Remerciements

Cette page est certainement la plus difficile à écrire. Non pas parce qu'il soit compliqué de remercier celles et ceux qui se sont impliqués dans ce travail, mais parce que souvent c'est la seule page d'un mémoire lue ou parcourue avec attention. De plus oublier quelqu'un dans les remerciements, c'est comme présenter une classification d'algèbres incomplète. C'est la catastrophe.

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Introduction

The memory consists of two main parts and an annexe. The first part essentially concerns the algebraic study of nonassociative and n-ary algebras: structure, identities, operads, cohomology. The second part is devoted to the study of some geometrical structures on Lie algebras corresponding to left invariant structures on a Lie group. We are interested in complex structures (existence of complex structures on a real Lie group), affine structures (existence of flat and torsion-free connections on a Lie group) and scalar products invariant under a finite abelian group of automorphisms (classification of pseudo-riemannian Γ-symmetric spaces).

The purpose of the annexe is to establish and solve the Gerstenhaber equations of deformations for a binary multiplication of algebras when the ring of coefficients of the deformations is a local ring of valuation. This case contains the classical deformations of Gerstenhaber.

Part I.

In Chapter 1 we classify nonassociative identities which can be satisfied by the associator of a multiplication of an algebra (binary algebra). Such an identity can be interpreted through an action of the group algebra $\mathbb{K}[\Sigma_3]$ on the symmetric group Σ_3. At first we show that

1) Any submodule of the Σ_3-module $\mathbb{K}[\Sigma_3]$ is of rank 1. We describe each submodule and give a generating vector for it.

2) Any submodule is a linear $\mathbb{K}$-subspace of $\mathbb{K}[\Sigma_3]$ of dimension at most 6. We give the classification of these linear subspaces and so we classify the generators of the submodules according to the dimension of their corresponding linear subspaces in the real case.

3) Each generator corresponds to an identity on the associator of a product of algebra. Two identities are equivalent if they correspond to the action of two vectors of $\mathbb{K}[\Sigma_3]$ generating the same submodule.

Then we obtain a classification of nonassociative identities including the most classical one i.e. corresponding to Lie-admissible, pre-Lie, Vinberg, associative, third power associative, alternating, flexible algebras, etc. We study separately the Lie-admissible and third power associative cases. More regular classes of identities are defined considering the subgroups G_i of Σ_3. Each of these subgroups determines two linear subspaces of $\mathbb{K}[G_i]$ invariant under G_i and 1-dimensional. The generators of these spaces generate submodule of $\mathbb{K}[\Sigma_3]$ of rank 1. The corresponding nonassociative relations, called G_i-associativity relations, are described. We finally extend this study to the half associators $(xy)z$ and $x(yz)$.

Chapter 2 is devoted to the study of Poisson algebras from a nonassociative point of view. We define a polarization-depolarization process to represent an algebra either with one operation without any specific symmetry or with one commutative and one anticommutative operations. We illustrate how this change of perspectives leads to some new results on two main examples: we apply the depolarization to a Poisson algebra in order to study it as a nonassociative algebra and we use the polarization in Chapter 4 to show that the operad for Lie-admissible algebras is Koszul. Speaking about Poisson algebras we prove that

1) There is a biunivoque correspondance between Poisson algebras and nonassociative algebras called Poisson-admissible algebras whose associator A satisfies

$$3A(X,Y,Z) = (X \cdot Z) \cdot Y + (Y \cdot Z) \cdot X - (Y \cdot X) \cdot Z - (Z \cdot X) \cdot Y.$$

Thus a Poisson algebra is defined either by the classical Lie and associative products satisfying Leibniz rule or by the nonassociative product. Considering a Poisson algebra as a nonassociative algebra we show that

2) Considering a Poisson algebra as a nonassociative algebra we show that a Poisson algebra is flexible and it admits a Pierce decomposition. We deduce the classifications of 2 and 3-dimensional Poisson algebras. To obtain them, we use the classifications of 2 and 3-dimensional commutative associative algebras obtained in Chapter 10.

3) The study of deformations of Poisson-admissible algebras (i.e. Poisson algebras considered as nonassociative algebras) leads to another cohomology that the Lichnerowicz cohomology. In fact, this cohomology corresponds to deformations of a (classical) Poisson algebra with an unchanged associative product. Considering the algebra as a nonassociative algebra, we deform simultaneously the associative product and the Lie bracket (called Poisson bracket) while Leibniz's rule is preserved. So we describe a cohomology associated to the nonassociative identity. We compare the components of degree 2 (i.e. H^2)of this cohomology with ones of Lichnerowicz's cohomology (parametrizing only the deformations of the Poisson bracket) and Hochschild's cohomology (parametrizing the deformations of the associative product).

Chapter 3 extends the notion of coassociative coalgebra to the G_i-associative algebras. We prove that the basic properties linking associative algebras and coassociative coalgebras are similar. We define, for example, the Lie-admissible coalgebras, the Lie-admissible bialgebras and show that the dual of a Lie-admissible coalgebra is provided with a structure of Lie-admissible algebra.

Chapter 4 develops the study of nonassociative algebras from the point of view of quadratic operads. After some recalling on operads, quadratic operads and duality mostly in the context of operads for binary algebras,

1) we define the operads for Lie-admissible algebras and other G-associative algebras;

2) we define their dual operads;

3) we prove that the operad $\mathcal{L}ieAdm$ is Koszul using the polarization process of Chapter 1;

4) we prove that the operad $\mathcal{G}_4$-*ass* for G_4-associative algebras is not Koszul computing the generating functions of $\mathcal{G}_4$-*ass* and its dual. The generating series of a Koszul operad and its dual operad are connected by a functional equation which is not satisfied for $\mathcal{G}_4$-*ass* and its dual, so both operads are non Koszul.

Chapter 5 is devoted to third-power associative algebras. Among these algebras we find flexible, alternative, left alternative algebras. For each subgroup G of Σ_3 we define a class of G-p^3-associative algebras which is a subclass of third-power associative algebras similarly to G-associative algebras for the Lie-admissible case. As in Chapter 4 we define the operad $\mathcal{G}$-p^3ass and its dual. We prove that *the operad for alternative algebras is non Koszul*. A consequence is that the deformations of alternative algebras can not be parametrized by the operadic cohomology and this contradicts some recent works studying only the operadic cohomology. We prove that the space of 4-cochains of the operadic cohomology is trivial and that the sequence of the operadic cohomology is a short sequence. The deformations are then parametrized by a cohomology associated to a minimal model of the initial operad. After computing the generating function of $\mathcal{A}lt$ we explicit the first terms of the complex associated to the minimal model. This permits to describe the obstructions to extensions of partial deformations of alternative algebras.

In Chapter 6 we define and study some properties attached to a quadratic operad $\mathcal{P}$ with one binary generating operation, that is, we define the current operad $\widetilde{P}$ of $\mathcal{P}$, we study if $\mathcal{P}$ is an Hopf or/and a cyclic operad and finally we define dihedrality of $\mathcal{P}$. First we define for $\mathcal{P}$ a quadratic operad $\widetilde{\mathcal{P}}$ also with one binary generating operation which has the property that we can summarize as: $\mathcal{P} \otimes \widetilde{\mathcal{P}} = \mathcal{P}$ (the tensor product of a $\mathcal{P}$-algebra with a $\widetilde{\mathcal{P}}$-algebra is a $\mathcal{P}$-algebra). We show that the relations describing $\widetilde{\mathcal{P}}$ are minimal for this property. Then we give the description of $\widetilde{\mathcal{P}}$ when $\mathcal{P}$ is a classical operad ($\mathcal{A}ss, \mathcal{L}ie, \mathcal{P}oiss$, etc.) or one of the operads defined in the previous chapters ($\mathcal{G}_i$-*ass*, $\mathcal{G}_i$-p^3ass) and we show that in general $\widetilde{\mathcal{P}}$ is not the dual operad of $\mathcal{P}$. Then we investigate if $\mathcal{P}$ is an Hopf operad and give explicit conditions for $\mathcal{P}$ to be Hopf. To obtain them we use the polarization-depolarization process. We also introduce a new kind of symmetry for operads, the *dihedrality* responsible for the existence of dihedral cohomology. We study cyclicity and dihedrality on some examples and summarize this properties in Figure 6.1.

Chapters 7 and 8 concern n-ary associatives algebras. The study of n-ary associatives algebras have already been done in [47] from the operadic and (co)homological point of view. But we show that the even and odd cases behave in a completely different way so can not be treated simultaneously. Chapter 7 mostly concerns the algebraic level and Chapter 8 the operadic level, although both are linked.

In Chapter 7, we focus on the case $n = 3$ and describe the free algebras on finite-dimensional 3-ary partially associative algebras. Since these algebras are graded, we describe explicitly the basis of the first homogeneous components. This result is obtained using an original method of coding the vectors of the free algebra. The operadic approach in [47] contains also some misunderstandings of the odd case coming from the interpretation of the dual of an operad for n-ary algebras. Contrary to what we find in this paper, the operads associated to 3-ary partially associative algebras are non Koszul so the operadic cohomology does not capture deformations. Then we consider 3-ary partially associative algebras with operation of degree 1 whose associated operad is Koszul (Chapter 8 and the correct definition of the dual will light up this study). We explicit a cohomology (which is the operadic cohomology) which governs deformations. We also give an extension of the notion of coassociative algebras for n-ary algebras. Then we generalize the notion of n-ary partially and totally associative algebras and define n-ary σ-partially and σ-totally associative algebras because they appear naturally when we define a $(2k+1)$-ary product on the space $T_q^p(E)$ of tensors which are contravariant of order p and covariant of order q. The main reason for introducing these products is that they can be interpreted as products of "hypercubic matrices", that is, square tables of length $p+q$. This generalizes in a natural way the classical associative product of matrices. We finish this chapter by computing the current 3-ary τ_{13}-totally associative algebra.

In Chapter 8 we give explicitly the definition of the dual of an operad for n-ary algebras with multiplication of degree d. So the dual of an operad for $(n-odd)$-ary algebras with operation in degree 0 is an operad for n-ary algebras with operation in degree d *different from* 0 and we can not get ride of it in the computations. We apply this definition to some lightning examples of operads for various algebras with an n-linear operation satisfying a specific version of associativity regrouped in four families. We compute the generating functions and investigate Koszulity of these operads. We then focus on the operad for binary anti-associative algebras (associativity is replaced by $x(yz) = -(xy)z$). Since it is non Koszul, the deformation cohomology differs from the standard one and we are conduced to describe the relevant part of the deformation cohomology for this type of algebras using the minimal model for the anti-associative operad. We then discuss free partially associative algebras and prove that the description of Gnedbaye ([47]) is true for the even case.

To study Koszulness we adapt the definition of dual operad of Ginzburg-Kapranov to our case which conduces to introduce the operations with a degree in the dual (in [47] all operations are of degree 0).

Part II.

This second part of this work (Part II) is devoted to the study of some geometrical structures on Lie algebras. These geometrical structures correspond to left invariant geometrical structure on connected Lie groups. We consider three types of structures:

- Complex structures on nilpotent Lie algebras. These structures are in correspondence with the left invariant complex structures on real nilpotent Lie groups or on real nilmanifolds. Such a structure on a Lie group permits to consider holomorphic maps and when these structure are biinvariant in the associated Lie algebra, the Lie group can be provided with an holomorphic structure.

- Affine structures on nilpotent Lie algebras. These structures correspond to flat and torsion-free left invariant connections on nilmanifolds. Since there exist some nilmanifolds with no affine structure, the problem of existence is natural.

- Pseudo-riemannian Γ-manifolds. In case of a symmetric homogeneous manifold, the group of symmetries is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Here we consider a homogeneous manifold admitting a group of symmetries isomorphic to Klein's group. We consider riemannian tensors whose considered symmetries are isometries. Inspired by the classification of riemannian compact symmetric manifolds of Elie Cartan, we determine, when the homogeneous manifold is the quotient of a compact simple Lie group provided with riemannian and pseudo-riemannian metrics adapted to the symmetries.

In Chapter 9, we consider the problem of existence of complex structures on nilpotent Lie algebras. The classification of 6-dimensional nilpotent real Lie algebras provided with a complex structure is known. But this classification is based on the general classification of real and complex Lie algebras. Such a classification does not exist for dimensions greater than or equal to 8. Thus our approach to this problem must be different. First of all we prove that

A filiform algebra (i.e a nilpotent Lie algebra with maximal nilindex) has no complex structures.

In terms of characteristic sequence (also called Goze's invariant), filifom algebras correspond to nilpotent Lie algebras with the characteristic sequence equal to $(n-1,1)$, where n is the dimension of the Lie algebra. Then we are interested in quasi filiform algebras, that is, with Goze's invariant equal to $(n-2,1,1)$. In this case, the approach is based on the notion of generalized complex structures of Gualtieri and Cavalcanti (in their works, the authors find the main result for filiform algebras again). In this case we prove that

Any quasi filiform algebra provided with a complex structure is 6*-dimensional.*

Moreover, we prove that

Any 6*-dimensional quasi filiform algebra provided with a complex structure is isomorphic to the Lie algebra*

$$\begin{cases} [X_0, X_i] = X_{i+1}, \quad i = 1,2,3, \\ [X_1, X_2] = X_5, \\ [X_1, X_5] = X_4. \end{cases}$$

In Chapter 10 we consider the problem of existence of an affine structure on nilpotent Lie algebras. We approach this problem considering pre-Lie algebras. In fact, a pre-Lie algebra is a Lie-admissible algebra whose corresponding Lie algebra admits an affine structure. The problem of existence of affine structures on nilpotent Lie algebras is still open. Contrary to what people expected, the counter-example of Benoist shows that there are nilpotent Lie algebras with no affine structure. In Chapter 10 we are interested in the abelian case. More precisely,

We classify all the affine structures (complete or not) on $\mathbb{R}^3$ *and on the* 3*-torus.*

This completes the classical work of Goldmann concerning the classification of complete complex structures on $\mathbb{R}^2$ and on the 2-dimensional torus. We then obtain general results on existence problems. We first prove that

Any non characteristically (i.e with a non nilpotent derivation) filiform algebra admits an affine structure.

This permits to prove that

There exists a non complete affine representation on the Heisenberg group.

In fact, the existence of an affine structure permits to produce an affine representation of dimension equal to the dimension of the Lie algebra or to this dimension plus one. And, from this result, someone can make a conjecture that any affine representation of filiform algebra is complete. Thus, this conjecture is wrong. We also construct current affine Lie algebras, that is, Lie algebras with an affine structure obtained by a tensor product of a pre-Lie algebra with an algebra associated with the current operad. At the end of this chapter we are interested in contact Lie algebras with affine structures. This is motivated by the classical result that any symplectic Lie algebra admits a natural affine structure. This is not the case in the contact situation (cf the counter-example of Benoist). We describe an obstruction to extend a symplectic affine structure to a contact Lie algebra obtained by a central extension.

Chapter 11 is devoted to the determination of pseudo-riemannian metrics on compact simple $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$-symmetric spaces. In a recent paper, Bathurin and Goze generalize the notion of symmetric spaces and propose a new class of reductive spaces called Γ-symmetric spaces. The most interesting examples are the flag manifolds. It's not a symmetric space but we can find in this space a lot of commuting symmetries which are in correspondance with an abelian group (Γ). Here we are interested in the riemannian case. The main result is the classification of compact $\mathbb{Z}_2^2$-symmetric spaces.

Chapter 12 concerns valued deformations of algebras. Classically the deformations of Gerstenhaber have been studied in a more general framework that the initial one, that is, people take coefficients in a local ring A, then there is a maximal ideal $\mathfrak{m}$, and they assume that $A/\mathfrak{m}$ is a field isomorphic to $\mathbb{K}$. Two concrete examples of deformations in a local ring are given by the deformations of Gerstenhaber with coefficients in the ring of formal series $\mathbb{K}[[t]]$ and perturbations which give results on the rigidity. This leads to deformations with coefficients in a valuation ring (i.e defined by a valuation) which is a local ring because both previous examples appear as particular cases. In this context, we prove that we can write the equations of deformations of Gerstenhaber in a finite and unique way. The main reason for taking a valuation ring instead of a local ring is that, if we consider an element x that belongs to the complementary of the valuation ring in its field of fractions, then x^{-1} belongs to the maximal ideal $\mathfrak{m}$. We prove that the infinite system, associated with Gerstenhaber's deformations and giving the conditions to obtain a deformation, can be reduced to a system of finite rank.

Part I

Nonassociative algebras, Operads, n-ary algebras

Chapter 1

Nonassociative Algebras

In this chapter, we recall the principal notions related to nonassociative algebras: simplicity, center, solvability, nilpotency. Alternative algebras are also recalled because they will be studied from the operadic viewpoint in Chapter 5. The main result of this chapter is the classification of nonassociative identities with respect to a natural action of Σ_3 on the associator of a general binary multiplication, where Σ_3 denotes the symmetric group on three elements. At first we prove that any invariant linear subspace of the group algebra $\mathbb{K}[\Sigma_3]$ is a Σ_3-module of rank 1. Thus, we can associate with an invariant linear subspace of $\mathbb{K}[\Sigma_3]$ a vector generating the corresponding Σ_3-module. With each of these vectors, we associate a relation on the associator of a multiplication. Two relations associated with two vectors that are generators of the same module are, by definition, equivalent. This leads to define natural classes of nonassociative relations. In particular this allows to distinguish two classes of nonassociative algebras: the class of Lie-admissible algebras and the class of 3-power associative algebras. Both classes correspond to an one-dimensional invariant subspace of $\mathbb{K}[\Sigma_3]$. We describe both classes by classifying all identities corresponding to an invariant subspace of $\mathbb{K}[\Sigma_3]$. Amongst these families, we are mostly interested in the families called G-associative and G-p^3-associative algebras, where G is a subgroup of Σ_3. If G is such a subgroup, then it determines naturally two vectors of $\mathbb{K}[\Sigma_3]$, $v_G = \sum_{\sigma\in G}(-1)^{\varepsilon(\sigma)}\sigma$ and $w_G = \sum_{\sigma\in G}\sigma$. Each of these vectors determines a classe of nonassociative algebras such as associative algebras, Pre-Lie, Vinberg, Lie-admissible algebras, 3-power associative algebras and left and right alternative algebras. We can then study these algebras with this "symmetric aspect".

1.1 Nonassociative algebras

In this section we recall principal notions and results about nonassociative algebras. The details can be read in [111].

1.1.1 Some examples

Let $\mathbb{K}$ be a field of characteristic 0 and A a $\mathbb{K}$-vector space. A structure of (binary) algebra on A is given by a linear map

$$\mu : A^{\otimes 2} \to A$$

which satisfies the distributive laws

$$\mu((x+y)\otimes z) = \mu(x\otimes z) + \mu(y\otimes z)$$
$$\mu(x\otimes(y+z)) = \mu(x\otimes y) + \mu(x\otimes z)$$

and

$$\alpha\mu(x\otimes y) = \mu(\alpha x\otimes y) = \mu(x\otimes \alpha y)$$

for all $x, y, z \in A$ and $\alpha \in \mathbb{K}$.

Remark and Notations. We use also μ for the corresponding bilinear map $A \times A \longrightarrow A$ and when no confusion is possible, we write $x \cdot y$ or xy in place of $\mu(x, y)$ or $\mu(x \otimes y)$.

In particular:

- An *associative algebra* is an algebra where the associative condition
$$(xy)z - x(yz) = 0$$
holds, for any $x, y, z \in A$. Thus algebra means nonassociative algebra.

- A *Lie algebra* is an algebra whose multiplication, classically denoted by $[x, y]$, is skew-symmetric and satisfies the Jacobi identity:
$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$$
for any $x, y, z \in A$. A Lie algebra is associative if and only if it is 2-step nilpotent, that is, if the product satisfies
$$[[x, y], z] = 0.$$

- A *Jordan algebra* is an algebra whose multiplication is commutative and satisfies
$$(xy)x^2 = x(yx^2).$$

- An *alternative algebra* is defined by the identities
$$\begin{cases} x^2y = x(xy), \\ yx^2 = (yx)x. \end{cases}$$

The definition of subalgebras, ideals, homomorphisms, quotient algebras are classical.

If μ is the multiplication of the algebra $\mathcal{A}$, we define

- the *commutator* of μ by
$$[x, y]_\mu = \mu(x, y) - \mu(y, x).$$

- the *associator* of μ by
$$A_\mu(x, y, z) = \mu(\mu(x, y), z) - \mu(x, \mu(y, z)).$$

Thus an alternative multiplication is given by
$$A_\mu(x, x, y) = A_\mu(y, x, x) = 0, \quad \text{for all } x, y \in \mathcal{A}$$
and a *Jordan algebra* by
$$[x, y]_\mu = A_\mu(x, y, x^2) = 0.$$
Note that any multiplication μ satisfies
$$\mu(a, A_\mu(x, y, z)) + \mu(A_\mu(a, x, y), z) = A_\mu(\mu(a, x), y, z) - A_\mu(a, \mu(x, y), z) + A_\mu(a, x, \mu(y, z))$$
for all $x, y, z, a \in \mathcal{A}$.

