimes n} \to V$

Operad $\mathcal{E}nd(V)$: $\mathcal{E}nd(V)(n) = Hom(V^{\otimes n}, V)$

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$\Gamma(E)$ Free Operad generated by a $\mathbb{K}[\Sigma_2]$-module $E = \mathbb{K}[\Sigma_2]$

$$E = \mathbb{K}[\Sigma_2] \simeq \{x_1x_2, x_2x_1\} = \Gamma(E)(2)$$

$$\Gamma(E)(3) = \Big\{(x_ix_j)x_k, x_i(x_jx_k), \{i, j, k\} = \{1, 2, 3\}\Big\}.$$

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$E^\vee = Sgn_2 \otimes E^\sharp$

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For associative algebras:
standard cohomology = deformation cohomo. = Hochschild cohomo.

For associative commutative algebras:
standard cohomology = deformation cohomo. = Harrison cohomo.

For Lie algebras:
standard cohomology = deformation cohomo. = Chevalley-Eilenberg cohomo.

4. Operads and deformations of some binary Non-associative algebras in the Non-Koszul case

- Anti-associative algebras. $(ab)c = -a(bc)$

Operad $\widetilde{\mathcal{A}ss} = \Gamma(\mu)/(\mu \circ_1 \mu + \mu \circ_2 \mu)$
$\widetilde{\mathcal{A}ss}^! = \widetilde{\mathcal{A}ss}$
$g_{\widetilde{\mathcal{A}ss}}(t) = t + t^2 + t^3$ do not satisfy the functional equation

$\Rightarrow \widetilde{\mathcal{A}ss}$ is not Koszul.

Standard Cohomolgy of an $\widetilde{\mathcal{A}ss}$-algebra. $H^*_{\widetilde{\mathcal{A}ss}}(A,A)_{st}$

$$C^1_{\widetilde{\mathcal{A}ss}}(A,A)_{st} \xrightarrow{\delta^1_{st}} C^2_{\widetilde{\mathcal{A}ss}}(A,A)_{st} \xrightarrow{\delta^2_{st}} C^3_{\widetilde{\mathcal{A}ss}}(A,A)_{st} \xrightarrow{\delta^3_{st}} 0$$

$$C^p_{\widetilde{\mathcal{A}ss}}(A,A)_{st} := Hom(V^{\otimes p}, V), p = 1,2,3$$

$$C^p_{\widetilde{\mathcal{A}ss}}(A,A)_{st} := 0, p \geq 4$$

$$\delta^1_{st}(\varphi)(a,b) := a\varphi(b) - \varphi(ab) + \varphi(a)b,$$

$$\delta^2_{st}(f)(a,b,c) := af(b,c) + f(ab,c) + f(a,bc) + f(a,b)c,$$

Deformation Cohomolgy of an $\widetilde{\mathcal{A}ss}$-algebra. $H^*_{\mathcal{A}ss}(A,A)_{defo}$

The deformation cohomology governs deformations of $\widetilde{\mathcal{A}ss}$-algebras: $H^2_{\widetilde{\mathcal{A}ss}}(A,A)_{defo}$ parametrizes isomorphism classes of infinitesimal deformations.
$H^3_{\widetilde{\mathcal{A}ss}}(A,A)_{defo}$ contains obstructions to extensions of partial deformations.

$$C^1_{\widetilde{\mathcal{A}ss}}(A,A)_{defo} \overset{\delta^1_{defo}}{\longrightarrow} C^2_{\widetilde{\mathcal{A}ss}}(A,A)_{defo} \overset{\delta^2_{defo}}{\longrightarrow} C^3_{\widetilde{\mathcal{A}ss}}(A,A)_{defo} \overset{\delta^3_{defo}}{\longrightarrow} \cdots$$

$C^p_{\widetilde{\mathcal{A}ss}}(A,A)_{defo} = Hom(V^{\otimes p},V),\ p=1,2,3$
$C^4_{\widetilde{\mathcal{A}ss}}(A,A)_{defo} =$
$Hom(V^{\otimes 5},V) \oplus Hom(V^{\otimes 5},V) \oplus Hom(V^{\otimes 5},V) \oplus Hom(V^{\otimes 5},V),$
$\delta^p_{defo} = \delta^p_{st},\ p=1,2$
$\delta^3_{defo}(g) = \delta^3(g) = (\delta^3_1(g), \delta^3_2(g), \delta^3_3(g), \delta^3_4(g)),$
where

$$\delta_1^3(g)(a,b,c,d,e) := ag(b,c,de) - g(a,b,c(de)) + (ab)g(c,d,e) - g(ab,cd,e)$$
$$+ g(ab,c,d)e - g((ab)c,d,e) + g(a,b,c)(de) - g(a,bc,de),$$

$$\delta_2^3(g)(a,b,c,d,e) := g((ab)c,d,e) - g(ab,c,d)e + g(a,b,cd)e - g(a,b(cd),e)$$
$$+ ag(b,cd,e) - g(a,b,(cd)e) + (ab)g(c,d,e) - g(ab,c,de),$$

$$\delta_3^3(g)(a,b,c,d,e) := g(a,bc,de) - ag(bc,d,e) + g(a,(bc)d,e) - a(g(b,c,d)e)$$
$$+ g(a,b,cd)e - g(ab,c,d)e + (g(a,b,c)d)e - g(a(bc),d,e),$$ and

$$\delta_4^3(g)(a,b,c,d,e) := g(ab,cd,e) - g(a,b,(cd)e) + ag(b,cd,e) - g(a,b(cd),e)$$
$$+ (ag(b,c,d))e - g(a,bc,d)e + (g(a,b,c)d)e - g(ab,c,d)e,$$

• Left-Alternative algebras.