The *nucleus* $\mathcal{N}(\mathcal{A})$ of $\mathcal{A}$ is
$$\mathcal{N}(\mathcal{A}) = \{g \in \mathcal{A} \mid A_\mu(g, x, y) = A_\mu(x, g, y) = A_\mu(x, y, g) = 0 \ \text{ for all } \ x, y \in \mathcal{A}\}.$$

The *center* $\mathcal{C}(\mathcal{A})$ of $\mathcal{A}$ is
$$\mathcal{C}(\mathcal{A}) = \{c \in \mathcal{N}(\mathcal{A}) \mid \mu(x, c) - \mu(c, x) = [x, c]_\mu = 0 \ \text{ for any } \ x \in \mathcal{A}\}.$$
It is easy to verify that $\mathcal{N}(\mathcal{A})$ and $\mathcal{C}(\mathcal{A})$ are associative subalgebras of $\mathcal{A}$.

1.1.2 Simple algebras

For any $a \in \mathcal{A}$, we denote by
$$R_a : x \to \mu(x, a)$$
and
$$L_a : x \to \mu(a, x)$$
the linear operators called respectively right and left multiplications.

If $\mathcal{M}(\mathcal{A})$ is the associative subalgebra of $gl(\mathcal{A})$ generated by the left and the right multiplications on $\mathcal{A}$, any element of $\mathcal{M}(\mathcal{A})$ is written as $\sum S_1 \cdots S_k$, where S_i are right or left multiplications.

Definition 1 *The algebra $\mathcal{A}$ is simple if $\{0\}$ and $\mathcal{A}$ are the only ideals of $\mathcal{A}$ and $\mathcal{A} \neq \{0\}$.*

Then if $\mathcal{A}$ is simple, $\mathcal{A}^2 = \mathcal{A}$ because $\mathcal{A}^2$ is an ideal of $\mathcal{A}$. Since an ideal of $\mathcal{A}$ is an invariant subspace under $\mathcal{M}(\mathcal{A})$ thus $\mathcal{A}$ is simple if and only if $\mathcal{M}(\mathcal{A}) \neq \{0\}$ and it is an irreducible set of $gl(\mathcal{A})$.

Definition 2 *The algebra $\mathcal{A}$ is a division algebra if $\mathcal{A} \neq \{0\}$ and if for any $a \in \mathcal{A} \setminus 0$, L_a and R_a have inverses L_a^{-1} and R_a^{-1}.*

This is equivalent to the existence of an unique solution in $\mathcal{A}$ for the equation $a \cdot x = b$ and $y \cdot a = b$ when $a \neq 0$. If $\mathcal{A}$ is finite dimensional then $\mathcal{A}$ is a division algebra if and only if it is without zero divisors.

Definition 3 *The algebra $\mathcal{A}$ is central simple if for any field extension $\mathbb{F}$ of $\mathbb{K}$, the $\mathbb{F}$-algebra $\mathcal{A} \otimes_{\mathbb{K}} \mathbb{F} = \mathcal{A}_{\mathbb{F}}$ is simple.*

Such an algebra is necessarily simple ($\mathbb{F} = \mathbb{K}$).

Example([65]). We consider the centralizer $\mathcal{C}'$ of $\mathcal{M}(\mathcal{A})$ in $gl(\mathcal{A})$, it is generated by the endomorphisms T of $gl(\mathcal{A})$ satisfying
$$\forall x, y \in \mathcal{A},\ R_y \circ T = T \circ R_y,\ L_x \circ T = T \circ L_x.$$
This implies
$$T(x \cdot y) = T(x) \cdot y = x \cdot T(y)$$
and, if $T, S \in \mathcal{C}'$
$$S \circ T(x \cdot y) = S(T(x \cdot y)) = S(T(x) \cdot y) = T(x) \cdot S(y) = T(x \cdot S(y)) = T(S(x \cdot y)) = T \circ S(x \cdot y),$$
that is,
$$S \circ T = T \circ S \quad \text{on } \mathcal{A}^2.$$
If $\mathcal{A}$ is simple, $\mathcal{A} = \mathcal{A}^2$ and $S \circ T = T \circ S$. We deduce ([65]) that $\mathcal{C}'$ is a field and $\mathcal{A}$, regarded as a $\mathcal{C}'$-algebra, is central simple.

1.1.3 Solvable and nilpotent algebras

For any algebra $\mathcal{A}$, a descending sequence of subalgebras of $\mathcal{A}$ is defined by
$$\begin{cases} \mathcal{A}^{(1)} = \mathcal{A}, \\ \mathcal{A}^{(k+1)} = \mathcal{A}^{(k)} \cdot \mathcal{A}^{(k)}, \end{cases}$$
and one has
$$\mathcal{A} = \mathcal{A}^{(1)} \supseteq \mathcal{A}^{(2)} \supseteq \cdots \supseteq \mathcal{A}^{(k)} \supseteq \cdots$$

Definition 4 *The algebra $\mathcal{A}$ is solvable if $\mathcal{A}^{(k)} = 0$ for some integer k. The algebra $\mathcal{A}$ is nilpotent if there exists an integer k such that any product of k elements is 0.*

Thus any nilpotent algebra is solvable. It is clear that if I and J are solvable ideals of an algebra $\mathcal{A}$, then $I + J$ is also solvable. Then if $\dim \mathcal{A} < \infty$, there is a unique maximal solvable ideal of $\mathcal{A}$ denoted $\mathcal{R}(\mathcal{A})$ and called the radical of $\mathcal{A}$.

Theorem 5 *The algebra $\mathcal{A}$ is nilpotent if and only if the associative algebra $\mathcal{M}(\mathcal{A})$ is nilpotent.*

1.1.4 Lie algebra associated to $\mathcal{A}$

Let $\mathcal{A}$ be an algebra. We consider $gl(\mathcal{A})$ with its Lie algebra structure. The Lie algebra $\mathcal{L}(\mathcal{A})$ associated to $\mathcal{A}$ is the Lie subalgebra of $gl(\mathcal{A})$ generated by the operations L_a and R_a.

If we consider $\mathfrak{h} = R(\mathcal{A}) + L(\mathcal{A})$, where $R(\mathcal{A})$ (resp. $L(\mathcal{A})$) is the space generated by the right (resp. left) multiplications, then taking

$$\mathfrak{h}_1 = \mathfrak{h},\ \mathfrak{h}_2 = [\mathfrak{h}, \mathfrak{h}],\ \mathfrak{h}_k = [\mathfrak{h}, \mathfrak{h}_{k-1}]$$

we obtain

$$\mathcal{L}(\mathcal{A}) = \oplus_{i=1}^{+\infty} \mathfrak{h}_i,$$

with $[\mathfrak{h}_i, \mathfrak{h}_j] \subset \mathfrak{h}_{i+j}$.

We denote by $Der(\mathcal{A})$ the Lie subalgebra of $gl(\mathcal{A})$ constituted by derivations of $\mathcal{A}$, that is, $D \in gl(\mathcal{A})$ and D satisfies

$$\forall x, y \in \mathcal{A},\ D(x \cdot y) = D(x) \cdot y + x \cdot D(y),$$

which means that D satisfies

$$\begin{cases} [R_y, D] = R_{D(y)}, & \text{for all } y \in \mathcal{A}, \\ [L_x, D] = L_{D(x)}, & \text{for all } x \in \mathcal{A}. \end{cases}$$

A derivation D is *inner* if $D \in \mathcal{L}(\mathcal{A})$. The set $In(\mathcal{A}) = \mathcal{L}(\mathcal{A}) \cap Der(\mathcal{A})$ of inner derivations is an ideal of $Der(\mathcal{A})$.

1.1.5 Alternative algebras

We have defined an alternative algebra as an algebra whose associator satisfies

$$A_\mu(x, x, y) = A_\mu(y, x, x) = 0 \tag{1.1}$$

for any elements x, y of the algebra. Linearizing this identity we obtain

$$A_\mu(x_1, x_2, x_3) = (-1)^{\varepsilon(\sigma)} A_\mu(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}) \tag{1.2}$$

for any σ in the symmetric group on 3 elements Σ_3, where $(-1)^{\varepsilon(\sigma)}$ denotes the signature of the permutation σ.

An algebra is *flexible* if the associator satisfies (1.2) with $\sigma = \tau_{13}$, where τ_{13} is the transposition exchanging 1 and 3. Thus any alternative algebra is flexible. We deduce some natural relations of order 4, called Moufang identities

$$\begin{cases} (xax)y = x[a(xy)], \\ y(xax) = [(yx)a]x, \\ (xy)(ax) = x(ya)x. \end{cases}$$

(Recall that xbx is well defined because $\mathcal{A}$ is flexible so $xbx = (xb)x = x(bx)$). For example the linearized form of the last Moufang identity is

$$A_\mu(x_2, \mu(x_1, x_4), x_3) + A_\mu(x_2, \mu(x_3, x_4), x_1) = -\mu(A_\mu(x_2, x_1, x_4), x_3) - \mu(A_\mu(x_2, x_3, x_4), x_1).$$

The relation between alternative algebras and associative algebras is given by Artin theorem:

Theorem 6 *The subalgebra generated by any two elements of an alternative algebra $\mathcal{A}$ is associative.*

In particular the subalgebra generated by an element $x \in \mathcal{A}$ is associative. This means that the power of x is well defined by the relation $x^{i+1} = x \cdot x^i$ and $x^i \cdot x^j = x^j \cdot x^i = x^{i+j}$ for any natural numbers $i, j \geq 1$. Such an algebra is called *power-associative*. An element x in a power-associative algebra is called nilpotent if there is an integer k such that $x^k = 0$. If any element of a power-associative algebra $\mathcal{A}$ is nilpotent, then $\mathcal{A}$ is a *nilalgebra*. In finite dimensional cases, the concept of nilpotent, solvable and nil-algebra coincide because any finite dimensional alternative nilalgebra $\mathcal{A}$ is nilpotent.

We can then define the concept of radical $\mathcal{R}(\mathcal{A})$ of $\mathcal{A}$ as the unique maximal solvable ideal (= nilpotent ideal = nilideal).

Definition 7 *An alternative algebra $\mathcal{A}$ is semi-simple if $\mathcal{R}(\mathcal{A}) = \{0\}$.*

If $\mathcal{A}$ is a finite dimensional alternative algebra then it is semisimple, it is a direct sum of simple algebras. This fact is based on Pierce decomposition. One establishes that any finite dimensional power-associative algebra which is not a nilalgebra contains a non trivial idempotent e ($e^2 = e$). Thus the operators L_e and R_e are commuting idempotent operators and $\mathcal{A}$ admits the decomposition

$$\mathcal{A} = \mathcal{A}_{0,0} + \mathcal{A}_{1,1} + \mathcal{A}_{1,0} + \mathcal{A}_{0,1},$$

where

$$\mathcal{A}_{i,j} = \{x_{ij} \mid L_e(x_{ij}) = ix_{ij}, R_e(x_{ij}) = jx_{ij}\},$$

and the decomposition is written

$$x = exe + (ex - exe) + (xe - exe) + (x - ex - xe + exe).$$

Finally recall Wedderburn Theorem which generalizes the classical Wedderburn Theorem for associative algebras to alternative algebras . A finite-dimensional algebra $\mathcal{A}$ over $\mathbb{K}$ is *separable* if, for any extension $\mathbb{F}$ of $\mathbb{K}$, the algebra $\mathcal{A}_\mathbb{F}$ is a direct sum of simple ideals. If $\mathcal{A}$ is alternative, this is equivalent to say that $\mathcal{A}$ is semi-simple.

Theorem 8 *Let $\mathcal{A}$ be a finite dimensional alternative algebra over $\mathbb{K}$ with radical $\mathcal{R}(\mathcal{A})$. If $\mathcal{A}/\mathcal{R}(\mathcal{A})$ is separable then $\mathcal{A} = S \oplus \mathcal{R}(\mathcal{A})$ where S is a subalgebra of $\mathcal{A}$ isomorphic to $\mathcal{A}/\mathcal{R}(\mathcal{A})$.*

1.2 $\mathbb{K}[\Sigma_3]$-associative algebras

In this section we define the concept of $\mathbb{K}[\Sigma_3]$-associative algebras (see [56]), that is, algebras whose associator of the corresponding product satisfies some relations associated to invariant subspaces of the group algebra of the symmetric group Σ_3 when we consider a natural right action of Σ_3. This approach permits to have, practically, all the classical nonassociative algebras in a same class. At first we recall classical notions related to $\mathbb{K}[\Sigma_3]$. In particular, we prove that any invariant linear subspace of $\mathbb{K}[\Sigma_3]$ is a Σ_3-module of rank 1. This means that any invariant subspace determines a vector of $\mathbb{K}[\Sigma_3]$ which is a basis of this space but considered as a Σ_3-module. Thus we will determine all its invariant subspaces and define, for each one of these spaces a corresponding generator of the associated module. This permits to associate with an invariant subspace of $\mathbb{K}[\Sigma_3]$ a relation concerning the associator of an algebra product, determined by the generator of this module.

1.2.1 The Σ_3-module $\mathbb{K}[\Sigma_3]$

We denote by τ_{ij} the transposition exchanging i and j and by c the cycle (123). Let $\mathbb{K}[\Sigma_3]$ be the algebra-group of the symmetric group on three elements Σ_3, that is, the 6-dimensional $\mathbb{K}$-vector space with basis

$$\left\{\sigma_1 = Id, \sigma_2 = \tau_{12}, \sigma_3 = \tau_{13}, \sigma_4 = \tau_{23}, \sigma_5 = c = (123), \sigma_6 = c^2\right\},$$

with the associative multiplication

$$(\sum_{i=1}^{6} a_i\sigma_i) \cdot (\sum_{j=1}^{6} b_j\sigma_j) = \sum_{i,j=1}^{6} a_i a_j \sigma_i \circ \sigma_j.$$

This algebra is provided with a structure of right Σ_3-module considering

$$\begin{array}{ccc} \Sigma_3 \times \mathbb{K}[\Sigma_3] & \rightarrow & \mathbb{K}[\Sigma_3] \\ (\sigma, \sum_{i=1}^{6} a_i\sigma_i) & \mapsto & \sigma \cdot (\sum_{i=1}^{6} a_i\sigma_i) = \sum_{i=1}^{6} a_i \sigma^{-1} \circ \sigma_i, \end{array}$$

where $a_i \in \mathbb{K}$ for for any $i \in \{1, 2, \cdots, 6\}$. Indeed we have

$$\begin{array}{rl} \sigma \cdot (\sigma' \cdot \sum_{i=1}^{6} a_i \sigma_i) = & \sigma \cdot (\sum_{i=1}^{6} a_i \sigma'^{-1} \circ \sigma_i) = \sum_{i=1}^{6} a_i \sigma^{-1} \circ \sigma'^{-1} \circ \sigma_i \\ & \sum_{i=1}^{6} a_i (\sigma' \circ \sigma)^{-1} \circ \sigma_i = (\sigma' \circ \sigma) \cdot (\sum_{i=1}^{6} a_i \sigma_i). \end{array}$$

For any element $v \in \mathbb{K}[\Sigma_3]$, we denote by $\mathcal{O}(v)$ the orbit of v with respect to this action. Similarly we denote by $\mathcal{O}(E)$ the orbit of any subset E of $\mathbb{K}[\Sigma_3]$. We denote by $\mathbb{K}\{v_1, v_2, \cdots, v_p\}$ the linear span of the vectors $v_1, v_2 \cdots, v_p$ and $F_v = \mathbb{K}[\mathcal{O}(v)]$ the linear span of the vectors of $\mathcal{O}(v)$.

A subset F of $\mathbb{K}[\Sigma_3]$ is invariant if $F = \mathcal{O}(F)$. By Maschke theorem, every invariant subspace is a direct sum of irreducible invariant subspaces and the dimension of an irreducible invariant subspace is one or two.

Proposition 1 *The one-dimensional invariant subspaces of* $\mathbb{K}[\Sigma_3]$ *are* $F_{v_{\Sigma_3}}$ *and* $F_{w_{\Sigma_3}}$ *respectively generated by the vectors*

$$\begin{array}{l} v_{\Sigma_3} = V = Id - \tau_{12} - \tau_{13} - \tau_{23} + c + c^2 \text{ and} \\ w_{\Sigma_3} = W = Id + \tau_{12} + \tau_{13} + \tau_{23} + c + c^2. \end{array} \tag{1.3}$$

Of course, these vector spaces are irreducible.

Each subgroup of Σ_3 determines an invariant non-irreducible subspace. In fact, let

$$\begin{array}{l} G_1 = \{Id\}, \\ G_2 = \{Id, \tau_{12}\}, \\ G_3 = \{Id, \tau_{23}\}, \\ G_4 = \{Id, \tau_{13}\}, \\ G_5 = \{Id, c, c^2\} \text{ (the alternating group)}, \\ G_6 = \Sigma_3 \end{array} \tag{1.4}$$

be the subgroups of Σ_3. The linear spaces $\mathbb{K}[G_i]$ (which are the linear span of the vectors of G_i) are not invariant subspaces of $\mathbb{K}[\Sigma_3]$ for $i \neq 6$ because we have $\mathbb{K}[\mathcal{O}(\mathbb{K}[G_i])] = \mathbb{K}[\Sigma_3]$ for any $i = 1, \cdots, 6$. But to any subgroup G_i we can associate two vectors

$$\left\{ \begin{array}{l} v_{G_i} = \sum_{\sigma_j \in G_i} (-1)^{\varepsilon(\sigma_j)} \sigma_j, \\ w_{G_i} = \sum_{\sigma_j \in G_i} \sigma_j. \end{array} \right.$$

Each of these vectors defines an invariant subspace $F_{v_{G_i}} = \mathbb{K}[\mathcal{O}(v_{G_i})]$ and $F_{w_{G_i}} = \mathbb{K}[\mathcal{O}(w_{G_i})]$. For example if we consider $G_6 = \Sigma_3$ we get

$$\begin{array}{l} v_{\Sigma_3} = V = Id - \tau_{12} - \tau_{13} - \tau_{23} + c + c^2 \text{ and} \\ w_{\Sigma_3} = W = Id + \tau_{12} + \tau_{13} + \tau_{23} + c + c^2. \end{array} \tag{1.5}$$

The first one is related to the character of Σ_3 given by the signature and the second to the trivial one. It is easy to prove that

1. For $v_{G_1} = Id$, $\dim F_{v_{G_1}} = 6$,
2. For $v_{G_2} = Id - \tau_{12}$, $\dim F_{v_{G_2}} = 3$,
3. For $v_{G_3} = Id - \tau_{23}$, $\dim F_{v_{G_3}} = 3$,
4. For $v_{G_4} = Id - \tau_{13}$, $\dim F_{v_{G_4}} = 3$,
5. For $v_{G_5} = Id + c + c^2$, $\dim F_{v_{G_5}} = 2$.
6. Since $w_{G_1} = v_{G_1} = Id$, the dimension of $F_{w_{G_1}}$ is 6,
7. For $w_{G_2} = Id + \tau_{12}, w_{G_3} = Id + \tau_{23}$ and $w_{G_4} = Id + \tau_{13}$, the dimension of $F_{w_{G_i}}$ is 3,
8. Since $w_{G_5} = v_{G_5}$, the dimension of $F_{w_{G_5}}$ is 2.

In fact, $\dim F_{G_i} = \frac{|\Sigma_3|}{|G_i|}$ because G_i leads the vector v_{G_i} invariant.

1.2.2 The rank of the sub-modules of $\mathbb{K}[\Sigma_3]$

The aim of this section is to prove

Theorem 9 *Every Σ_3-submodule of $\mathbb{K}[\Sigma_3]$ is of rank one.*

This means that, if F is an invariant linear subspace of $\mathbb{K}[\Sigma_3]$, there is a vector v_F such that

$$F = \mathbb{K}[\mathcal{O}(v_F)].$$

For example, each subgroup of Σ_3 determines two modules of rank one generated by the vectors v_{G_i} and w_{G_i}.
To prove the theorem we investigate the structure of all invariant subspaces of $\mathbb{K}[\Sigma_3]$.

- **One dimensional invariant subspaces.** This case has been studied in the previous section and $F = F_V$ or $F = F_W$.

- **Two dimensional invariant subspaces**.

Proposition 2 *Every irreducible two-dimensional invariant subspace F is of type $F = F_{u_2^1}$ with*

$$u_2^1 = \lambda_1 Id - \lambda_1\tau_{12} + (\lambda_1+\lambda_3)\tau_{13} - \lambda_3\tau_{23} + \lambda_3 c - (\lambda_1+\lambda_3)c^2. \tag{1.6}$$

Every non irreducible 2-dimensional invariant subspace F is of type $F = F_{u_2^2} = F_V \oplus F_W$ with

$$u_2^2 = Id + c + c^2.$$

Proof. Let F be an irreducible 2-dimensional invariant subspace of $\mathbb{K}[\Sigma_3]$. Then neither V nor W belongs to F and if $v \in F$, then $F = F_v$. We first study a 2-dimensional invariant space F_v generated, as vector space, by v and τ_{12} and then the general case.

Lemma 1 *Let $v \in \mathbb{K}[\Sigma_3]$. If $F_v = F$ is a 2-dimensional irreducible invariant space generated by v and $\tau_{12}(v)$ then*

$$\begin{aligned} v &= a_1 Id + a_2\tau_{12} + (\alpha a_1 + \beta a_2)\tau_{13} - (\alpha a_1 + (1+\beta)a_2)\tau_{23} \\ &\quad + (\beta a_1 + \alpha a_2)c - ((1+\beta)a_1 + \alpha a_2)c^2 \end{aligned}$$

with $\alpha^2 = 1 + \beta + \beta^2$.

Proof of 1. Let $v = a_1 Id + a_2\tau_{12} + a_3\tau_{13} + a_4\tau_{23} + a_5 c + a_6 c^2$ be in $\mathbb{K}[\Sigma_3]$ and suppose that F is generated by v and $\tau_{12}(v)$. Since $\tau_{ij}(v) \in F$ the vector v satisfies

$$\sum_{i=1}^{6} a_i = 0.$$

Since $\{\tau_{1i}\}_{i=2,3}$ generates Σ_3, the $\mathbb{K}[\Sigma_3]$-module F is 2-dimensional if and only if $\{v, \tau_{12}(v)\}$ are independent and $\{v, \tau_{12}(v), \tau_{13}(v)\}$ are related. Let $\tau_{13}(v) = \alpha v + \beta\tau_{12}(v)$. This is equivalent to the following system

$$\begin{cases} Aa_1 = Ba_2,\ Aa_3 = Ba_6,\ Aa_4 = Ba_5, \\ Aa_2 = Ba_1,\ Aa_6 = Ba_3,\ Aa_5 = Ba_4, \\ A = 1 - \alpha^2 + \alpha^2\beta - \beta^3 \\ B = \alpha - \alpha^3 + \alpha\beta^2 + \alpha\beta, \end{cases}$$

which implies that $(A^2 - B^2)a_1 = (A^2 - B^2)a_2 = 0$.

1^{srt} case. $A^2 - B^2 \neq 0$. Then $a_1 = a_2 = 0$. Since $\tau_{ij}(v) \in F$ for $(i,j) \neq (1,2)$, we deduce that $v = 0$, which is impossible.

2^{nd} case. $A^2 - B^2 = 0$.

i) If $A = B$, that is, $1-\alpha^2+\alpha^2\beta-\beta^3 = \alpha(1-\alpha^2+\beta^2+\beta)$ we have $A(a_1-a_2)=0$. Then if $A \neq 0$, we obtain $a_1 = a_2, a_3 = a_6, a_4 = a_5$ and it implies that $v = \tau_{12}(v)$ which contradicts the hypothesis. Thus $A = B = 0$ and the coefficients α and β satisfy

$$\begin{cases} 1-\alpha^2+\alpha^2\beta-\beta^3 = 0, \\ \alpha(1-\alpha^2+\beta^2+\beta) = 0. \end{cases}$$

If $\alpha = 0,\ \beta = 1$ (here we choose $\mathbb{K} = \mathbb{R}$), we have $W \in F$ and F is not irreducible. Thus $\alpha \neq 0$ and

$$\begin{cases} 1-\alpha^2 = -\beta-\beta^2, \\ 1-\alpha^2 = -\beta(\alpha^2-\beta^2). \end{cases}$$

We will study successively the cases $\beta = 0$ and $\beta \neq 0$.

If $\beta = 0,\ \alpha = -1$ or $\alpha = 1$.

- If $\beta = 0$ and $\alpha = 1$ then $v = (a_1, a_2, a_1, -a_1-a_2, a_2, -a_1-a_2)$ in the basis $\left\{Id, \tau_{12}, \tau_{13}, \tau_{23}, c, c^2\right\}$.
- If $\beta = 0$ and $\alpha = -1$ then $v = (a_1, a_2, -a_1, a_1-a_2, -a_2, -a_1+a_2)$.

In fact, we write in this case $\tau_{13}(v) = v$ which implies

$$v = (a_1, a_2, a_1, a_4, a_2, a_4).$$

Since $\tau_{23}(v) = \alpha' v + \beta' \tau_{12}(v)$, we deduce that

$$(\alpha'-\beta')(a_2-a_1) = 0.$$

If $a_1 = a_2$, then $v = (a_1, a_1, a_1, a_4, a_1, a_4)$ with

$$\begin{cases} a_4 = (\alpha'+\beta')a_1, \\ a_1 = \alpha' a_1 + \beta' a_4 = \alpha' a_4 + \beta' a_1, \end{cases}$$

which gives that $\beta' = -1$ or $a_4 = a_1$. The second case corresponds to the vector W. Thus we have

$$v = (a_1, a_1, a_1, -2a_1, a_1, -2a_1),$$

because $-2-\alpha'+\alpha'^2 = 0$, which means that $\alpha' = -1$ (if $\alpha' = 2$ we would have $v = W$).

If $a_1 \neq a_2$ then $\alpha' = \beta'$ and we deduce that

$$\begin{cases} a_4 = \alpha'(a_1+a_2), \\ a_2 = \alpha'(a_1+a_4), \\ a_1 = \alpha'(a_2+a_4), \end{cases}$$

which implies that $(a_1+a_2+a_4)(1-2\alpha') = 0$. If $\alpha' = \frac{1}{2}$ we get $v = W$ so the system also implies $a_4 = -a_1-a_2$and the vector v has the expecting form.

The second case is similar to the previous one.

Suppose now that $\beta \neq 0$. Then the system between α and β is equivalent to a single equation

$$\alpha^2 = 1+\beta+\beta^2.$$

We deduce that $a_3 = \alpha a_1 + \beta a_2$, $a_5 = \alpha a_2 + \beta a_1$ and $\beta a_6 = (1-\alpha^2)a_1 - \alpha\beta a_2$. Since $\alpha^2 = 1+\beta+\beta^2$ and $\beta \neq 0$, we deduce that $a_6 = -(1+\beta)a_1 - \alpha a_2$. Likewise we have $a_4 = -\alpha a_1 - (1+\beta)a_2$. So

$$v = (a_1, a_2, \alpha a_1 + \beta a_2, -\alpha a_1 - (1+\beta)a_2, \beta a_1 + \alpha a_2, -(1+\beta)a_1 - \alpha a_2)$$

with $\alpha^2 = 1+\beta+\beta^2$.

ii) $A = -B$. If $A \neq 0$, then $a_1 = -a_2$, $a_3 = -a_6$ and $a_4 = -a_5$. This implies that $v = \tau_{12}(v)$, which contradicts the hypothesis. Then $A = 0$ and we come back to the previous case. $\square$

Let us investigate the general case. Assume that F is an irreducible two-dimensional subspace. Then

$$F = \mathbb{K}\{v, \tau(v)\} \text{ or } F = \mathbb{K}\{v, \gamma(v)\}$$

where τ is a transposition and γ a 3-cycle. But all the transpositions are conjugated in Σ_3. Then if $F = \mathbb{K}\{v, \tau(v)\}$ there exists $v' \in F$ such that $F_{v'} = F = \mathbb{K}\{v', \tau_{12}(v')\}$. In fact, $F = \mathbb{K}\{v, \tau_{23}(v)\} = \mathbb{K}\{c(v), \tau_{12}(c(v))\}$ and $v' = c(v)$ or $F = \mathbb{K}\{v, \tau_{13}(v)\} = \mathbb{K}\left\{c^2(v), \tau_{12}(c^2(v))\right\}$ and $v' = c^2(v)$. In this case the expression of v' is given by Proposition 2. Assume now that $F = \mathbb{K}\{v, \gamma(v)\}$. Since the transpositions generate Σ_3, the dependence of the vectors $\{v, \tau(v)\}$ for every transposition would implies the dependence of the vectors $\{v, \gamma(v)\}$ for every 3-cycle. Then $F = \mathbb{K}\{v, \gamma(v)\} = \mathbb{K}\{v, \tau(v)\}$ and we can find v' such that $F = F_{v'} = \mathbb{K}\{v', \tau_{12}(v')\}$. Thus we can suppose that any 2-dimensional invariant space F satisfies $F = F_v = \mathbb{K}\{v, \gamma(v)\}$ with v given by Proposition 2:

$$\begin{aligned} v \; &= a_1 Id + a_2\tau_{12} + (\alpha a_1 + \beta a_2)\tau_{13} - (\alpha a_1 + (1+\beta)a_2)\tau_{23} \\ &+(\beta a_1 + \alpha a_2)c - ((1+\beta)a_1 + \alpha a_2)c^2, \end{aligned}$$

with $\alpha^2 = 1 + \beta + \beta^2$. We now want to find a vector generating F as σ_3-module but whose components are "nicer". We have

$$\begin{aligned} \tau_{12}(v) \; &= a_2 Id + a_1\tau_{12} - ((1+\beta)a_1 + \alpha a_2)\tau_{13} + (\beta a_1 + \alpha a_2)\tau_{23} \\ &-(\alpha a_1 + (1+\beta)a_2)c + (\alpha a_1 + \beta a_2)c^2. \end{aligned}$$

So considering the vector $u = v - \tau_{12}(v)$ we observe that

$$\begin{aligned} \tau_{12}(u) &= -u, \\ \tau_{13}(u) &= \tau_{13}(v) - c(v) \\ &= \alpha v + \beta\tau_{12}(v) - \beta v - \alpha\tau_{12}(v) = (\alpha - \beta)(u) \\ \tau_{23}(u) &= \tau_{23}(v) - c^2(v) = -\alpha v - (1+\beta)\tau_{12}(v) - (1+\beta)v - \alpha\tau_{12}(v) \\ &= -(\alpha + \beta + 1)(v + \tau_{12}(v)). \end{aligned}$$

Thus $\{u, \tau_{23}(u)\}$ generate F_v if $\alpha + \beta + 1 \neq 0$. If $\alpha + \beta + 1 = 0$, we have that $\{u, c(u)\}$ generate F_v. So, in any case, $F_v = F_u$. But the vector u has the simplified following form

$$\begin{aligned} u \; &= (a_1 - a_2, a_2 - a_1, (\alpha+\beta+1)a_1 + (\alpha+\beta)a_2, -(\alpha+\beta)a_1 - (\alpha+\beta+1)a_2, \\ &(\alpha+\beta)a_1 + (\alpha+\beta+1)a_2, -(\alpha+\beta+1)a_1 - (\alpha+\beta)a_2) \\ &= (\lambda_1, -\lambda_1, \lambda_2, -\lambda_3, \lambda_3, -\lambda_2) \end{aligned}$$

with $\lambda_1 + \lambda_3 - \lambda_2 = 0$, that is,

$$u = (\lambda_1, -\lambda_1, \lambda_1 + \lambda_3, -\lambda_3, \lambda_3, -\lambda_1 - \lambda_3)$$

in the basis $\left\{Id, \tau_{12}, \tau_{13}, \tau_{23}, c, c^2\right\}$. Since

$$\begin{cases} \lambda_1 = a_1 - a_2, \\ \lambda_3 = (\alpha + \beta)\lambda_1 + a_1, \end{cases}$$

the condition $\alpha^2 = 1 + \beta + \beta^2$ does not give conditions on λ_1 and λ_3

• **Three dimensional invariant subspaces.** Since any invariant irreducible subspace is of dimension at most 2, every invariant space of dimension greater than or equal to 3 is a sum of irreducible spaces. Since $\dim F = 3$, then $F = F_V \oplus F_{u_2^1}$ or $F = F_W \oplus F_{u_2^1}$ with $F_{u_2^1}$ defined by (1.6).

Assume that $F = F_V \oplus F_{u_2^1}$ with $u_2^1 = \lambda_1 Id - \lambda_1\tau_{12} + (\lambda_1 + \lambda_3)\tau_{13} - \lambda_3\tau_{23} + \lambda_3 c - (\lambda_1 + \lambda_3)c^2$. If $\lambda_1 \neq 0$, considering the vector

$$v' = u_2^1 - \lambda_1 V = (0, 0, 2\lambda_1 + \lambda_3, \lambda_1 - \lambda_3, -\lambda_1 + \lambda_3, -2\lambda_1 - \lambda_3)$$

we have $\mathbb{K}[\mathcal{O}(v')] = \mathbb{K}\{v', \tau_{13}(v'), \tau_{23}(v')\}$. This space is 3-dimensional if $(b \neq a)$ or $(a = -b \neq 0)$ with $a = 2\lambda_1 + \lambda_3$ and $b = -\lambda_1 + \lambda_3$. which is the case as $\lambda_1 \neq 0$. So we obtain $F = F_{v'}$ with

$$v' = (0, 0, a, -b, b, -a).$$

Considering $v" = \tau_{13}(v') = (a, b, 0, -a, 0, -b)$, we have $\mathbb{K}\ [\mathcal{O}(v")] = \mathbb{K}[\mathcal{O}(v')] = F$. So, if $a \neq 0$, dividing by a and putting $t = \frac{b}{a}$, we obtain $F = F_v$ with $v = (1, t, 0, -1, 0, -t)$ and $t \neq 1$ ($t = 1$ corresponds to $\lambda_1 = 0$). If $a = 0$ and $b \neq 0$, we have $F = F_{u_3^1}$ with $u_3^1 = (0, -1, 0, 0, 0, 1)$. Finally if $\lambda_1 = 0$ we get $u_2^1 = (0, 0, \lambda_3, -\lambda_3, \lambda_3, -\lambda_3) = \lambda_3(0, 0, 1, -1, 1, -1)$ so $F = F_{u_3^2}$ with $u_3^2 = (1, -1, 0, -2, 2, 0)$. Moreover $F_{u_3^1} \neq F_{u_3^2}$.

Assume that $F = F_W \oplus F_{u_2^1}$. If $\lambda_1 \neq 0$, then for $v = u_2^1 - \lambda_1 W$ we have $\dim \mathbb{K}[\mathcal{O}(v)] = 3$ and $F = F_{u_3^3}$ with $u_3^3 = (0, -2, t, -1-t, -1+t, -2-t)$.

If $\lambda_1 = 0$, then $u_2^1 = (0, 0, \lambda_3, -\lambda_3, \lambda_3, -\lambda_3)$ and $F = F_{u_3^4}$ with $u_3^4 = \frac{u_2^1}{\lambda_3} + W = (1, 1, 2, 0, 2, 0)$. Moreover $F_{u_3^3} \neq F_{u_3^4}$.

Proposition 3 *Every 3-dimensional invariant subspace is of type $F_{u_3^i}$ with $i \in \{1, 2, 3\}$ and*

$$\begin{cases} u_3^1 = Id + t\tau_{12} - \tau_{23} - tc^2, & t \neq 1, \\ u_3^2 = Id - \tau_{12} - 2\tau_{23} + 2c, \\ u_3^3 = -2\tau_{12} + t\tau_{13} - (1+t)\tau_{23} + (-1+t)c - (2+t)c^2, \\ u_3^4 = Id + \tau_{12} + 2\tau_{23} + 2c. \end{cases}$$

Moreover $V \in F_{u_3^i}$ for $i = 1, 2$ and $W \in F_{u_3^i}$ for $i = 3, 4$.

- **Four dimensional invariant subspaces.** If $\dim F = 4$, then F is of type $F_{u_2^1} \oplus F_{u'^1_2}$ with $F_{u_2^1} \neq F_{u'^1_2}$ or $F_{u_2^1} \oplus F_V \oplus F_W = F_{u_2^1} \oplus F_{u_2^2}$.

In the first case, we consider $E = \{(a_1, a_2, a_3, -a_2 - a_3, a_5, -a_1 - a_5)\}$. If $v = (a_1, a_2, a_3, a_4, a_5, a_6)$ is in $F = F_{u_2^1} \oplus F_{u'^1_2}$, then $\sum \sigma(v) = \sum_{i=1}^6 a_i W \in F$ and , as $V \notin F$, it implies that $\sum_{i=1}^6 a_i = 0$. Similarly $Id(v) - \tau_{12}(v) - \tau_{13}(v) - \tau_{23}(v) + c(v) + c^2(v) = (a_1 - a_2 - a_3 - a - 4 + a_5 + a_6)V \in F$ but $V \notin F$ so $a_1 - a_2 - a_3 - a - 4 + a_5 + a_6 = 0$. Thus $v = (a_1, a_2, a_3, -a_2 - a_3, a_5, -a_1 - a_5)$ and $F \subset E$. But $\dim E = \dim F = 5$ so

$$F = \{(a_1, a_2, a_3, -a_2 - a_3, a_5, -a_1 - a_5)\}.$$

Considering $v = (2, 0, 1, -1, 0, -2)$ we verify that $v \in F$ and $\dim(\mathbb{K}[\mathcal{O}(v)] = 5$ thus $F = F_v$.

In the second case $F = F_u \oplus F_V \oplus F_W$ with F_u irreducible and 2-dimensional. The subspace $F_u \oplus F_V$ has been studied before. Let $v' = (1, t, 0, -1, 0, -t)$, $t \neq 1$ and $v = v' + W = (2, 1+t, 1, 0, 1, 1-t)$. We have that $\dim F_{v'} = 4$ (because $t \neq 1$) and $W = \frac{1}{2}(\tau_{23}(v) + v), V = (1-t)(-\frac{1}{2}v - \frac{3}{2}\tau_{12}(v) + \tau_{13}(v) + \tau_{23}(v))$. Thus $F = F_v$ with

$$v = 2Id + (1+t)\tau_{12} + \tau_{13} + c + (1-t)c^2.$$

Similar calculations applied to $v' = (1, -1, 0, -2, 2, 0)$ imply that $F = F_v$ with $v = (2, 0, 1, -1, 3, 1)$.

Proposition 4 *Every 4-dimensional invariant subspace is of type $F_{u_4^i}$ with $i \in \{1, 2, 3\}$ and*

$$\begin{cases} u_4^1 = 2Id + \tau_{13} - \tau_{23} - 2c^2, \\ u_4^2 = 2Id + (1+t)\tau_{12} + \tau_{13} + c + (1-t)c^2, & t \neq 1 \\ u_4^3 = 2Id + \tau_{13} - \tau_{23} + 3c - c^2. \end{cases}$$

Moreover V and W are in $F_{u_4^i}$ for $i = 2, 3$ and they do not belong to $F_{u_4^1}$.

- **Five dimensional invariant subspaces.** If $\dim F = 5$, then $F = F_{u_2^1} \oplus F_{u'^1_2} \oplus F_V$ or $F_{u_2^1} \oplus F_{u'^1_2} \oplus F_W$.

Assume that $F = F_{u_2^1} \oplus F_{u'^1_2} \oplus F_V$ with irreducible and 2-dimensional $F_{u_2^1}$ and $F_{u'^1_2}$. If $v \in F$ such that $v = \alpha_1 Id + \alpha_2\tau_{12} + \alpha_3\tau_{13} + \alpha_4\tau_{23} + \alpha_5 c + \alpha_6 c^2$, we have obviously

$$v + \tau_{12}(v) + \tau_{13}(v) + \tau_{23}(v) + c(v) + c^2(v) = \sum_{i=1}^{6} \alpha_i W \in F.$$

But $W \notin F$ then $\sum_{i=1}^{6} \alpha_i = 0$. So $v \in E = \{(a_1, a_2, a_3, a_4, a_5, a_6) \ /a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 0\}$ and $F \subset E$. But both F and E are 5-dimensional spaces so

$$F = \{(a_1, a_2, a_3, a_4, a_5, a_6) \ /a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 0\}.$$

Then there exists only one type of such subspace. A basis of F can be formed by the vectors

$$\{e_1 = (1, 0, 0, 0, 0, -1), e_2 = (0, 1, 0, 0, 0, -1), e_3 = (0, 0, 1, 0, 0, -1), \\ e_4 = (0, 0, 0, 1, 0, -1), e_5 = (0, 0, 0, 0, 1, 1)\}.$$

Consider for example the vector e_1 and compute $\mathcal{O}(e_1)$. We obtain that

$$\mathcal{O}(e_1) = \{e_1, \tau_{12}(e_1), \tau_{13}(e_1), c(e_1)\}$$

and dim $\mathbb{K}[\mathcal{O}(e_1)] = 4$. Since $V \notin \mathbb{K}[\mathcal{O}(e_1)]$ and dim $\mathbb{K}[\mathcal{O}(e_1 + V)] = 5$, then $F = F_v$ with $v = e_1 + V = (2, -1, -1, -1, 1, 0)$.