The operad $\mathcal{A}lt_l$ for Left-alternative algebras is non-Koszul.

Standard Cohomology of an $\mathcal{A}lt_l$-algebra.

$$\mathcal{C}^1_{\mathcal{P}}(A,A)_{st} \xrightarrow{\delta^1_{st}} \mathcal{C}^2_{\mathcal{P}}(A,A)_{st} \xrightarrow{\delta^2_{st}} \mathcal{C}^3_{\mathcal{P}}(A,A)_{st} \xrightarrow{0} 0.$$

The coboundary operators are given by

$$\begin{cases} \delta^1 f(a,b) & = & f(a)b + af(b) - f(ab), \\ \delta^2\varphi(a,b,c) & = & \varphi(ab,c) + \varphi(ba,c) - \varphi(a,bc) - \varphi(b,ac) \\ & & \varphi(a,b)c + \varphi(b,a)c - a\varphi(b,c) - b\varphi(a,c). \end{cases}$$

Deformation Cohomology of an $\mathcal{A}lt_l$-algebra. $H^*_{\mathcal{A}lt_l}(A,A)_{defo}$

$$\mathcal{C}^1_{\mathcal{P}}(A,A)_{defo} \xrightarrow{\delta^1} \mathcal{C}^2_{\mathcal{P}}(A,A)_{defo} \xrightarrow{\delta^2} \mathcal{C}^3_{\mathcal{P}}(A,A)_{defo} \xrightarrow{\delta^3} \mathcal{C}^4_{\mathcal{P}}(A,A)_{defo} \rightarrow \cdots$$

with

$$\begin{array}{l}
\mathcal{C}^1_{\mathcal{P}}(A,A)_{defo} = Hom(V^{\otimes 1}, V), \\
\mathcal{C}^2_{\mathcal{P}}(A,A)_{defo} = Hom(V^{\otimes 2}, V), \\
\mathcal{C}^3_{\mathcal{P}}(A,A)_{defo} = Hom(V^{\otimes 3}, V), \\
\mathcal{C}^4_{\mathcal{P}}(A,A)_{defo} = Hom(V^{\otimes 5}, V) \oplus \cdots \oplus Hom(V^{\otimes 5}, V),
\end{array}$$

In particular, any 4-cochains consists of 5-linear maps.

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anticommutative bracket: $[v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}] = \varepsilon(\sigma)[v_1 \otimes \cdots \otimes v_n]$,

Jacobi identity

$$\begin{aligned}
[[u_1, \cdots, u_n], v_1, \cdots, v_{n-1}] &= [[u_1, v_1, \cdots, v_{n-1}], u_2, \cdots, u_n] \\
&\quad + [u_1, [u_2, v_1, \cdots, v_{n-1}], u_3, \cdots, u_n] \\
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$$\sum_{\sigma\in Sh(n,n-1)} (-1)^{\epsilon(\sigma)}[[x_{\sigma(1)},\cdots,x_{\sigma(n)}],x_{\sigma(n+1)},\cdots,x_{\sigma(2n-1)}] = 0,$$

$$Sh(n,n-1) = \{\sigma \in \Sigma_{2n-1}, \sigma(1) < \cdots < \sigma(n), \sigma(n+1) < \cdots < \sigma(2n-1)\}.$$

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2. Associative n-ary algebras.

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Totally associative algebras satisfy

$$\mu\left(1\!\!1^{\otimes i-1}\otimes\mu\otimes 1\!\!1^{\otimes n-i}\right)=\mu\left(1\!\!1^{\otimes j-1}\otimes\mu\otimes 1\!\!1^{\otimes n-j}\right),$$

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Partially associative algebras satisfy
$\sum_{i=1}^{n}(-1)^{(i+1)(n-1)}\mu\left(1\!\!1^{\otimes i-1}\otimes\mu\otimes 1\!\!1^{\otimes n-i}\right)=0$
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Other generalizations of associativity for n-ary algebras are σ-totally and σ-partially associative algebras adapted to the products of tensors and hypercubic matrices.

3. Free 3-ary partially associative algebra $L(V)$.

$L(V) = \oplus_{p \geq 0} L^{2p+1}(V)$
$L^1(V) = V$
$L^3(V) = V^{\otimes 3}$

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$L^{2p+1}(V) = F^{2p+1}(V)/R_{2p+1}(V)$

with $F(V) = \oplus_{p \geq 0} F^{2p+1}(V)$ free algebra corresponding to a 3-product.