Assume that $F = F_{u_2^1} \oplus F_{u'^1_2} \oplus F_W$ and $v = (a_1, a_2, a_3, a_4, a_5, a_6) \in F$. Then

$$v - \tau_{12}(v) - \tau_{13}(v) - \tau_{23}(v) + c(v) + c^2(v) = (a_1 - a_2 - a_3 - a_4 + a_5 + a_6)V$$

is in F. Since $V \notin F$, we have that $a_1 - a_2 - a_3 - a_4 + a_5 + a_6 = 0$. It implies that

$$F = \{(a_1, a_2, a_3, a_4, a_5, a_6) \ /a_1 - a_2 - a_3 - a_4 + a_5 + a_6 = 0\}.$$

Let $e_1 = (1, 0, 0, 0, 0, -1)$ be in F_v. It is easy to see that $\mathbb{K}[\mathcal{O}(e_1)]$ is 4-dimensional. Then, considering $v = e_1 + W$, we verify that $\dim(\mathbb{K}[\mathcal{O}(v)]) = 5$ and

$$v = 2Id + \tau_{12} + \tau_{13} + \tau_{23} + c$$

is a generator of the module F.

Proposition 5 *Every 5-dimensional invariant subspace is of type $F_{u_5^i}$ with $i \in \{1, 2\}$ and*

$$\begin{cases} u_5^1 = 2Id - \tau_{12} - \tau_{13} - \tau_{23} + c, \\ u_5^2 = 2Id + \tau_{12} + \tau_{13} + \tau_{23} + c. \end{cases}$$

Moreover $V \in F_{u_5^1}$ and $W \in F_{u_5^2}$.

- **Six dimensional invariant subspaces.** If dim $F = 6$, then $F = \mathbb{K}[\Sigma_3]$. Let us consider the vector $e_1 = (1, 0, 0, 0, 0, 0)$. We have $\mathbb{K}[\mathcal{O}(e_1)] = \mathbb{K}[\Sigma_3] = F$. Then $F = F_v$ with $v = e_1$.

1.2.3 $\mathbb{K}[\Sigma_3]$-associative algebras

Let $(\mathcal{A}, \mu)$ be a $\mathbb{K}$-algebra. We have denoted by A_μ the associator of the product μ. Let v be a vector of $\mathbb{K}[\Sigma_3]$. We define the linear map

$$\Phi_v^{\mathcal{A}} : \mathcal{A}^{\otimes 3} \to \mathcal{A}^{\otimes 3}$$

by

$$\Phi_v^{\mathcal{A}}(x_1 \otimes x_2 \otimes x_3) = x_{\sigma^{-1}(1)} \otimes x_{\sigma^{-1}(2)} \otimes x_{\sigma^{-1}(3)}.$$

For example, if $v = Id$ then $\Phi_v^{\mathcal{A}} = Id_{\mathcal{A}^{\otimes 3}}$; if $v = Id - \tau_{12}$ then

$$\Phi_v^{\mathcal{A}}(x_1 \otimes x_2 \otimes x_3) = x_1 \otimes x_2 \otimes x_3 - x_2 \otimes x_1 \otimes x_3.$$

Definition 10 *An algebra $(\mathcal{A},\mu)$ is $\mathbb{K}[\Sigma_3]$-associative if there exists a nonzero vector v in $\mathbb{K}[\Sigma_3]$ such that*

$$A_\mu \circ \Phi_v^{\mathcal{A}} = 0.$$

Such a relation concerning a (non)-trivial associator will be called *a symmetric nonassociative relation.*
Example.

1. If $v = Id$ then $A_\mu \circ \Phi_v^{\mathcal{A}} = A_\mu = 0$ is the associativity. The corresponding algebra is associative.
2. If $v = Id - \tau_{12}$ then $A_\mu \circ \Phi_v^{\mathcal{A}} = 0$ leads to

$$\mu(\mu(x_1,x_2),x_3) - \mu(x_1,\mu(x_2,x_3)) - \mu(\mu(x_2,x_1),x_3) + \mu(x_2,\mu(x_1,x_3)) = 0$$

and $(\mathcal{A},\mu)$ is a Vinberg algebra.

Definition 11 *The symmetric relations of the associators of two $\mathbb{K}[\Sigma_3]$-associative algebras $(\mathcal{A},\mu)$ and $(\mathcal{B},\rho)$ are called equivalent if these relations $A_\mu \circ \Phi_v^{\mathcal{A}} = 0$ and $A_\rho \circ \Phi_w^{\mathcal{B}} = 0$ are given by vectors $v,w \in \mathbb{K}[\Sigma_3]$ such that*

$$\mathbb{K}[\mathcal{O}(v)] = \mathbb{K}[\mathcal{O}(w)].$$

From the previous paragraph each class corresponds to a symmetric relation on the associator and is associated to an invariant subspace of $\mathbb{K}[\Sigma_3]$.

Example. The Kosier-Osborn nonassociative algebras. The nonassociative algebras studied in [72] by Kosier and Osborn satisfy either

$$A_\mu \circ \Phi_V^{\mathcal{A}} = 0,$$

with

$$V = Id - \tau_{12} - \tau_{13} - \tau_{23} + c + c^2$$

or

$$A_\mu \circ \Phi_v^{\mathcal{A}} = 0,$$

with $v = \alpha_1 Id - \alpha_2\tau_{12} - \alpha_1\tau_{13} - \alpha_3\tau_{23} + \alpha_2 c + \alpha_3 c^2$.

1.2.4 G-associative algebras

Let $G_i,\ i = 1,\cdots,6$ be the subgroups of Σ_3 defined in (1.4). We can associated with each subgroup G_i of Σ_3 a Σ_3-invariant subspace of $\mathbb{K}[\Sigma_3]$ denoted by $F_{v_{G_i}}$ and $F_{w_{G_i}}$ and generated by the vectors v_{G_i} and w_{G_i} defined by (1.5).

Definition 12 *A G-associative algebra is a $\mathbb{K}$-algebra whose associator satisfies*

$$A_\mu \circ \Phi_{v_{G_i}}^{\mathcal{A}} = 0$$

for some $i \in \{1,\cdots,6\}$.

Then we have the following classes of G-associative algebras:

1. G_1-associative algebras = associative algebras,
2. G_2-associative algebras = Vinberg algebras,
3. G_3-associative algebras = pre-Lie algebras,
4. G_6-associative algebras = Lie-admissible algebras.

Recall that an algebra (A, μ) is called Lie-admissible if the skew symmetric map

$$[\ ,\ ] = \mu - \mu \circ \tau_{12}$$

is a Lie bracket. Jacobi identity is equivalent to

$$A_\mu \circ \Phi^{\mathcal{A}}_{v_{G_6}} = 0.$$

Remark. The G_5-associativity identity writes

$$A_\mu(X_1, X_2, X_3) + A_\mu(X_2, X_3, X_1) + A_\mu(X_3, X_1, X_2) = 0.$$

If μ is skew symmetric, this reduces to

$$\mu(\mu(X_1 \otimes X_2) \otimes X_3) + \mu(\mu(X_2 \otimes X_3) \otimes X_1) + \mu(\mu(X_3 \otimes X_1) \otimes X_2) = 0$$

and μ is a Lie algebra product. Then G_5-associative algebras can be considered as a non skew-symmetric version of Lie algebras.

1.2.5 Classification of $\mathbb{K}[\Sigma_3]$-associative Lie-admissible identities

A $\mathbb{K}[\Sigma_3]$-associative algebra defined by the identity

$$A_\mu \circ \Phi^{\mathcal{A}}_v = 0$$

is Lie-admissible if and only if $V = v_{G_6} \in F_v$. From the determination of invariant subspaces of $\mathbb{K}[\Sigma_3]$ given in 1.2 we deduce

Theorem 13 *Every Lie-admissible $\mathbb{K}[\Sigma_3]$-algebra belongs to one of the following cases:*

- *(I):* $A_\mu(x,y,z) - A_\mu(y,x,z) - A_\mu(z,y,x) - A_\mu(x,z,y) + A_\mu(y,z,x) + A_\mu(z,x,y) = 0.$
- *(II):* $A_\mu(x,y,z) + A_\mu(y,z,x) + A_\mu(z,x,y) = 0.$
- *(III):* $\alpha A_\mu(x,y,z) - \alpha A_\mu(y,x,z) + (\alpha+\beta-3)A_\mu(z,y,x) - \beta A_\mu(x,z,y) + \beta A_\mu(y,z,x)$ $+(3-\alpha-\beta)A_\mu(z,x,y) = 0,$ *with* $(\alpha,\beta) \neq (1,1).$
- *(IV_1):* $2A_\mu(x,y,z) + (1+\alpha)A_\mu(y,x,z) + A_\mu(z,y,x) + A_\mu(y,z,x) + (1-\alpha)A_\mu(z,x,y) = 0$ *with* $\alpha \neq 1.$
- *(IV_2):* $2A_\mu(x,y,z) + A_\mu(z,y,x) - A_\mu(x,z,y) + 3A_\mu(y,z,x) + A_\mu(z,x,y) = 0.$
- *(V):* $2A_\mu(x,y,z) - A_\mu(y,x,z) - A_\mu(z,y,x) - A_\mu(x,z,y) + A_\mu(y,z,x) = 0.$
- *(VI):* $A_\mu(x,y,z) = 0.$

Let us note that the case (III) contains, in this relation, the relations associated with the vectors u_3^1 and u_3^2. In particular, we have

1. G_2-associative algebras are of type (III) with $(\alpha,\beta) = (3,0)$.
2. G_3-associative algebras are of type (III) with $(\alpha,\beta) = (0,0)$.
3. G_4-associative algebras are of type (III) with $(\alpha,\beta) = (0,3)$.
4. G_5-associative algebras are of type (II).

1.2.6 Third-power associative algebras

Definition 14 *An algebra (A,μ) is called third-power associative if*

$$A_\mu \circ \Phi_W^{\mathcal{A}} = 0,$$

where $W = w_{\Sigma_3} = \sum_{\sigma\in\Sigma_3}\sigma$.

The class of third-power associative algebras have been introduced by Albert in [3]. An equivalent definition of $A_\mu \circ \Phi_W = 0$ is to write

$$A_\mu(x,x,x) = 0.$$

In [69] and [72] such an algebra is called "associative dependent algebra". An interesting case of third-power associative is the class of power-associative algebras. For such an algebra, every element generates an associative subalgebra.

A $\mathbb{K}[\Sigma_3]$-algebra defined by an identity

$$A_\mu \circ \Phi_v^{\mathcal{A}} = 0$$

is third-power associative if and only if $W \in F_v$.

Theorem 15 *Every third-power associative $\mathbb{K}[\Sigma_3]$-associative algebra corresponds to one of the following cases:*

- (I') : $A_\mu(x,y,z) + A_\mu(y,x,z) + A_\mu(z,y,x) + A_\mu(x,z,y) + A_\mu(y,z,x) + A_\mu(z,x,y) = 0$. *This identity defines the category of third-power associative algebras.*

- $(II') = (II)$.,

- (III') :
$\alpha A_\mu(x,y,z) + (2-\alpha)A_\mu(y,x,z) + (\alpha+\beta-1)A_\mu(z,y,x) + (2-\beta)A_\mu(x,z,y) + \beta A_\mu(y,z,x) + (3-\alpha-\beta)A_\mu(z,x,y) = 0$ *with* $(\alpha,\beta) \neq (1,1)$.

- $(IV_1') = (IV_1)$.

- $(IV_2') = (IV_2)$.

- (V')*:* $2A_\mu(x,y,z) + A_\mu(y,x,z) + A_\mu(z,y,x) + A_\mu(x,z,y) + A_\mu(y,z,x) = 0$.

- $(VI)' = (VI)$.

Example: Alternative algebras. Let us examine more particularly algebras of type (V'). Recall that an algebra $(\mathcal{A},\mu)$ is alternative if its product satisfies

$$A_\mu(x,x,y) = A_\mu(y,x,x) = 0.$$

This condition is equivalent to the following system

$$A_\mu \circ \Phi_{v_1} = A_\mu \circ \Phi_{v_2} = 0,$$

with $v_1 = Id + \tau_{12}$ and $v_2 = Id + \tau_{23}$. These vectors v_i belong to $\mathbb{K}\left(\mathcal{O}\left(u_5^2\right)\right)$ and the class of algebras of type (V') corresponds to the class of alternative algebras. Since $v = Id + \tau_{13} \in F_{u_5^2}$, any alternative algebra is also flexible, that is, $A_\mu(x,y,x) = 0$. In Chapter 5 we will look at these algebras from the viewpoint of operads. If $\mathcal{A}lt = (\mathcal{A}lt(n))_n$ denotes the operad associated to alternative algebras, the Moufang identities correspond to the relations in $\mathcal{A}lt(4)$.

Example: Power associative algebras. An algebra $(\mathcal{A},\mu)$ is power-associative if any element $x \in \mathcal{A}$ generates an associative algebra. Any power-associative algebra is, of course, a third-power associative algebra. In the $\mathbb{K}[\Sigma_3]$-cases, the third-power associative algebras which are power-associative correspond to types (III'), (IV_1),(IV_2),(V') and (VI). A well-known class of power-associative algebras is the (γ,δ)-Type [75] which arises in the study of classes of algebras having the property that if I is an ideal of the algebra, then I^2 is also an ideal of the algebra $\mathcal{A}$. A power-associative algebra satisfying this property is either associative, or of (γ,δ)-Type. They are defined by

$$\begin{cases} A_\mu(x,x,x)=0, \\ A_\mu(x,y,z)+A_\mu(y,z,x)+A_\mu(z,x,y)=0, \\ A_\mu(x,y,z)+\gamma A_\mu(y,x,z)+\delta A_\mu(z,x,y)=0, \ \gamma^2-\delta^2+\delta-1=0, \end{cases}$$

and correspond to the algebras of type (IV_1). They are non-flexible power-associative and Lie-admissible algebras.

1.3 $\mathbb{K}[\Sigma_3]^2$-associative algebras

The associator A_μ of μ can be decomposed as

$$A_\mu = A_\mu^L - A_\mu^R,$$

where $A_\mu^L(x_1,x_2,x_3)=\mu(\mu(x_1,x_2),x_3)$ and $A_\mu^R(x_1,x_2,x_3)=\mu(x_1,\mu(x_2,x_3))$.

Now, instead of considering an action of Σ_3 on the associator we can consider it independently on A_μ^L and A_μ^R which will induce different symmetries.

Definition 16 *The category of $(\mathbb{K}[\Sigma_3])^2$-associative algebras corresponds to the $\mathbb{K}$-algebras $(\mathcal{A},\mu)$ for which there exist $v,w\in\mathbb{K}[\Sigma_3],(v,w)\neq(0,0)$, such that*

$$A_\mu^L\circ\Phi_v=0, \quad A_\mu^R\circ\Phi_w=0, \tag{1.7}$$

or

$$A_\mu^L\circ\Phi_v - A_\mu^R\circ\Phi_w=0 \tag{1.8}$$

The reduction of these identities was first studied in [72]. The main result is

Proposition 6 *[72] Let $\mathcal{A}$ be an unitary $(\mathbb{K}[\Sigma_3])^2$-associative algebra. Then $\mathcal{A}$ is either Lie-admissible or satisfies*

$$A_\mu^L\circ\Phi_v - A_\mu^R\circ\Phi_w=0$$

with

$$v=\lambda_1 Id-\lambda_1\tau_{12}+(\lambda_1+\lambda_2)\tau_{13}-\lambda_2\tau_{23}+\lambda_2 c-(\lambda_1+\lambda_2)c^2$$

and

$$w=\mu_1 Id-\mu_1\tau_{12}+(\mu_1+\lambda_2)\tau_{13}-\lambda_2\tau_{23}+\lambda_2 c-(\mu_1+\lambda_2)c^2.$$

The vectors v and w generate two dimensional invariant spaces F_v and F_w. In the following we denote by $\sigma(v)$ the vector $\sum a_i\sigma\circ\sigma_i$ with $v=\sum a_i\sigma_i$.

Let $(\mathcal{A},\mu)$ be a $(\mathbb{K}[\Sigma_3])^2$-associative algebra given by the relation

$$A_\mu^L\circ\Phi_v - A_\mu^R\circ\Phi_w=0$$

with $v\neq 0$, $w\neq 0$. Suppose that $(\mathcal{A},\mu)$ is a Lie-admissible algebra. It is easy to see that $V=v_{\Sigma_3}\in F_v\cap F_w$. In this case the product μ satisfies

$$A_\mu\circ\Phi_V=0$$

and $(\mathcal{A},\mu)$ is a $(\mathbb{K}[\Sigma_3])$-associative algebra. Then there exists $\chi\in\mathbb{K}[\Sigma_3]$ such that

$$\chi(v)=\chi(w)=V.$$

Examples

1. Consider the $(\mathbb{K}[\Sigma_3])^2$-associative algebra whose product satisfies

$$x \cdot (y \cdot z) - x \cdot (z \cdot y) - (x \cdot y) \cdot z + (y \cdot x) \cdot z = 0.$$

Here $v = Id - \tau_{12}$ and $w = Id - \tau_{23}$. If $\chi = Id + c + c^2$, we have $\chi(v) = \chi(w) = V$ and the product is Lie-admissible. This example plays a particular role in the study of Hopf operad.

2. Consider the $(\mathbb{K}[\Sigma_3])^2$-associative algebra defined by

$$2(x \cdot y) \cdot z - (y \cdot x) \cdot z - (z \cdot y) \cdot x - (x \cdot z) \cdot y + (y \cdot z) \cdot x - x \cdot (y \cdot z) - y \cdot (z \cdot x) - z \cdot (x \cdot y) = 0.$$

Now $v = 2Id - \tau_{12} - \tau_{13} - \tau_{23} + c$ and $w = Id + c + c^2$. Here $\chi(w) = V$ implies $\chi = a_1 Id + a_2\tau_{12} + a_3\tau_{13} - (1 + a_2 + a_3)\tau_{23} + a_5 c + (1 - a_1 - a_5)c^2$. For this vector we have $\chi(v) \neq V$ and the $(\mathbb{K}[\Sigma_3])^2$-associative algebra is not Lie-admissible.

3. A Leibniz algebra [77] is a $\mathbb{K}[\Sigma_3]$-associative algebra A satisfying

$$x \cdot (y \cdot z) - (x \cdot y) \cdot z + (x \cdot z) \cdot y = 0$$

for all $x, y, z \in A$. Here $v = Id - \tau_{23}$ and $w = Id$. Such an algebra is neither Lie-admissible nor third-power associative.

Remarks.

1. The existence of $\chi \in \mathbb{K}[\Sigma_3]$ such that $\chi(v) = V$ and $\chi(w) \neq V$ doesn't imply the non Lie-admissibility of the algebra. For example, if we consider $v = Id - \tau_{12}$ then every χ such that $\chi(v) = V$ has the form $\chi = a_1 Id + (a_1 - 1)\tau_{12} + a_3\tau_{13} + a_4\tau_{23} + (a_3 + 1)c + (a_4 + 1)c^2$. Now consider $w = Id - \tau_{13}$. Every χ such that $\chi(w) = V$ is written $\chi = a_1 Id + (a_1 - 1)\tau_{12} + (a_1 - 1)\tau_{13} + (a_1 - 1)\tau_{23} + a_1 c + a_1 c^2$. In particular if $\chi = Id + \tau_{13} + 2c + c^2$ this vector satisfies $\chi(v) = V$ and $\chi(w) \neq V$. But the algebra $(\mathcal{A}, \mu)$ defined by

$$A^L_\mu \circ \Phi_v - A^R_\mu \circ \Phi_w = 0$$

is Lie-admissible.

2. Replacing V by W we obtain similar results for power-associative algebras.

Chapter 2

Poisson algebras viewed as Non Associative algebras

In this chapter, we describe an interesting example of nonassociative algebra where the nonassociative identity is equivalent to the three classical identities of a Poisson algebra. Then we have an original viewpoint of Poisson algebras. Recall that Poisson algebra is usually defined as a commutative associative algebra with a Lie bracket, these operations satisfying the Leibniz rule. The main idea is to use a polarization-depolarization method to associate with a general binary multiplication, an algebra provided with two products, the first one commutative and the second one skew-symmetric. Here, starting with the nonassociative identity $3A(X,Y,Z) = (X \cdot Z) \cdot Y + (Y \cdot Z) \cdot X - (Y \cdot X) \cdot Z - (Z \cdot X) \cdot Y$ and using this process we find the classical Poisson structure. The description of Poisson structures in terms of a single bilinear operation enables us to explore Poisson algebras in the realm of nonassociative algebras. To distinguish the classical Poisson algebras with these nonassociative algebras, we call these last Poisson-admissible algebras (although they are the same). Let us note that this method will be used in Chapter 4 to prove some (non)Koszul properties of the operad for Lie-admissible algebras.

In this chapter, we describe the polarization-depolarization process and we illustrate it considering classical algebras (associative algebras, G-associative algebras). Then we focus on Poisson algebras. Starting from the Poisson-admissible identity, we prove that Poisson-admissible algebras are flexible algebras. So we adapte Pierce decompositions. We have a natural notion of Radical and simplicity. We give the classification in dimension 2 and 3.