$L^5(V) = 2V^{\otimes 5}$
$F^5(V) = Span\{(v_1v_2v_3)v_4v_5, v_1(v_2v_3v_4)v_5, v_1v_2(v_3v_4v_5)\}$
$R_5(V) = Span\{(v_1v_2v_3)v_4v_5 + v_1(v_2v_3v_4)v_5 + v_1v_2(v_3v_4v_5)\}$
$= Span\{(123)45 + 1(234)5 + 12(345)\}$

For 7 : 8 (independent) relations on 12 vectors.

$((123)45)67 + (123)(456)7 + (123)4(567)$

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For 9 : 80 relations on 55 vectors. $\cdots$

$\Rightarrow$ Coding

$(123)45 := \{1\}$ (position of left parenthesis)

$1(234)5 := \{2\}$

$12(345) := \{3\}$

$((123)45)67 := \{11\}$

$(123)45 + 1(234)5 + 12(345) = 1 + 2 + 3$

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$1{+}2{+}3 \rightarrow$

$11{+}12{+}13$

$22{+}23{+}24$

33+34+35
11+14+15
12+22+25
13+23+33
14+24+34
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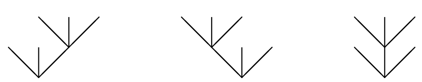
We can also consider a representation by trees: The vectors $\{3\}, \{1\} = s(\{3\}), \{2\}$ are represented by



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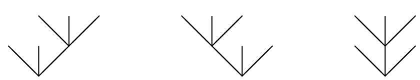

We observe that

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$\dim L^3(V) = m^3, \quad \dim L^5(V) = 2m^5, \quad \dim L^7(V) = 4m^7,$
$\dim L^9(V) = 5m^9, \quad \dim L^{11}(V) = 6m^{11}, \quad \dim L^{13}(V) = 7m^{13}.$

if $\dim V = m$.

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$\to g_{p\mathcal{A}ss_0^3}(t) = t + t^3 + 2t^5 + 4t^7 + 5t^9 + 6t^{11} + 7t^{13} + \cdots$

The function h satisfying $g_{p\mathcal{A}ss_0^3}(-h(-t)) = t$ is not the generating function of an operad (i.e $h \neq g_{\mathcal{P}}$) $\Rightarrow p\mathcal{A}ss_0^3$ is non Koszul.

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n-ary quadratic Operad. $\mathcal{P} = \Gamma(E)/(R)$

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$E^\vee = Sgn_n \otimes \uparrow^{n-2} E^\sharp$

$<,>: \Gamma(E^\vee)(2n-1) \otimes \Gamma(E)(2n-1) \to \mathbb{K}$

$R^\perp \subset \Gamma(E^\vee)(2n-1)$ annihilator of $R \subset \Gamma(E)(2n-1)$

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Definition

Operad for totally associative n-ary algebras with operation in degree d

$$tAss_d^n := \Gamma(\mu)/(R)$$

with μ an arity n generator of degree d and

$$R := Span\left\{\mu \circ_i \mu - \mu \circ_j \mu, \text{ for } i,j = 1,\dots,n\right\}.$$

Operad for partially associative n-ary algebras with operation in degree d.

$$pAss_d^n := \Gamma(\mu)/\left(\sum_{i=1}^{n} (-1)^{(i+1)(n-1)} \mu \circ_i \mu\right)$$

with μ a generator of degree d and arity n.

Proposition For each $n \geq 2$ and d,

$$(t\mathcal{A}ss^n_d)^! = p\mathcal{A}ss^n_{-d+n-2},$$

$$(p\mathcal{A}ss^n_d)^! = t\mathcal{A}ss^n_{-d+n-2},$$

Koszulity:

$t\mathcal{A}ss^n_d$	n even	n odd
d even	yes	yes
d odd	no	no

$p\mathcal{A}ss^n_d$	n even	n odd
d even	yes	no
d odd	no	yes

We define $t\widetilde{\mathcal{A}ss}^n_d := \mathbf{s}\, t\mathcal{A}ss^n_{d-n+1}$ and $p\widetilde{\mathcal{A}ss}^n_d := \mathbf{s}^{-1} p\mathcal{A}ss^n_{d+n-1}$.

$t\widetilde{\mathcal{A}ss}^n_d$-algebras satisfy:

$$(-1)^{i(n+1)}\mu \circ_i \mu = (-1)^{j(n+1)}\mu \circ_j \mu.$$

$p\widetilde{\mathcal{A}ss}^n_d$-algebras satisfy:

$$\sum_{i=1}^{n} \mu \circ_i \mu = 0.$$

$(t\widetilde{\mathcal{A}ss}^n_d)^! = p\widetilde{\mathcal{A}ss}^n{}_{-d+n-2}$ and $(p\widetilde{\mathcal{A}ss}^n_d)^! = t\widetilde{\mathcal{A}ss}^n{}_{-d+n-2}$

Koszulity:

$t\widetilde{\mathcal{A}ss}^n_d$	n even	n odd
d even	no	yes
d odd	yes	no

$p\widetilde{\mathcal{A}ss}^n_d$	n even	n odd
d even	no	no
d odd	yes	yes