The classical cohomology deformation of Poisson algebras, classically called Lichnerowicz cohomology, considers only deformations of the Lie structure, leaving invariant the associative structure. Since in this chapter Poisson algebras are considered as nonassociative algebras, we consider the cohomology that governs general deformations of Poisson-admissible structures (both deformations of the Lie bracket and the associative product) and give the description of the spaces of degree 2 and 3. In fact, this cohomology corresponds to the operadic cohomology associated with the operad for Poisson-admissible algebras. We compare this cohomology with the Lichnerowicz one.

2.1 Introduction. Polarization, Depolarization

2.1.1 Polarization, Depolarization

Let $\mathbb{K}$ be a commutative field of characteristic different from 2 and 3. We are going to study classes of *algebras with one operation* $\cdot : V \otimes V \to V$ and axioms given as linear combinations of terms of the form $v_{\sigma(1)} \cdot (v_{\sigma(2)} \cdot v_{\sigma(3)})$ and/or $(v_{\sigma(1)} \cdot v_{\sigma(2)}) \cdot v_{\sigma(3)}$, where $\sigma \in \Sigma_3$ is a permutation. All classical examples of algebras, such as associative, commutative, Lie and, quite surprisingly, Poisson algebras, are of this type. The multiplication of $\cdot : V \otimes V \to V$ is

1. commutative, that is, $x \cdot y = y \cdot x$ for all $x, y \in V$,
2. anticommutative, that is, $x \cdot y = -y \cdot x$ for all $x, y \in V$,
3. without any symmetry, which means that there is no relation between $x \cdot y$ and $y \cdot x$.

Any multiplication $\cdot : V \otimes V \to V$ of type (3) can be decomposed into the sum of a commutative multiplication $\bullet$ and an anti-commutative one $[-,-]$ via the *polarization* given by

$$x \bullet y := \frac{1}{\sqrt{2}}(x \cdot y + y \cdot x) \quad \text{and} \quad [x, y] := \frac{1}{\sqrt{2}}(x \cdot y - y \cdot x), \tag{2.1}$$

for $x, y \in V$. The inverse process of *depolarization* assembles a type (1) multiplication $\bullet$ with a type (2) multiplication $[-,-]$ into

$$x \cdot y := \frac{1}{\sqrt{2}}(x \bullet y + [x, y]), \quad \text{for} \quad x, y \in V. \tag{2.2}$$

The coefficient $\frac{1}{\sqrt{2}}$ was chosen so that the polarization followed by the depolarization (and vice versa) is the identity. There are however other, less 'symmetric' choices of the coefficients with the same property, such as 1 in (2.1) and $\frac{1}{2}$ in (2.2), which do not need the assumption $\sqrt{2} \in \mathbb{K}$.

The polarization enables one to view structures with a type (3) multiplication (such as associative algebras in Example 2.1.2) as structures with one commutative and one anticommutative operation, while the depolarization interprets structures with one commutative and one anticommutative operation (such as Poisson algebras in Example 2.2.1) as structures with one type (3) operation. We will try to convince the reader that this change of perspective might sometimes lead to new insights and results.

2.1.2 Examples to warm up

In this section we give a couple of examples to illustrate the (de)polarization trick. We will usually omit the $\bullet$ denoting a commutative multiplication and write simply xy instead of $x \bullet y$.

Example 1. *Associative algebras* are traditionally understood as structures with one operation of type (3). If we polarize the multiplication $\cdot : V \otimes V \to V$, the associativity

$$(x \cdot y) \cdot z = x \cdot (y \cdot z), \quad \text{for } x, y, z \in V, \tag{2.3}$$

becomes equivalent to the following two axioms:

$$[x, yz] = [x, y]z + y[x, z], \tag{2.4}$$
$$[y, [x, z]] = (xy)z - x(yz). \tag{2.5}$$

Let us remark that the summation of (2.5) over cyclic permutations gives the Jacobi identity

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0. \tag{2.6}$$

Example 2. Recall [55] that a type (3) product $\cdot : V \otimes V \to V$ is called *Lie-admissible* if the commutator of this product is a Lie bracket or, equivalently, if the antisymmetric part $[-,-]$ of its polarization fulfills the Jacobi identity (2.6). This observation suggests that the polarization might be particularly suited for various types of Lie-admissible algebras.

Some important classes of Lie-admissible algebras where studied in Chapter 1. G_1*-associative algebras* are clearly associative algebras whose polarization we discussed in Example 2.1.2.

G_2-associative algebras and G_3-associative algebras are structures opposite to each other. We will therefore treat them as two versions of the same structure most often called a *pre-Lie algebra* in the literature [21, 41], although the G_2-version is sometimes called more specifically a *Vinberg* or *left-symmetric algebra* [112], while the G_3-version a *right-symmetric algebra* [94].

In the polarized form, G_2-associative algebras are structures with a commutative multiplication and a Lie bracket related by the axiom:

$$2[x,y]z+[[x,y],z]-x(yz)+y(xz)-x[y,z]+y[x,z]-[x,yz]+[y,xz]=0.$$

This axiom can be written as the sum

$$\begin{aligned} 0 &= \Big\{(xz)y - x(zy) - [z,[x,y]]\Big\} + \Big\{[x,y]z + y[x,z] - [x,yz]\Big\} \\ &\quad + \Big\{[y,xz] - x[y,z] - [y,x]z\Big\} \end{aligned}$$

of three terms which vanish separately if the multiplication is associative, see (2.4) and (2.5). The G_3-versions of the above formulas can be obtained by applying the transformation

$$xy \mapsto xy, \quad [x,y] \mapsto -[x,y]\,.$$

After polarizing, we identify G_4*-associative algebras* as structures satisfying

$$(xy)z - x(yz) = [[x,z],y],$$

which clearly implies the Jacobi (2.6). G_5*-associative algebras* have a commutative multiplication and a Lie bracket tied together by

$$[xy,z]+[yz,x]+[zx,y]=0. \tag{2.7}$$

The polarization of G_6*-associative algebras*, which are sometimes confusingly called just *Lie-admissible* algebras, reveals that the category of these objects consists of structures with a commutative multiplication and a Lie bracket, with no relation between these two operations.

Example 3. Lie-admissible structures mentioned in Example 2.1.2 are rather important. As it was shown in the seminal paper [41], there exists a natural pre-Lie structure on the Hochschild cochain complex of every associative algebra induced by a structure christened later, in [43], a *brace algebra*. This pre-Lie structure is responsible for the existence of the *intrinsic bracket* (also known as the *Gerstenhaber bracket*) on the Hochschild cohomology, see again [41].

We offer the following generalization of this structure. For a vector space V, denote by

$$X := \bigoplus_{m,n\geq 1} \mathrm{Lin}(V^{\otimes m}, V^{\otimes n})$$

the space of all multilinear maps. For $f \in \mathrm{Lin}(V^{\otimes b}, V^{\otimes a})$ and $g \in \mathrm{Lin}(V^{\otimes d}, V^{\otimes c})$ define

$$f \circ^i_j g \in \mathrm{Lin}(V^{\otimes a+c-1}, V^{\otimes b+d-1})$$

to be the map obtained by composing the j-th output of g into the i-th input of f and arranging the remaining outputs and inputs as indicated in Figure 2.1.2. Define finally

$$f \circ g := \sum_{1\leq i\leq b,\ 1\leq j\leq c} (-1)^{i(b+1)+j(c+1)} f \circ^i_j g.$$

We leave to the reader to verify that $(X,\circ)$ *is a* G_6*-associative algebra.*

Let $Y_{\mathrm{Hoch}} \subset X$ be the subspace $Y_{\mathrm{Hoch}} := \bigoplus_{m\geq 1} \mathrm{Lin}(V^{\otimes m}, V)$ and let dually $Y_{\mathrm{coHoch}} := \bigoplus_{n\geq 1} \mathrm{Lin}(V, V^{\otimes n})$. Clearly both Y_{Hoch} and Y_{coHoch} are $\circ$-closed. It turns out that $(Y_{\mathrm{Hoch}},\circ)$ is a G_3*-associative algebra* and $(Y_{\mathrm{coHoch}},\circ)$ a G_2*-associative algebra*. We recognize $(Y_{\mathrm{Hoch}},\circ)$ as the underlying space of the Hochschild cochain complex $C^*_{Hoch}(A;A)$ of an associative algebra $A=(V,\cdot)$ with the classical pre-Lie structure [41]. The space $(Y_{\mathrm{coHoch}},\circ)$ has a similar interpretation in terms of the Cartier cohomology of coassociative coalgebras [19].

To interpret X in a similar way, wee need to recall that an *infinitesimal bialgebra* [2] (also called a *mock bialgebra* in [33]) is a triple (V,μ,δ), where μ is an associative multiplication, δ is a coassociative comultiplication and

$$\delta(\mu(u,v)) = \delta_{(1)}(u)\otimes\mu(\delta_{(2)}(u),v) + \mu(u,\delta_{(1)}(v))\otimes\delta_{(2)}(v)$$

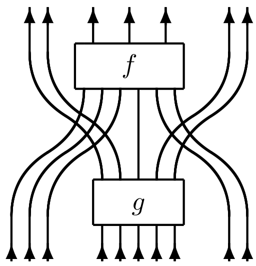


Figure 2.1: The composition $f \circ_3^4 g \in \mathrm{Lin}(V^{\otimes 10}, V^{\otimes 7})$ of functions $f \in \mathrm{Lin}(V^{\otimes 6}, V^{\otimes 3})$ and $g \in \mathrm{Lin}(V^{\otimes 5}, V^{\otimes 5})$.

for each $u, v \in V$, with the standard Sweedler's notation for the comultiplication used. It turns out that X is the underlying space of the cochain complex defining the cohomology of an infinitesimal bialgebra and $[f, g] := f \circ g - g \circ f$ is the intrinsic bracket, see [90], on this cochain complex. The reader is encouraged to verify that the axioms of infinitesimal bialgebras can be written as the 'master equation'

$$[\mu + \delta, \mu + \delta] = 0,$$

with $\mu : V \otimes V \to V$ and $\delta : V \to V \otimes V$ interpreted as elements of X.

Now we will develop the case of Poisson algebras so we use the usual notation $\{\ ,\ \}$ for the Poisson bracket instead of $[\ ,\ ]$. We consider (2.1) with the coefficient $\frac{1}{2}$, that is, $x \bullet y := \frac{1}{2}(x \cdot y + y \cdot x)$ and $[x, y] := \frac{1}{2}(x \cdot y - y \cdot x)$, and (2.2) with 1, i.e. $x \cdot y := x \bullet y + [x, y]$.

2.2 Nonassociative algebra associated to a Poisson algebra

2.2.1 Depolarization of Poisson identities

Poisson algebras are usually defined as structures with two operations, a commutative associative one, denoted by $\bullet$ and an anti-commutative one satisfying the Jacobi identity, denoted by $\{,\}$. These operations are tied up by a distributive law

$$\{x, y \bullet z\} = \{x, y\} \bullet z + y \bullet \{x, z\}$$

which we already saw in (2.4). The depolarization reinterprets Poisson algebras as structures with one type (3) operation $\cdot : V \otimes V \to V$ and one axiom:

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z - \frac{1}{3}((x \cdot z) \cdot y + (y \cdot z) \cdot x - (y \cdot x) \cdot z - (z \cdot x) \cdot y)).$$

Remark. In this example, the ground field will be the complex numbers $\mathbb{C}$. In an unpublished note, Livernet and Loday considered a one-parameter family of algebras with the axioms

$$\begin{aligned} [x,[y,z]] + [y,[z,x]] + [z,[x,y]] &= 0, \\ [x, yz] &= [x,y]z + y[x,z], && (2.8) \\ (xy)z - x(yz) &= q[y,[x,z]], && (2.9) \end{aligned}$$

depending on a complex parameter q. Observe that, for $q \neq 0$, the first axiom (the Jacobi identity) is implied by the third one. Let us call algebras satisfying the above axioms LL_q*-algebras*.

For $q = 0$, (2.9) becomes the associativity and we recognize the usual definition of Poisson algebras. If $q = 1$, we get associative algebras, in the polarized form of Example 2.1.2. Furthermore, one may also consider the limit for $q \to \infty$:

$$\begin{aligned}
[x,[y,z]]+[y,[z,x]]+[z,[x,y]] &= 0,\\
[x,yz] &= [x,y]z+y[x,z],\\
[y,[x,z]] &= 0.
\end{aligned}$$

In this case, the first identity trivially follows from the last one. These LL_∞-algebras are algebras with a two-step-nilpotent anticommutative bracket and a commutative multiplication, related by a distributive law (the second equation).

The depolarization allows one to interpret LL_q-algebras as algebras with one type (3) operation $\cdot : V \otimes V \to V$. One must distinguish two cases. For $q \neq -3$ we get the axiom

$$x\cdot(y\cdot z) = (x\cdot y)\cdot z + \frac{q-1}{q+3}\{(x\cdot z)\cdot y + (y\cdot z)\cdot x - (y\cdot x)\cdot z - (z\cdot x)\cdot y)\},$$

while for $q = -3$ we get a structure with three axioms

$$\begin{aligned}
(x\cdot z)\cdot y + (y\cdot z)\cdot x - (y\cdot x)\cdot z - (z\cdot x)\cdot y) &= 0,\\
A(x,y,z)+A(z,y,x) &= 0,\\
A(x,y,z)+A(y,z,x)+A(z,x,y) &= 0.
\end{aligned}$$

It can be easily verified that the formula

$$x\star y := \frac{1+\sqrt{q}}{2}x\cdot y + \frac{1-\sqrt{q}}{2}y\cdot x \tag{2.10}$$

converts LL_q-algebras, for $q \neq 0, \infty$, into associative algebras.

Observe that axiom (2.8) of LL_q-algebras implies axiom (2.7) of G_5-associative algebras, therefore LL_q-algebras form, for each q, a subcategory of the category of G_5-associative algebras. An equally simple observation is that the polarized product of G_4-associative algebras satisfies the first and the third identities of LL_{-1}-algebras but not the distributive law.

Remark. LL-algebras can be used to interpret deformation quantization of Poisson algebras. The ground ring here will be the ring $\mathbb{K}[[t]]$ of formal power series in t. Let us recall [9] that a $*$-*product* on a $\mathbb{K}$-vector space A is a $\mathbb{K}[[t]]$-linear associative unital multiplication $* : A[[t]] \otimes A[[t]] \to A[[t]]$ which is commutative mod t. Expanding, for $u, v \in A$,

$$u * v = u *_0 v + t\, u *_1 v + t^2\, u *_2 v + \cdots, \quad \text{with } u *_i v \in A \text{ for } i \geq 0,$$

one easily verifies that the operations $\cdot_0$ and $[-,-]_0$ defined by

$$u \cdot_0 v := u *_0 v \ \text{ and } \ [u,v]_0 := u *_1 v - v *_1 u,\ u, v \in A,$$

are such that $P := (A, \cdot_0, [-,-]_0)$ is a Poisson algebra. The object $(A[[t]], *)$ is sometimes also called the *deformation quantization* of the Poisson algebra P. In applications, P is the $\mathbb{R}$-algebra $C^\infty(M)$ of smooth functions on a Poisson manifold M that represents the phase space of a classical physical system. One moreover assumes that all products $*_i$, $i \geq 0$, in (2.2.1) are bilinear differential operators, see again [9] for details. The relevance of LL_q-algebras for quantization is explained in the following theorem.

Theorem 17 *A $*$-product on a $\mathbb{K}$-vector space A is the same as an LL_{t^2}-algebra structure on the $\mathbb{K}[[t]]$-module $V := A[[t]]$.*

Proof. Given a $*$-product, define $\bullet : V \otimes V \to V$ and $[-,-] : V \otimes V \to V$ as the polarization (2.1) of $* : V \otimes V \to V$. Commutativity of $*$ mod t means that $[-,-] = 0$ mod t, therefore there exists a bilinear antisymmetric map $\{-,-\} : V \otimes V \to V$ such that $[-,-] = t\{-,-\}$. It is immediate to check that $(V, \bullet, \{-,-\})$ is an LL_{t^2}-algebra. On the other hand, given an LL_{t^2}-algebra $(V, \bullet, \{-,-\})$, then

$$u * v := \frac{1}{\sqrt{2}}(u \bullet v + t\{u,v\}), \quad \text{for} \quad u, v \in V,$$

clearly defines a $*$-product on A.

2.2.2 Poisson-admissible algebras

To distinguish the classical Poisson algebras and their nonassociative version we introduce the following definition:

Definition 18 *A nonassociative $\mathbb{K}$-algebra $(\mathcal{P}, \cdot)$ whose associator satisfies*

$$3A(X,Y,Z) = (X \cdot Z) \cdot Y + (Y \cdot Z) \cdot X - (Y \cdot X) \cdot Z - (Z \cdot X) \cdot Y. \tag{2.11}$$

is called a Poisson-admissible algebra.

Let $(\mathcal{P}, \cdot)$ and $(\mathcal{P}, \star)$ be Poisson-admissible algebras defining the same Poisson algebra $(\mathcal{P}, \{,\}, \bullet)$. Then

$$\begin{array}{c} X \cdot Y - Y \cdot X = X \star Y - Y \star X = 2\{X,Y\} \\ X \cdot Y + Y \cdot X = X \star Y + Y \star X = 2X \bullet Y \end{array}$$

and $X \cdot Y = X \star Y$ because the characteristic of $\mathbb{K}$ is different from 2.
Notation. We will denote (when no confusion is possible) the Poisson-admissible product by XY instead of $X \cdot Y$.

Proposition 7 *A Poisson-admissible algebra $(\mathcal{P}, \cdot)$ is flexible, that is, the associator satisfies*

$$A(X,Y,X) = 0$$

for every $X, Y \in \mathcal{P}$.

Proof. From (2.11) we have

$$3A(X,Y,X) = X^2Y + (YX)X - (YX)X - X^2Y = 0$$

where $X^2 = XX$. Then $(\mathcal{P}, \cdot)$ is flexible. □

We deduce easily that the associator of the multiplication $\cdot$ satisfies

$$A(X,Y,Z) + A(Z,Y,X) = 0 \quad \text{(flexibility)} \tag{2.12}$$

$$A(X,Y,Z) + A(Y,Z,X) - A(Y,X,Z) = 0. \tag{2.13}$$

Last relation is obtained by writing identity (2.11) for the triples (X,Y,Z), (Y,Z,X) and (Y,X,Z).
Remark. The system $\{(2.12),(2.13)\}$ is equivalent to the equation

$$2A(X,Y,Z) + \frac{1}{2}A(Y,X,Z) + A(Z,Y,X) + A(Y,Z,X) + \frac{3}{2}A(Z,X,Y) = 0. \tag{2.14}$$

In fact, (2.12) + (2.13) implies (2.14). Conversely if (2.14) is satisfied, then (2.14) applied to the triple (X,Y,X) gives

$$2A(X,Y,X) + A(Y,X,X) + A(X,X,Y) = 0$$

and to the triple (X,X,Y)

$$5A(X,X,Y) + 5A(Y,X,X) + 2A(X,Y,X) = 0.$$

We deduce (2.12) and (2.13). It is worth noting that a nonassociative algebra satisfying (2.14) is not always a Poisson-admissible algebra.

Proposition 8 *A Poisson-admissible algebra $(\mathcal{P}, \cdot)$ is a power associative algebra.*

Proof. Recall that a nonassociative algebra is power associative if every element generates an associative subalgebra. Let X be in $(\mathcal{P},\cdot)$. We define the power of X by $X^1 = X$, $X^{i+1} = X \cdot X^i$. We will prove that $X^{i+n}X^{j-n} = X^{i-p}X^{j+p} = X^{i+j}$ for all $i,j \geq 1$ and $1 \leq p \leq i$, $1 \leq n \leq j$. Let $j \geq 1$. Since $(\mathcal{P},\cdot)$ is flexible, we have $A(X, X^{j-1}, X) = 0$ and Equation (2.11) gives $XX^j = X^jX$. Now we shall use induction over i to prove that $X^iX^j = X^jX^i$. This identity is trivial for $i = 1$. Suppose that it is satisfied for some $i \geq 1$. Then relation (2.11) gives

$$3A(X, X^i, X^j) - (XX^j)X^i + (X^iX^j)X - (X^iX)X^j - (X^jX)X^i = 0$$

and as $X^iX^j = X^jX^i$, we obtain

$$4X^{i+1}X^j = 3X(X^iX^j) + (X^iX^j)X.$$

Similarly, (2.11) applied to the triple (X, X^j, X^i) gives

$$4X^{j+1}X^i = 3X(X^jX^i) + (X^jX^i)X.$$

From the assumption $X^iX^j = X^jX^i$, we obtain $X^{j+1}X^i = X^{i+1}X^j$. By (2.11), this implies $A(X^j, X, X^i) = 0$. Thus,

$$X^{j+1}X^i = X^jX^{i+1} = X^{i+1}X^j$$

and $X^iX^j = X^jX^i$ for all i, j. Finally, we prove that for fixed i the relation $X^{i-p}X^p = X^i$ is satisfied for any $1 \leq p \leq i$. It is evident for $p = 1$. Take p such that $1 < p < i$, and suppose that we have $X^{i-p}X^p = X^i$. Then

$$\begin{aligned} 3A(X^{i-p-1}, X, X^p) &= (X^{i-p-1}X^p)X + (X^{p+1})X^{i-p-1} - (X^{i-p})X^p \\ &-(X^pX^{i-p-1})X. \end{aligned}$$

By assumption $X^{i-p}X^p = X^i = X^pX^{i-p}$, thus this relation gives

$$X^{p+1}X^{i-p-1} = X^{i-p}X^p = X^i$$

and the algebra $(\mathcal{P},\cdot)$ is power associative. □

Remark. *Poisson algebras as $\mathbb{K}[\Sigma_3]$-associative algebras.*

In [56], we show that Poisson-admissible algebras belong to some class of $\mathbb{K}[\Sigma_3]$-associative algebras.

If $(\mathcal{P},\cdot)$ is a Poisson-admissible algebra, we see from (2.11) that the associator of the multiplication satisfies

$$A_\mu \circ \Phi_{v_1} = 0$$

for $v_1 = Id - \tau_{12} + c$. The flexibility identity (2.12) can be written as $A_\mu \circ \Phi_{v_2} = 0$ for $v_2 = Id + \tau_{13}$. From the classification of Theorem 13 , we deduce that any Poisson algebra is an algebra of type (IV_1) for $\alpha = -\frac{1}{2}$ (we have $v = 2Id + \frac{1}{2}\tau_{12} + \tau_{13} + c + \frac{3}{2}c^2$ and F_v is 4-dimensional).

2.3 Algebraic study of Poisson-admissible algebras

If $(\mathcal{P},\cdot)$ is a Poisson-admissible algebra, we denote by $\mathcal{A}_{\mathcal{P}}$ the commutative associative algebra with multiplication $x \bullet y := \frac{1}{2}(x \cdot y + y \cdot x)$ and $\mathfrak{g}_{\mathcal{P}}$ the Lie algebra with bracket $[x, y] := \frac{1}{2}(x \cdot y - y \cdot x)$.

2.3.1 Pierce decomposition

A power associative algebra $(\mathcal{P},\cdot)$ is a nilalgebra if any element X is nilpotent, i.e.

$$\forall X \in \mathcal{P}, \exists r \in \mathbb{N} \text{ such that } X^r = 0.$$

Proposition 9 *Any finite dimensional Poisson-admissible algebra which is not a nilalgebra contains a non-zero idempotent.*

Proof. This is a consequence of the power associativity of a Poisson algebra.

Let e be a non-zero idempotent, i.e. $e^2 = e$. But $e \bullet e = \frac{1}{2}(e \cdot e + e \cdot e)$ so $e \bullet e = e$ and e is also an idempotent of the associative algebra $\mathcal{A}_{\mathcal{P}}$. The Leibniz identity implies

$$\{e, x\} = \{e \bullet e, x\} = 2e \bullet \{e, x\}.$$

Therefore, $\{e, x\}$ is either zero or an eigenvector of the operator

$$L^{\bullet}_{e} : x \to e \bullet x$$

in $\mathcal{A}_{\mathcal{P}}$ associated to the eigenvalue $\frac{1}{2}$. Since e is an idempotent, the eigenvalues associated to $L^{\bullet}_{e}$ are 1 or 0. It follows that $\{e, x\} = 0$ which implies that e belongs to $Z(\mathfrak{g}_{\mathcal{P}})$, the center of the Lie algebra $\mathfrak{g}_{\mathcal{P}}$, and $e \cdot x = e \bullet x = x \bullet e = x \cdot e$.

Proposition 10 *Let $(\mathcal{P}, \cdot)$ be a Poisson-admissible algebra such that the center of the associated Lie algebra $\mathfrak{g}_{\mathcal{P}}$ is zero. Then $(\mathcal{P}, \cdot)$ has no idempotent different from zero. If $\mathcal{P}$ is finite dimensional then it is a nilalgebra.*

Suppose that there exists an idempotent $e \neq 0$. Since $\mathcal{P}$ is flexible, the operators $L^{\bullet}_{e}$ and $R^{\bullet}_{e}$ defined by $L^{\bullet}_{e}(x) = e \bullet x$ and $R^{\bullet}_{e}(x) = x \bullet e$ commute and $L^{\bullet}_{e} = L^{\cdot}_{e}, R^{\bullet}_{e} = R^{\cdot}_{e}$. Then $\mathcal{P}$ decomposes as

$$\mathcal{P} = \mathcal{P}_{0,0} \oplus \mathcal{P}_{0,1} \oplus \mathcal{P}_{1,0} \oplus \mathcal{P}_{1,1},$$

with $\mathcal{P}_{i,j} = \{x_{i,j} \in \mathcal{P} \text{ such that } ex_{i,j} = ix_{i,j},\, x_{i,j}e = jx_{i,j}\}$, $i, j \in \{0, 1\}$. From Proposition 9, $e \in Z(\mathfrak{g}_{\mathcal{P}})$. So $\{e, x\} = 0$ for any x, that is, $ex = xe$ and $\mathcal{P}_{0,1} = \mathcal{P}_{1,0} = \{0\}$.

Proposition 11 *If the Poisson-admissible algebra $(\mathcal{P}, \cdot)$ has a non-zero idempotent, it admits the Pierce decomposition*

$$\mathcal{P} = \mathcal{P}_{0,0} \oplus \mathcal{P}_{1,1},$$

where $\mathcal{P}_{0,0}$ and $\mathcal{P}_{1,1}$ are Poisson-admissible algebras with the induced product.

Proof. We have to show that $\mathcal{P}_{0,0}$ and $\mathcal{P}_{1,1}$ are Poisson subalgebras. Let $x, y \in \mathcal{P}_{0,0}$, then $ex = ey = xe = ye = 0$. From (2.11), we obtain

$$\begin{cases} -3e(xy) &= (xy)e, \\ 0 &= (xy)e - (yx)e, \\ 3(xy)e &= -(yx)e. \end{cases}$$

So $(xy)e = -3e(xy) = (yx)e = -3(xy)e$ and $(xy)e = e(xy) = 0$. Then $xy \in \mathcal{P}_{0,0}$. Similarly if $x, y \in \mathcal{P}_{1,1}$, then (2.11) applied to the triple (e, x, y) gives $xy = e(xy)$. The same equation applied to (x, e, y) and (x, y, e) gives

$$\begin{cases} (xy)e + yx - xy - (yx)e &= 0, \\ 3(xy)e - 3xy - yx + (yx)e &= 0. \end{cases}$$

Thus, $4(xy)e - 4xy = 0$ which means that $(xy)e = xy$ and $\mathcal{P}_{1,1}$ is a Poisson subalgebra of $(\mathcal{P}, \cdot)$ □

Remark. Poisson algebras are Lie-admissible power-associative algebras. In [73] Kosier gave examples of simple Lie-admissible power-associative finite-dimensional algebras called anti-flexible algebras. These algebras also have the property $A = A_{0,0} \oplus A_{1,1}$ in every Pierce decomposition.

2.3.2 Pierce decomposition associated to orthogonal idempotents

Let e_1 and e_2 be non-zero independent orthogonal idempotents, $e_1 e_2 = e_2 e_1 = 0$. Let $\mathcal{P} = \mathcal{P}^1_{0,0} \oplus \mathcal{P}^1_{1,1} = \mathcal{P}^2_{0,0} \oplus \mathcal{P}^2_{1,1}$ be the corresponding Pierce decompositions. Let us suppose that $x \in \mathcal{P}^1_{0,0}$. Applying (2.11) to the triples associated to the elements $\{e_1, e_2, x\}$, we obtain the condition

$$(xe_2)e_1 = (e_2 x)e_1 = e_1(e_2 x) = e_1(xe_2) = 0,$$

for the elements xe_2 and e_2x in $\mathcal{P}^1_{0,0}$. In other words,

$$L_{e_2}(\mathcal{P}^1_{0,0}) \subset \mathcal{P}^1_{0,0}, \quad R_{e_2}(\mathcal{P}^1_{0,0}) \subset \mathcal{P}^1_{0,0},$$

where $L_{e_2}(x) = e_2x$ and $R_{e_2}x = xe_2$. So, e_2 is an idempotent of the Poisson algebra $(\mathcal{P}^1_{0,0}, \cdot)$. Thus we have

$$\mathcal{P}^1_{0,0} = \mathcal{P}^1_{0,0} \cap \mathcal{P}^2_{0,0} \oplus \mathcal{P}^1_{0,0} \cap \mathcal{P}^2_{1,1}.$$

Using the same reasoning, we can show that if $x \in \mathcal{P}^1_{1,1}$ then, $e_2x = xe_2 = 0$ and

$$\mathcal{P}^1_{1,1} = \mathcal{P}^1_{1,1} \cap \mathcal{P}^2_{0,0} \oplus \mathcal{P}^1_{1,1} \cap \mathcal{P}^2_{1,1}.$$

But $\mathcal{P}^2_{1,1} \subset \mathcal{P}^1_{0,0}$ so that $\mathcal{P}^1_{1,1} \cap \mathcal{P}^2_{1,1} = \{0\}$ and $\mathcal{P}^1_{1,1} \cap \mathcal{P}^2_{0,0} = \mathcal{P}^1_{1,1}$. Then,

$$\mathcal{P} = \mathcal{P}^1_{0,0} \cap \mathcal{P}^2_{0,0} \oplus \mathcal{P}^1_{1,1} \oplus \mathcal{P}^2_{1,1}.$$

Proposition 12 *If e_1 and e_2 are non-zero orthogonal idempotents, then $\mathcal{P}$ decomposes into a direct sum of Poisson subalgebras,*

$$\mathcal{P} = \mathcal{P}^1_{0,0} \cap \mathcal{P}^2_{0,0} \oplus \mathcal{P}^1_{1,1} \oplus \mathcal{P}^2_{1,1}.$$

Proposition 6.19 can be easily generalized to a family of orthogonal idempotents $\{e_1, \cdots, e_k\}$. The corresponding decomposition can then be written as

$$\mathcal{P} = \cap_{i=1}^k \mathcal{P}^i_{0,0} \oplus_{i=1}^k \mathcal{P}^j_{1,1}.$$

2.3.3 Radical of a Poisson algebra

We already know that a Poisson algebra $(\mathcal{P}, \cdot)$ is power associative. Recall that an element $x \in \mathcal{P}$ is nilpotent if there is an integer r such that $x^r = 0$. An algebra (resp. a two-sided ideal) consisting only of nilpotent elements is called a nilalgebra (resp. a nilideal). If $\mathcal{P}$ is a finite dimensional Poisson algebra, then there is a unique maximal nilideal $\mathcal{N}(\mathcal{P})$ called the nilradical. Let $\mathcal{A}_\mathcal{P}$ be the commutative associative algebra associated to $(\mathcal{P}, \cdot)$. Then, the Jacobson radical $J(\mathcal{A}_\mathcal{P})$ of $\mathcal{A}_\mathcal{P}$ contains $\mathcal{N}(\mathcal{P})$. Since $\mathcal{N}(\mathcal{P})$ is a two-sided ideal of $(\mathcal{P}, \cdot)$, it is also a Lie ideal of $\mathfrak{g}_\mathcal{P}$. One can easily prove:

Proposition 13 *The nilradical $\mathcal{N}(\mathcal{P})$ of $(\mathcal{P}, \cdot)$ coincides with the maximal Lie ideal of $\mathfrak{g}_\mathcal{P}$ contained in $\mathcal{J}(\mathcal{A}_\mathcal{P})$.*

Remarks.

- In the category of associative algebras, or more generally, of alternative algebras, any nilalgebra is nilpotent. This is no longer true in the category of Poisson algebras as the following example shows.

  Let $(\mathcal{P}, \cdot)$ be the 3-dimensional algebra defined by

  $$\begin{cases} e_i^2 = 0, \\ e_1e_2 = -e_2e_1 = e_2, \\ e_1e_3 = -e_3e_1 = -e_3, \\ e_2e_3 = -e_3e_2 = e_1. \end{cases}$$

  The corresponding algebra $\mathcal{A}_\mathcal{P}$ is abelian and any element of $\mathcal{P}$ is nilpotent. The Poisson algebra $\mathcal{P}$ is a nilalgebra. But $\mathcal{P}^2 = \mathcal{P}$ so $\mathcal{P}$ is not a nilpotent algebra. This algebra is an example of simple nilalgebra.

- An element $x \in \mathcal{P}$ is *properly nilpotent* if it is nilpotent and xy and yx are nilpotent for any $y \in \mathcal{P}$. The Jacobson radical of $\mathcal{A}_\mathcal{P}$ coincides with the set of properly nilpotent elements of $\mathcal{A}_\mathcal{P}$. Let x be a properly nilpotent element of $\mathcal{P}$ and suppose that $x \notin \mathcal{N}(\mathcal{P})$. We know that $x \in \mathcal{J}(\mathcal{A}_\mathcal{P})$. By Proposition 13, there exists $y \in \mathcal{P}$ such that $\{x, y\} \notin \mathcal{N}(\mathcal{P})$. We have $x \bullet y \in \mathcal{J}(\mathcal{A}_\mathcal{P})$. This implies that $\{x, y\} \notin \mathcal{J}(\mathcal{A}_\mathcal{P})$, otherwise $xy \in \mathcal{J}(\mathcal{A}_\mathcal{P})$ and $\mathcal{N}(\mathcal{P})$ would not be maximal. But $x \in \mathcal{J}(\mathcal{A}_\mathcal{P})$, so xy is nilpotent and $xy \in \mathcal{J}(\mathcal{A}_\mathcal{P})$. This is a contradiction and the nilradical coincides with the set of properly nilpotent elements. Zorn's theorem concerning nilalgebra still holds in the framework of Poisson algebras.

- We have seen that any finite dimensional Poisson algebra which is not a nilalgebra contains a non-zero idempotent. An idempotent e is *principal* if there is no idempotent u orthogonal to e (i.e. $ue = eu = 0$ with $u^2 = u \neq 0$). If $(\mathcal{P}, \cdot)$ is not a nilalgebra, $\mathcal{A}_{\mathcal{P}}$ is not a nilalgebra and it has a principal idempotent element. Let e be such an element. As $e^2 = e \bullet e = e$, it is an idempotent element of $\mathcal{P}$. If one can find u such that $u^2 = u \bullet u = u$ with $ue = eu = 0$, then $u \bullet e = e \bullet u = 0$ which is impossible. Therefore we have:

  Proposition 14 *Any finite dimensional Poisson-admissible algebra which is not a nilalgebra contains a principal idempotent element.*

- Let us assume that $\mathcal{P}$ is a unitary algebra. If x is an invertible element of $\mathcal{P}$, there exists $x^{-1} \in \mathcal{P}$ such that $xx^{-1} = x^{-1}x = 1$. In particular $x \bullet x^{-1} = x^{-1} \bullet x = 1$ and x^{-1} is the inverse of x in $\mathcal{A}_{\mathcal{P}}$. Thus the inverse of an invertible element of $\mathcal{P}$ is unique. Let us note that if $\mathcal{P}$ is unitary, finite dimensional and if the unit is the only idempotent element, any non-nilpotent element is invertible. In fact, such an element x generates an associative algebra which admits an idempotent. Then $1 \in \mathcal{P}$, which turns out to be the only idempotent and can be expressed as

$$1 = \sum \alpha_i x^i = x(\sum \alpha_i x^{i-1}).$$

  It follows that $\sum \alpha_i x^{i-1}$ is the inverse of x.

2.3.4 Simple Poisson algebras

A Poisson-admissible algebra $(\mathcal{P}, \cdot)$ is *simple* if it has not some proper ideal and if $\mathcal{P}^2 \neq \{0\}$. Let L_x and R_x be the left and right translations by $x \in \mathcal{P}$. Let $\mathcal{M}(\mathcal{P})$ be the associative subalgebra of $End(\mathcal{P})$ generated by L_x, R_x for $x \in \mathcal{P}$. In this algebra, we have the following relations

$$\begin{cases} L_x \bullet R_x = R_x \bullet L_x, \\ 4L_{x^2} = 3(L_x)^2 - (R_x)^2 + 2R_x \bullet L_x, \\ 4R_{x^2} = 3(R_x)^2 - (L_x)^2 + 2R_x \bullet L_x. \end{cases}$$

The algebra $\mathcal{P}$ is simple if and only if $\mathcal{P}$ is a non-trivial irreducible $\mathcal{M}(\mathcal{P})$-module.

One can consider the centralizer $\tilde{\mathcal{C}}$ of $\mathcal{M}(\mathcal{P})$ in $End(\mathcal{P})$. If $\mathcal{P}$ is simple and if $\tilde{\mathcal{C}}$ is non-trivial, then $\tilde{\mathcal{C}}$ is a field which is a central simple Poisson algebra over itself.

Remark. We saw in Remarks in 2.3.3 that there are Poisson-admissible algebras which are nilalgebras. In this case $\mathcal{N}(\mathcal{P})$ is non-zero. We can consider the Albert radical $\mathcal{R}(\mathcal{P})$ defined as the intersection of all maximal ideals $\mathcal{M}$ of $\mathcal{P}$ such that $\mathcal{P}^2 \not\subset \mathcal{M}$. In the algebra defined in Remark 2.3.3, $\mathcal{P}^2 = \mathcal{P}$. If $\mathcal{M}$ is maximal and satisfies $\mathcal{M} \subseteq \mathcal{P}^2$ and $\mathcal{M} \neq \mathcal{P}^2$, then $\mathcal{M} = \{0\}$. The Albert radical is $\{0\}$ which implies the semi-simplicity of $\mathcal{P}$.

Proposition 15 *If $(\mathcal{P}, \cdot)$ is a simple nilalgebra such that $x^2 = 0$ for all $x \in \mathcal{P}$ then $\mathcal{A}_{\mathcal{P}}$ is an associative nilalgebra satisfying $(\mathcal{A}_{\mathcal{P}})^2 = 0$.*

Proof. The subalgebra $\mathcal{P}^2 = \{xy, x, y \in \mathcal{P}\}$ is an ideal of $\mathcal{P}$, so $\mathcal{P}^2 = \mathcal{P}$. By hypothesis, for every $x \in \mathcal{P}$ we have $x^2 = 0$. Then

$$(x + y)^2 = x^2 + y^2 + xy + yx = xy + yx = 0$$

for all $x, y \in \mathcal{P}^2$. This implies

$$x \bullet y = \frac{1}{2}(xy + yx) = 0$$

thus the associative algebra $\mathcal{A}_{\mathcal{P}}$ is trivial.

We can also consider simple Poisson-admissible algebras which are not nilalgebras. In this case the Albert radical is $\{0\}$ and $\mathcal{P}^2 \neq 0$.

Proposition 16 *Let $(\mathcal{P},\cdot)$ be a finite dimensional simple Poisson-admissible algebra which is not a nilalgebra. Then it has a unit element.*

Proof. In fact, $\mathcal{P}$ has a principal idempotent e. Its Pierce decomposition $\mathcal{P}=\mathcal{P}_{0,0}\oplus\mathcal{P}_{1,1}$ is such that $\mathcal{P}_{0,0}\subset\mathcal{R}(\mathcal{P})$. Then $\mathcal{P}_{0,0}=\{0\}$ and $\mathcal{P}=\mathcal{P}_{1,1}$. Therefore, $e=1$.

2.4 Simple complex Poisson algebras such that $\mathfrak{g}_{\mathcal{P}}$ is simple

Lemma 2 *Let $(\mathcal{P},\cdot)$ be a Poisson-admissible algebra. If $\mathfrak{g}_{\mathcal{P}}$ is a simple Lie algebra then $\mathcal{P}$ is a simple algebra.*

Proof. If $I\subsetneq\mathcal{P}$ is an ideal of $\mathcal{P}$, then I is also an ideal of $\mathfrak{g}_{\mathcal{P}}$ so I must be trivial.

Proposition 17 *If $\mathfrak{g}_{\mathcal{P}}$ is a simple complex Lie algebra, then $UV=\{U,V\}$ for all $U,V\in\mathcal{P}$, that is, the associative algebra $\mathcal{A}_{\mathcal{P}}$ satisfies $\mathcal{A}_{\mathcal{P}}^2=\{0\}$.*

Proof. Let $\mathfrak{g}_{\mathcal{P}}$ be a simple complex Lie algebra of rank r. Let $\mathfrak{n}_-\oplus\mathfrak{h}\oplus\mathfrak{n}_+$ be its root-decomposition, where $\mathfrak{h}$ is a Cartan subalgebra. Let $\{Y_j,H_i,X_j\}$ be the corresponding Weyl basis. Since $\{H_k^2,H_i\}=0$ for all $i=1,\cdots,r$ we deduce that

$$H_k^2\in\mathfrak{h},\quad k=1,\cdots,r.$$

Thus, $H_k^2=\sum_{i=1}^r\alpha_k^iH_i$. Let us put $\{H_k,X_j\}=\rho_{k,j}X_j$. We obtain

$$\{H_k,X_j^2\}=2\rho_{k,j}X_j^2$$

for all $k=1,\cdots,r$. Thus $2\rho_{k,j}$ are also roots of $\mathfrak{g}_{\mathcal{P}}$, but this is impossible so $X_j^2=0$ for every j. Similarly we have for all $k=1,\cdots,r$

$$\{H_k,X_j\bullet X_i\}=(\rho_{k,j}+\rho_{k,j})X_j\bullet X_i$$

so $(\rho_{k,j}+\rho_{k,j})$ are roots. This implies

$$X_j\bullet X_i=0.$$

In the same way we have

$$Y_j\bullet Y_i=0$$

for all i,j. It turns out that

$$\{H_k^2,X_j\}=2H_k\bullet\{H_k,X_j\}=2\rho_{k,j}H_k\bullet X_j=\sum_{i=1}^r\alpha_k^i\rho_{i,j}X_j$$

and

$$\rho_{k,i}H_k\bullet X_j=\frac{1}{2}(\sum_{i=1}^r\alpha_k^i\rho_{i,j})X_j.$$

For any j there is k such that $\rho_{k,j}\neq 0$. Thus

$$\{H_k\bullet X_j,X_j\}=0=H_k\bullet\{X_j,x_j\}+\{H_k,X_j\}\bullet X_j=\rho_{k,j}X_j^2$$

and

$$X_j^2=0,\ \forall j.$$

By similar arguments, the identities $Y_j^2=0$ hold. For $i=1,\cdots,r$ we have

$$\{X_i^2,Y_i\}=0=2X_i\bullet\{X_i,Y_i\}=-4X_i\bullet H_i.$$

Thus, $\sum\alpha_i^j\rho_{j,i}=0$. As the matrix $(\rho_{j,i})$ is non-singular, we deduce that $\alpha_i^j=0$, i.e,

$$H_i^2=0,\forall i=1,\cdots,r.$$

The Poisson algebra $\mathcal{P}$ is a nilalgebra. Moreover, $H_i\bullet X_j=H_i\bullet Y_j=0$ and we conclude that $U\bullet V=0$ for all $U,V\in\mathcal{A}_{\mathcal{P}}$. □

2.5 Classification of finite dimensional complex Poisson algebras

Let $\mathcal{P}$ be a finite dimensional complex Poisson algebra.

Lemma 3 *If there is a non-zero vector $X \in \mathfrak{g}_{\mathcal{P}}$ such that $ad\,X$ is diagonalizable with 0 as a simple root, then $\mathcal{A}^2_{\mathcal{P}} = \{0\}$.*

Proof. Let $\{e_1, \cdots, e_n\}$ be a basis of $\mathfrak{g}_{\mathcal{P}}$ such that $ad\,e_1$ is diagonal with respect to this basis. By assumption, $\{e_1, e_i\} = \lambda_i e_i$ with $\lambda_i \neq 0$ for $i \geq 2$. Since $\{e_1^2, e_1\} = 2e_1 \bullet \{e_1, e_1\} = 0$, it follows that $e_1^2 = ae_1$. But for any $i \neq 1$, $\{e_1^2, e_i\} = 2e_1 \bullet \{e_1, e_i\} = 2\lambda_i e_1 \bullet e_i$ and $\{e_1^2, e_i\} = a\lambda_i e_i$, thus $e_1 \bullet e_i = \frac{a}{2} e_i$. The associativity of the product $\bullet$ implies that $(e_1 \bullet e_1) \bullet e_i = ae_1 \bullet e_i = \frac{a^2}{2} e_i = e_1 \bullet (e_1 \bullet e_i) = \frac{a^2}{4} e_i$. Therefore $a = 0$ and $e_1^2 = 0 = e_1 \bullet e_i$ for any i. Finally, $0 = \{e_1 \bullet e_j, e_i\} = e_1 \bullet \{e_j, e_i\} + e_j \bullet \{e_1, e_i\} = \lambda_i e_j \bullet e_i$, which implies $e_i \bullet e_j = 0, \ \forall i, j \geq 1$. $\square$

2.5.1 Classification of 2 dimensional Poisson algebras

• If $\mathfrak{g}_{\mathcal{P}}$ is abelian then $\mathcal{A}_{\mathcal{P}}$ can be any complex associative commutative algebra and $XY = X \bullet Y$. In this case the classification of Poisson algebras boils down to the classification of commutative associative complex algebras [36].

• If $\mathfrak{g}_{\mathcal{P}}$ is not abelian, it is solvable and isomorphic to the Lie algebra given by $\{e_1, e_2\} = e_2$. From Lemma 3 we know that $\mathcal{A}_{\mathcal{P}}$ is trivial and $e_i e_j = \{e_i, e_j\}$ for $i, j = 1, 2$.

2.5.2 Classification of 3 dimensional Poisson algebras

• If $\mathfrak{g}_{\mathcal{P}}$ is abelian then $\mathcal{A}_{\mathcal{P}}$ can be an arbitrary associative commutative algebra and $XY = X \bullet Y$. In this case the classification is given in [36].

• If $\mathfrak{g}_{\mathcal{P}}$ is nilpotent but not abelian it is isomorphic to the Heisenberg algebra. Let us consider a basis $\{e_i\}_{i=1,2,3}$ of $\mathfrak{g}_{\mathcal{P}}$ such that $\{e_1, e_2\} = e_3$. It follows from the Leibniz identities that $e_1^2 = ae_1 + be_3$. But $\{e_1^2, e_2\} = 2e_1 \bullet e_3 = ae_3$ and $\{e_1 \bullet e_3, e_2\} = e_3 \bullet e_3 = \{ae_3, e_2\} = 0$. The associativity of $\bullet$ implies that $a = 0$. We see that

$$e_1^2 = \alpha e_3, \ e_1 \bullet e_3 = e_3^2 = 0.$$

Similarly,

$$e_2^2 = \beta e_3, \ e_2 \bullet e_3 = 0.$$

Finally, $\{e_1 \bullet e_2, e_i\} = 0$ for $i = 1, 2, 3$ implies $e_1 \bullet e_2 = \gamma e_3$. Thus $\mathcal{A}_{\mathcal{P}}$ is isomorphic to the algebra:

$$\begin{cases} e_1^2 = \alpha e_3, \\ e_2^2 = \beta e_3, \\ e_1 \bullet e_2 = e_2 \bullet e_1 = \gamma e_3. \end{cases}$$

We obtain the following Poisson algebra

$$\begin{cases} e_1^2 = \alpha e_3, \\ e_2^2 = \beta e_3, \\ e_1 \cdot e_2 = (\gamma + 1) e_3, \\ e_2 \cdot e_1 = (\gamma - 1) e_3. \end{cases}$$

The base change

$$\begin{cases} e_1' = ae_1 + be_2 \\ e_2' = ce_1 + de_2 \end{cases}$$

gives

$$\begin{cases} (e_1')^2 = (a^2\alpha + 2ab\gamma + b^2\beta) e_3, \\ (e_2')^2 = (c^2\alpha + 2cd\gamma + d^2\beta) e_3. \end{cases}$$

If $\gamma^2 - \alpha\beta \neq 0$, the equation $\alpha + 2x\gamma + x^2\beta = 0$ has two distinct roots and we can assume that e'_1 and e'_2 are linearly independent such that $(e'_1)^2 = (e'_2)^2 = 0$. In this case the only possible values of parameters α and β are $\alpha = \beta = 0$. We obtain the one-parametric family

$$\mathcal{P}_{3,1}(\gamma) = \begin{cases} e_1^2 = e_2^2 = e_3^2 = 0, \\ e_1 \cdot e_2 = (1+\gamma)e_3, \\ e_2 \cdot e_1 = (-1+\gamma)e_3, \\ e_1 \cdot e_3 = e_3 \cdot e_1 = e_3 \cdot e_2 = e_2 \cdot e_3 = 0. \end{cases}$$

If $\gamma^2 - \alpha\beta = 0$ and if $\beta \neq 0$, we can always choose c and d such that $e_2^2 = 0$. Then we can suppose that $\beta = 0$. This implies $\gamma = 0$. If $\alpha = 0$ we obtain $\mathcal{P}_{3,1}(0)$. If $\alpha \neq 0$, we can assume $\alpha = 1$ which gives the algebra:

$$\mathcal{P}_{3,2} = \begin{cases} e_1^2 = e_3, \\ e_2^2 = e_3^2 = 0, \\ e_1 \cdot e_2 = e_3, \\ e_2 \cdot e_1 = -e_3, \\ e_1 \cdot e_3 = e_3, \cdot e_1 = e_3 \cdot e_2 = e_2 \cdot e_3 = 0. \end{cases}$$

• Suppose that $\mathfrak{g}_{\mathcal{P}}$ is solvable but not nilpotent. Then the following three cases may happen.

i) The multiplication is defined by $\{e_1, e_2\} = e_2$. Then $(\mathcal{P}, \cdot)$ is isomorphic to one of the following Poisson algebras:

$$\begin{cases} e_1^2 = \alpha e_3, \\ e_1 \cdot e_2 = -e_2, \cdot e_1 = e_2, \\ e_1 \cdot e_3 = e_3 \cdot e_1 = \beta e_3, \\ e_2^2 = 0 \\ e_2 \cdot e_3 = e_3 \cdot e_2 = 0, \\ e_3^2 = \gamma e_3 \end{cases} \quad \text{with } \beta^2 = \alpha\gamma, \quad \begin{cases} e_1^2 = 0, \\ e_1 \cdot e_2 = e_2, \\ e_2 \cdot e_1 = -e_2, \\ e_1 \cdot e_3 = e_3, \cdot e_1 = \gamma e_1, \\ e_2^2 = 0, \\ e_2 \cdot e_3 = e_3 \cdot e_2 = \gamma e_2, \\ e_3^2 = \gamma e_3. \end{cases}$$

The first family give the Poisson algebras

$$\mathcal{P}_{3,3}(\alpha) = \begin{cases} e_1^2 = \alpha^2 e_3, \\ e_1 \cdot e_2 = -e_2 \cdot e_1 = e_2, \\ e_1 \cdot e_3 = e_3 \cdot e_1 = \alpha e_3, \\ e_2^2 = 0, \\ e_2 \cdot e_3 = e_3 \cdot e_2 = 0, \\ e_3^2 = e_3 \end{cases}, \quad \mathcal{P}_{3,4} = \begin{cases} e_1^2 = e_3, \\ e_1 \cdot e_2 = -e_2 \cdot e_1 = e_2, \\ e_1 \cdot e_3 = e_3 \cdot e_1 = 0, \\ e_2^2 = e_3^2 = 0, \\ e_2 \cdot e_3 = e_3 \cdot e_2 = 0, \end{cases} \quad \text{and}$$

$$\mathcal{P}_{3,5} = \begin{cases} e_1^2 = e_2^2 = e_3^2 = 0, \\ e_1 \cdot e_2 = -e_2 \cdot e_1 = e_2, \\ e_1 \cdot e_3 = e_3 \cdot e_1 = 0, \\ e_2 \cdot e_3 = e_3 \cdot e_2 = 0. \end{cases}$$

The second family reduces to

$$\mathcal{P}_{3,6} = \begin{cases} e_1^2 = e_2^2 = 0, \\ e_3^2 = e_3, \\ e_1 \cdot e_2 = e_2, \\ e_2 \cdot e_1 = -e_2, \\ e_1 \cdot e_3 = e_3 \cdot e_1 = e_1, \\ e_2 \cdot e_3 = e_3 \cdot e_2 = e_2. \end{cases}$$

ii) The multiplication is given by $\{e_1, e_2\} = e_2$ and $\{e_1, e_3\} = \alpha e_3$ with $\alpha \neq 0$. From Lemma 3, $(\mathcal{P}, \cdot)$ is isomorphic to

$$\mathcal{P}_{3,7}(\alpha) = \begin{cases} e_1^2 = e_2^2 = e_3^2 = 0, \\ e_1 \cdot e_2 = -e_2 \cdot e_1 = e_2, \\ e_1 \cdot e_3 = -e_3 \cdot e_1 = \alpha e_3, \\ e_2 \cdot e_3 = e_3 \cdot e_2 = 0. \end{cases}, \quad \alpha \neq 0.$$

iii) The multiplication is given by $\{e_1, e_2\} = e_2 + e_3$ and $\{e_1, e_3\} = e_3$. Since 1 is an eigenvalue of ad_{e_1} with multiplicity 2, by adapting the proof of Lemma 3, we can conclude that $\mathcal{A}_{\mathcal{P}}$ is trivial. We get the Poisson algebra:

$$\mathcal{P}_{3,8} = \begin{cases} e_1^2 = e_2^2 = e_3^2 = 0, \\ e_1 \cdot e_2 = -e_2 \cdot e_1 = e_2 + e_3, \\ e_1 \cdot e_3 = -e_3 \cdot e_1 = e_3, \\ e_2 \cdot e_3 = e_3 \cdot e_2 = 0. \end{cases}$$

• If $\mathfrak{g}_{\mathcal{P}}$ is simple, it is isomorphic to $sl(2)$. Therefore, it is rigid. We have already studied this case previously. We deduce that $\mathcal{P}$ is isomorphic to

$$\mathcal{P}_{3,9} = \begin{cases} e_1^2 = e_2^2 = e_3^2 = 0, \\ e_1 \cdot e_2 = -e_2 \cdot e_1 = 2e_2, \\ e_1 \cdot e_3 = -e_3 \cdot e_1 = -2e_3, \\ e_2 \cdot e_3 = -e_3 \cdot e_2 = e_1. \end{cases}$$

2.6 Cohomology of Poisson algebras

In [79], A. Lichnerowicz introduced a cohomology for Poisson algebras. The k-cochains are skew-symmetric k-linear maps that are derivatives in each of their arguments. The coboundary operator denoted by δ_{LP} is given by

$$\delta_{LP}\varphi(X_0, \cdots, X_k) = \sum_{i=0}^{k} (-1)^i \left[X_i, \varphi(X_0, \cdots, \check{X}_i, \cdots, X_k)\right] +$$
$$\sum_{0 \leq i < j \leq k} (-1)^{i+j} \varphi(\{X_i, X_j\}, X_0, \cdots, \check{X}_i, \cdots, \check{X}_j, \cdots, X_k)$$

where $\check{X}_i$ means that the term X_i is omitted and $\{,\}$ is the Lie bracket of the Poisson multiplication. Note that if $f : \mathcal{P}_1 \to \mathcal{P}_2$ is a morphism of Poisson algebras, then f does not lead, in general, to a nontrivial functorial morphism between the cohomology groups. The functoriality question for Poisson cohomology has been addressed in the literature for instance in [66].

Since the Lichnerowicz cohomology pays attention only to the Lie part of a Poisson algebra, we need a better definition of cohomology that would, for instance, govern general deformations of Poisson algebras. Such a definition is provided by theory of quadratic Koszul operads. We describe it in details only in degrees 0, 1, 2 and 3. Our approach will be based on the definition of Poisson-admissible algebras.

2.6.1 The $k-$cochains

We proved in [56] that for any $\mathbb{K}[\Sigma_3]^2$-associative algebra $(\mathcal{A}, \mu)$ defined by the relation

$$A_\mu^L \circ \phi_v - A_\mu^R \circ \phi_w = 0,$$

with $v, w \in \mathbb{K}[\Sigma_3]$, the cochains $\varphi \in \mathcal{C}^i(\mathcal{A}, \mathcal{A})$ can be chosen invariant under $F_v^\perp \cap F_w^\perp$ (for the notations see [56]). For a Poisson algebra we have $v = Id$, $w = 3Id - \tau_{23} + \tau_{12} - c + c^2$. Then $F_v^\perp \cap F_w^\perp = \{0\}$ and if $\mathcal{C}^k(\mathcal{P}, \mathcal{P})$ is the space of k-cochains of $\mathcal{P}$, we obtain

$$\mathcal{C}^k(\mathcal{P}, \mathcal{P}) = End(\mathcal{P}^{\otimes^k}, \mathcal{P}).$$

Remark. In [91] an explicit presentation of the space of cochains is given using operads. More precisely, we have

$$\mathcal{C}^k(\mathcal{P}, \mathcal{P}) = \mathcal{L}in(\mathcal{P}oiss(n)^! \otimes_{\Sigma_n} V^{\otimes^n}, V)$$

where V is the underlying vector space (here $\mathbb{C}^n$). But $End(\mathcal{P}^{\otimes^k}, \mathcal{P})$ and $\mathcal{L}in(\mathcal{P}oiss(n)^! \otimes_{\Sigma_n} V^{\otimes^n}, V)$ are isomorphic.

2.6.2 The coboundary operators $\delta^k_{\mathcal{P}}$, $(k=0,1,2)$

Notation. Let $(\mathcal{P},\cdot)$ be a Poisson algebra, $\mathfrak{g}_{\mathcal{P}}$ and $\mathcal{A}_{\mathcal{P}}$ its corresponding Lie and associative algebras. We denote by $H^{\star}_C(\mathfrak{g}_{\mathcal{P}},\mathfrak{g}_{\mathcal{P}})=Z^{*}_C(\mathfrak{g}_{\mathcal{P}},\mathfrak{g}_{\mathcal{P}})/B^{*}_C(\mathfrak{g}_{\mathcal{P}},\mathfrak{g}_{\mathcal{P}})$ the Chevalley cohomology of $\mathfrak{g}_{\mathcal{P}}$ and by $H^{\star}_H(\mathcal{A}_{\mathcal{P}},\mathcal{A}_{\mathcal{P}})$ the Harrison cohomology of $\mathcal{A}_{\mathcal{P}}$. We will define coboundary operators $\delta^k_{\mathcal{P}}$ on $\mathcal{C}^k(\mathcal{P},\mathcal{P})$.

i) $k=0$. We put

$$H^0(\mathcal{P},\mathcal{P})=\{X\in\mathcal{P}\,\text{such that}\,\forall Y\in\mathcal{P}, X\cdot Y=0\}.$$

ii) $k=1$. For $f\in End(\mathcal{P},\mathcal{P})$, we put

$$\delta^1_{\mathcal{P}}f(X,Y)=f(X)\cdot Y+X\cdot f(Y)-f(X\cdot Y),$$

for any $X,Y\in\mathcal{P}$. Then we have

$$H^1(\mathcal{P},\mathcal{P})=H^1_C(\mathfrak{g}_{\mathcal{P}},\mathfrak{g}_{\mathcal{P}})\cap H^1_H(\mathcal{A}_{\mathcal{P}},\mathcal{A}_{\mathcal{P}}).$$

iii) $k=2$. For $\varphi\in\mathcal{C}^2(\mathcal{P},\mathcal{P})$ we define

$$\begin{array}{rl}\delta^2_{\mathcal{P}}\varphi(X,Y,Z)= & 3\varphi(X\cdot Y,Z)-3\varphi(X,Y\cdot Z)-\varphi(X\cdot Z,Y)-\varphi(Y\cdot Z,X)\\ & +\varphi(Y\cdot X,Z)+\varphi(Z\cdot X,Y)+3\varphi(X,Y)\cdot Z-3X\cdot\varphi(Y,Z)\\ & -\varphi(X,Z)\cdot Y-\varphi(Y,Z)\cdot X+\varphi(Y,X)\cdot Z+\varphi(Z,X)\cdot Y.\end{array}$$

The space $H^2(\mathcal{P},\mathcal{P})$ parametrizes deformations of the multiplication of $\mathcal{P}$. We saw in the previous sections that deformations of $(\mathcal{P},\cdot)$ induce deformations of $\mathfrak{g}_{\mathcal{P}}$ and of $\mathcal{A}_{\mathcal{P}}$. In contrast to $H^*(\mathcal{P},\mathcal{P})$, the Lichnerowicz-Poisson cohomology reflects deformations of the bracket only.

Suppose that the Poisson product satisfies $X\cdot Y=-Y\cdot X$. In this case $\{X,Y\}=X\cdot Y$ and $X\bullet Y=0$. If $\varphi\in\mathcal{C}^2(\mathcal{P},\mathcal{P})$ is also a skew-symmetric map, then

$$\begin{array}{rl}\delta^2_{\mathcal{P}}\varphi(X,Y,Z)= & 2\varphi(X\cdot Y,Z)+2\varphi(Y\cdot Z,X)-2\varphi(X\cdot Z,Y)\\ & +2\varphi(X,Y)\cdot Z+2\varphi(Y,Z)\cdot X-2\varphi(X,Z)\cdot Y\\ = & \delta^2_{L,P}\varphi(X,Y,Z).\end{array}$$

We recognize the formula as the Lichnerowicz-Poisson differential.

Proposition 18 *Let φ be in $\mathcal{C}^2(\mathcal{P},\mathcal{P})$. If φ_a and φ_s are respectively the skew-symmetric and the symmetric parts of φ then we have:*

$$\left\{\begin{array}{rl}12\delta^2_C\varphi_a(X,Y,Z)= & \delta^2_{\mathcal{P}}\varphi(X,Y,Z)-\delta^2_{\mathcal{P}}\varphi(Y,X,Z)-\delta^2_{\mathcal{P}}\varphi(Z,Y,X)\\ & -\delta^2_{\mathcal{P}}\varphi(X,Z,Y)+\delta^2_{\mathcal{P}}\varphi(Y,Z,X)+\delta^2_{\mathcal{P}}\varphi(Z,X,Y),\\ & \\ 12\delta^2_H\varphi_s(X,Y,Z)= & \delta^2_{\mathcal{P}}\varphi(X,Y,Z)-\delta^2_{\mathcal{P}}\varphi(Z,Y,X)+\delta^2_{\mathcal{P}}\varphi(X,Z,Y)\\ & -\delta^2_{\mathcal{P}}\varphi(Z,X,Y).\end{array}\right.$$

Proof. The proof is a straightforward calculation. Recall that if φ is a skew-symmetric bilinear map then the Chevalley coboundary operator is given by

$$\begin{array}{rcl}\delta_C(\varphi)(X,Y,Z) & = & \{\varphi(Y,Z)\,X\}+\{\varphi(Z,X),Y\}+\{\varphi(X,Y),Z\}\\ & & +\varphi(\{X,Y\},Z)+\varphi(\{Z,X\},Y)+\varphi(\{Y,Z\},X)\end{array}$$

and if φ is a symmetric bilinear map, the Harrison coboundary operator is given by

$$\begin{array}{rcl}\delta_H(\varphi)(X,Y,Z) & = & \varphi(X,Y)\bullet Z-X\bullet\varphi(Y,Z)+\varphi(X\bullet Y,Z)\\ & & -\varphi(X,Y\bullet Z).\end{array}$$

Now, to compute $\delta^2_C\varphi_a$ we replace $\varphi_a(X,Y)$ by $(\varphi(X,Y)-\varphi(Y,X))/2$ and $\{X,Y\}$ by $(X\cdot Y-Y\cdot X)/2$ in the expression of $\delta^2_C\varphi_a(X,Y,Z)$. We leave it to the reader.

Corollary 19 *Let φ_s and φ_a be the symmetric and skew-symmetric parts of $\varphi\in\mathcal{C}^2(\mathcal{P},\mathcal{P})$. If $\varphi\in Z^2(\mathcal{P},\mathcal{P})$, then $\varphi_s\in Z^2_H(\mathcal{A}_{\mathcal{P}},\mathcal{A}_{\mathcal{P}})$ and $\varphi_a\in Z^2_C(\mathfrak{g}_{\mathcal{P}},\mathfrak{g}_{\mathcal{P}})$.*

2.6.3 Relation between $Z_H^2(\mathcal{A}_\mathcal{P},\mathcal{A}_\mathcal{P})$, $Z_C^2(\mathfrak{g}_\mathcal{P},\mathfrak{g}_\mathcal{P})$ and $Z^2(\mathcal{P},\mathcal{P})$

To show the relation between $Z^2(\mathcal{P},\mathcal{P})$ and the classical Chevalley and Harrison cohomological spaces, we have to introduce the following operators

$$\mathcal{L}_1,\mathcal{L}_2 : \mathcal{C}^2(\mathcal{P},\mathcal{P}) \to \mathcal{C}^3(\mathcal{P},\mathcal{P}).$$

They are given by

$$\mathcal{L}_1(\varphi)(X,Y,Z) = \varphi(X \bullet Y,Z) - \varphi(X,Z) \bullet Y - X \bullet \varphi(Y,Z)$$

and

$$\mathcal{L}_2(\varphi)(X,Y,Z) = -3\varphi(X,\{Y,Z\}) + \{\varphi(X,Y),Z\} - \{\varphi(X,Z),Y\}.$$

Lemma 4 *Let $\varphi \in C^2(\mathcal{P},\mathcal{P})$. If φ_s and φ_a are the symmetric and skew-symmetric parts of φ, we have*

$$\delta^2_\mathcal{P}\varphi = \quad \delta^2_C\varphi_a + 2\delta^2_H\varphi_s + \tilde{\delta}^2_C\varphi_s + \tilde{\delta}^2_H\varphi_a + \mathcal{L}_1(\varphi_a) + \mathcal{L}_2(\varphi_s),$$

where $\tilde{\delta}_C$ and $\tilde{\delta}_H$ are the linear maps $C^2(\mathcal{P},\mathcal{P}) \to C^3(\mathcal{P},\mathcal{P})$ extending naturally δ_C and δ_H.

Proof. Starting from $\varphi = \varphi_a + \varphi_s$ and $X \cdot Y = \{X,Y\} + X \bullet Y$ we obtain

$$\begin{aligned}
\delta^2_\mathcal{P}\varphi(X,Y,Z) = \quad & 3\varphi_a(\{X,Y\},Z) - 3\varphi_a(X,\{Y,Z\}) - \varphi_a(\{X,Z\},Y) \\
& -\varphi_a(\{Y,Z\},X) + \varphi_a(\{Y,X\},Z) + \varphi_a(\{Z,X\},Y) + 3\{\varphi_a(X,Y),Z\} \\
& -3\{X,\varphi_a(Y,Z)\} - \{\varphi_a(X,Z),Y\} - \{\varphi_a(Y,Z),X\} + \{\varphi_a(Y,X),Z\} \\
& +\{\varphi_a(Z,X),Y\} + 3\varphi_a(X \bullet Y,Z) - 3\varphi_a(X,Y \bullet Z) - \varphi_a(X \bullet Z,Y) \\
& -\varphi_a(Y \bullet Z,X) + \varphi_a(Y \bullet X,Z) + \varphi_a(Z \bullet X,Y) + 3\varphi_a(X,Y) \bullet Z \\
& -3X \bullet \varphi_a(Y,Z) - \varphi_a(X,Z) \bullet Y - \varphi_a(Y,Z) \bullet X + \varphi_a(Y,X) \bullet Z \\
& +\varphi_a(Z,X) \bullet Y + 3\varphi_s(\{X,Y\},Z) - 3\varphi_s(X,\{Y,Z\}) - \varphi_s(\{X,Z\},Y) \\
& -\varphi_s(\{Y,Z\},X) + \varphi_s(\{Y,X\},Z) + \varphi_s(\{Z,X\},Y) + 3\{\varphi_s(X,Y),Z\} \\
& -3\{X,\varphi_s(Y,Z)\} - \{\varphi_s(X,Z),Y\} - \{\varphi_s(Y,Z),X\} + \{\varphi_s(Y,X),Z\} \\
& +\{\varphi_s(Z,X),Y\} + 3\varphi_s(X \bullet Y,Z) - 3\varphi_s(X,Y \bullet Z) - \varphi_s(X \bullet Z,Y) \\
& -\varphi_s(Y \bullet Z,X) + \varphi_s(Y \bullet X,Z) + \varphi_s(Z \bullet X,Y) + 3\varphi_s(X,Y) \bullet Z \\
& -3X \bullet \varphi_s(Y,Z) - \varphi_s(X,Z) \bullet Y - \varphi_s(Y,Z) \bullet X + \varphi_s(Y,X) \bullet Z \\
& +\varphi_s(Z,X) \bullet Y
\end{aligned}$$

Since φ_a is skew-symmetric and φ_s symmetric, this relation gives

$$\begin{aligned}
\delta^2_\mathcal{P}\varphi(X,Y,Z) = \quad & 2\varphi_a(\{X,Y\},Z) - 2\varphi_a(X,\{Y,Z\}) - 2\varphi_a(\{X,Z\},Y) \\
& +2\{\varphi_a(X,Y),Z\} - 2\{X,\varphi_a(Y,Z)\} - 2\{\varphi_a(X,Z),Y\} + 4\varphi_a(X \bullet Y,Z) \\
& -2\varphi_a(X,Y \bullet Z) + 2\varphi_a(X,Y) \bullet Z - 4X \bullet \varphi_a(Y,Z) - 2\varphi_a(X,Z) \bullet Y \\
& +2\varphi_s(\{X,Y\},Z) - 4\varphi_s(X,\{Y,Z\}) - 2\varphi_s(\{X,Z\},Y) + 4\{\varphi_s(X,Y),Z\} \\
& -2\{X,\varphi_s(Y,Z)\} + 4\varphi_s(X \bullet Y,Z) - 4\varphi_s(X,Y \bullet Z) + 4\varphi_s(X,Y) \bullet Z \\
& -4X \bullet \varphi_s(Y,Z),
\end{aligned}$$

that is,

$$\begin{aligned}
\delta^2_\mathcal{P}\varphi(X,Y,Z) = \quad & 2\delta_C\varphi_a(X,Y,Z) + 2\tilde{\delta}_H\varphi_a(X,Y,Z) + 2\mathcal{L}_1(\varphi_a)(X,Y,Z) \\
& +4\delta_H\varphi_s(X,Y,Z) + 2\tilde{\delta}_C\varphi_s(X,Y,Z) + 2\mathcal{L}_2(\varphi_s)(X,Y,Z).
\end{aligned}$$

Hence the lemma.

Theorem 20 *Let φ be in $C^2(\mathcal{P},\mathcal{P})$and let φ_s , φ_a be its symmetric and skew-symmetric parts. Then the following propositions are equivalent:*

1. $\delta^2_\mathcal{P}\varphi = 0$.
2. $\left\{\begin{array}{l} i)\ \delta^2_C\varphi_a = 0,\ \delta^2_H\varphi_s = 0, \\ ii)\ \tilde{\delta}^2_C\varphi_s + \tilde{\delta}^2_H\varphi_a + \mathcal{L}_1(\varphi_a) + \mathcal{L}_2(\varphi_s) = 0. \end{array}\right.$

Proof. $2 \Rightarrow 1$ is a consequence of Corollary 19. $1 \Rightarrow 2$ is a consequence of Corollary 19 and Lemma 4.

Applications.
Suppose that φ is skew-symmetric. Then $\varphi = \varphi_a$ and $\varphi_s = 0$. Then $\delta^2_{\mathcal{P}}\varphi = 0$ if and only if $\delta^2_C\varphi = 0$ and $\tilde{\delta}^2_H\varphi + \mathcal{L}_1(\varphi) = 0$. If we suppose moreover that φ is a biderivation on each argument, that is, $\mathcal{L}_1(\varphi) = 0$, then Theorem 20 implies that $\delta^2_{\mathcal{P}}\varphi = 0$ if and only if $\tilde{\delta}^2_H\varphi = 0$. But

$$\begin{array}{ll}\tilde{\delta}^2_H\varphi(X,Y,Z) & = \varphi(X,Y)\bullet Z - X\bullet\varphi(Y,Z) + \varphi(X\bullet Y,Z) - \varphi(X,Y\bullet Z)\\ & = \mathcal{L}_1(\varphi)(X,Y,Z) + \mathcal{L}_1(\varphi)(Y,Z,X).\end{array}$$

Thus $\tilde{\delta}^2_H\varphi = 0$ as soon as $\mathcal{L}_1(\varphi) = 0$.

Proposition 19 *Let φ be a skew-symmetric map which is a biderivation, that is, φ is a Lichnerowicz-Poisson 2-cochain. Then $\varphi \in Z^2_{LP}(\mathcal{P},\mathcal{P})$ if and only if $\varphi \in Z^2_{\mathcal{P}}(\mathcal{P},\mathcal{P})$.*

Similarly, if φ is symmetric, then $\delta^2_{\mathcal{P}}\varphi = 0$ if and only if $\delta^2_H\varphi = 0$ and $\tilde{\delta}^2_C\varphi + \mathcal{L}_2(\varphi) = 0$.

2.6.4 The case k=3

We need to define $\delta^3_{\mathcal{P}}\psi$ for $\psi \in \mathcal{C}^3(\mathcal{P},\mathcal{P})$ so that $H^3(\mathcal{P},\mathcal{P})$ represents obstructions to the integrability of infinitesimal deformations of the Poisson algebra $\mathcal{P}$. For each $\psi \in \mathcal{C}^3(\mathcal{P},\mathcal{P})$ we consider

$$\begin{array}{ll}\hat{\psi}(Z,T,X\cdot Y) = & \psi(Z,T,X\cdot Y) - \psi(Z,T\cdot X,Y) + \frac{1}{3}\psi(Z,T\cdot Y,X)\\ & +\frac{1}{3}\psi(Z,X\cdot Y,T) - \frac{1}{3}\psi(Z,X\cdot T,Y) - \frac{1}{3}\psi(Z,Y\cdot T,X).\end{array}$$

Suppose that $X\cdot\psi(Y,Z,T)$ appears in $\delta^3_{\mathcal{P}}\psi(X,Y,Z,T)$. Since $\delta^3_{\mathcal{P}}\circ\delta^2_{\mathcal{P}}\varphi = 0$, we see that the term $X\cdot\varphi(Y,Z\cdot T)$ occurs in $X\cdot\delta^2_{\mathcal{P}}\varphi(Y,Z,T)$. This term appears only once if φ is not skew-symmetric. Thus, in the general case, $\delta^3_{\mathcal{P}}\psi(X,Y,Z,T)$ cannot contain terms as $X\cdot\psi(Y,Z,T)$. We conclude that $\delta^3_{\mathcal{P}}\psi(X,Y,Z,T)$ can be written as:

$$\begin{array}{ll}\delta^3_{\mathcal{P}}\psi(X,Y,Z,T) = & \alpha_1\hat{\psi}(Z,T,X\cdot Y) + \alpha_2\hat{\psi}(Y,T,X\cdot Y) + \alpha_3\hat{\psi}(Y,Z,X\cdot T)\\ & +\alpha_4\hat{\psi}(X,T,Y\cdot Z) + \alpha_5\hat{\psi}(X,Z,Y\cdot T) + \alpha_6\hat{\psi}(X,Y,Z\cdot T).\end{array}$$

From the relations between $Z^2(\mathcal{P},\mathcal{P})$ and $Z^2_H(\mathcal{A}_{\mathcal{P}},\mathcal{A}_{\mathcal{P}})$, $Z^2_C(\mathfrak{g}_{\mathcal{P}},\mathfrak{g}_{\mathcal{P}})$, we have to assume that

$$\delta^3_{\mathcal{P}}\psi(X,Y,Z,T) = 0$$

as soon as ψ is a Lichnerowicz-Poisson cochain. This permits to compute the constants α_i.

2.7 Deformations of complex Poisson algebras

2.7.1 Generalities

By a deformation we understand a formal deformation in Gerstenhaber's sense. It turns out that formal deformations are equivalent to perturbations in the sense of Chapter 12 ([54]).

Let $\mathcal{P} = (V,\mu)$ be a Poisson algebra with multiplication μ and V the underlying complex vector space. Let $\mathbb{C}[[t]]$ be the ring of complex formal power series. A deformation of μ (or $\mathcal{P}$) is a $\mathbb{C}$-bilinear map:

$$\mu' : V\times V \longrightarrow V\otimes\mathbb{C}[[t]]$$

given by

$$\mu'(X,Y) = \mu(X,Y) + t\varphi_1(X,Y) + t^2\varphi_2(X,Y) + \cdots + t^n\varphi_n(X,Y) + \cdots$$

for all $X,Y \in V$ such that φ_i are bilinear maps satisfying, for $k \geq 1$,

$$\left\{\begin{array}{l}\sum_{i+j=2k+1}\varphi_i\circ\varphi_j + \varphi_j\circ\varphi_i + \delta(\varphi_{k+1}) = 0,\\ \\ \sum_{i+j=2k,i<j}\varphi_i\circ\varphi_j + \varphi_j\circ\varphi_i + \delta(\varphi_k) + \varphi_k\circ\varphi_k = 0,\end{array}\right.$$

with

$$\begin{array}{rl}\varphi_i \circ \varphi_j(X,Y,Z) = & \varphi_i(\varphi_j(X,Y),Z)+) - \varphi_i(X,\varphi_j(Y,Z)) - \frac{1}{3}\varphi_i(\varphi_j(X,Z),Y) \\ & -\frac{1}{3}\varphi_i(\varphi_j(Y,Z),X) + \frac{1}{3}\varphi_i(\varphi_j(Y,X),Z) + \frac{1}{3}\varphi_i(\varphi_j(Z,X),Y)\end{array}$$

and $\delta\varphi_i$ the coboundary operator of the Poisson cohomology defined in the previous section.

Definition 21 *A Poisson algebra $\mathcal{P} = (V,\mu)$ is rigid if every deformation μ' is isomorphic to μ, i.e., if there exists $f \in Gl(V \otimes \mathbb{C}[[t]])$ such that*

$$f^{-1}(\mu(f(X), f(Y))) = \mu'(X,Y),$$

for all $X, Y \in V$.

As for Lie or associative algebras, one can show, using similar arguments:

Proposition 20 *If $H^2(\mathcal{P},\mathcal{P}) = 0$, then $\mathcal{P} = (V,\mu)$ is rigid.*

The converse is not true. A rigid complex n-dimensional Poisson algebra with $H^2(\mathcal{P},\mathcal{P}) \neq 0$ corresponds to a point μ of the algebraic variety of Poisson structures on $\mathbb{C}^n$ such that the corresponding affine schema is not reduced at this point. We will see an example in the following section.

2.7.2 Finite dimensional complex rigid Poisson algebras

Let $\mathcal{P} = (\mathbb{C}^n,\mu)$ be an n-dimensional complex Poisson algebra and suppose that the associated Lie algebra $\mathfrak{g}_\mathcal{P}$ is a finite dimensional rigid solvable Lie algebra. It follows from [4] that $\mathfrak{g}_\mathcal{P}$ can be written as $\mathfrak{g}_\mathcal{P} = \mathfrak{t} \oplus \mathfrak{n}$, where $\mathfrak{n}$ is the nilradical and $\mathfrak{t}$ a maximal abelian subalgebra such that the operators adX are semi-simple for all X in $\mathfrak{t}$. The subalgebra $\mathfrak{t}$ is called the maximal exterior torus and its dimension the rank of $\mathfrak{g}_\mathcal{P}$.

Suppose that $\dim \mathfrak{t} = 1$ and for $X \in \mathfrak{g}_\mathcal{P}$, $X \neq 0$, the restriction of the operator adX on $\mathfrak{n}$ is invertible (all known solvable rigid Lie algebras satisfy this hypothesis). By Lemma 3, the associated algebra $\mathcal{A}_\mathcal{P}$ satisfies $\mathcal{A}^2_\mathcal{P} = \{0\}$.

Theorem 22 *Let $\mathcal{P}$ a complex Poisson algebra such that $\mathfrak{g}_\mathcal{P}$ is rigid solvable of rank 1 (i.e $\dim \mathfrak{t} = 1$) with non-zero roots. Then $\mathcal{P}$ is a rigid Poisson algebra.*

Proof. If μ' is a deformation of μ, then the corresponding Lie bracket $\{\ ,\ \}_{\mu'}$ is a deformation of the Lie bracket $\{\ ,\ \}_\mu$ of $\mathfrak{g}_\mathcal{P}$. Since $(\mathfrak{g}_\mathcal{P}, \{\ ,\ \}_\mu)$ is rigid, then $\{\ ,\ \}_{\mu'}$ is isomorphic to $\{\ ,\ \}_\mu$. If we denote by $\mathcal{P}' = (\mathbb{C}^n,\mu')$ the deformation of $\mathcal{P} = (\mathbb{C}^n,\mu)$, then $\mathcal{A}_{\mathcal{P}'}$ satisfies also $\mathcal{A}^2_{\mathcal{P}'} = \{0\}$. So, μ' is isomorphic to μ and $\mathcal{P}$ is rigid.

Proposition 21 *Let $\mathfrak{g}$ be a rigid solvable Lie algebra of rank 1 with non-zero roots. Then there is only one Poisson algebra $(\mathcal{P},\cdot)$ such that $\mathfrak{g}_\mathcal{P} = \mathfrak{g}$. It is defined by*

$$X_i \cdot X_j = \{X_i, X_j\}.$$

Example. The Poisson algebra $\mathcal{P}_{2,6}$ is rigid with $\dim H^2(\mathcal{P},\mathcal{P}) = 0$. In fact,

$$Z^2(\mathcal{P},\mathcal{P}) = \left\{\varphi \in \mathcal{C}^2(\mathcal{P},\mathcal{P}),\ \varphi(e_1,e_1) = \varphi(e_2,e_2) = 0,\ \varphi(e_1,e_2) = -\varphi(e_2,e_1)\right\}$$

and for every $f \in End(\mathcal{P})$ we have $\delta f(e_1,e_1) = 0 = \delta f(e_2,e_2)$ and $\delta f(e_1,e_2) = -\delta f(e_2,e_1) = ae_1 + be_2$. We observe that $H^2_C(\mathfrak{g}_\mathcal{P},\mathfrak{g}_\mathcal{P}) = 0$.

We can generalize the previous result to rigid solvable Lie algebras $(\mathfrak{g}_\mathcal{P}, \{\ ,\ \}_\mu)$ of rank r. In this case the nilradical $\mathfrak{n}$ is graded by the roots of $\mathfrak{t}$ [4]. If none of the roots is zero, then using the same arguments as in Lemma 14, we prove that $\mathcal{A}^2_\mathcal{P} = \{0\}$ and $\mathcal{P}$ is rigid. Then we have

Proposition 22 *Let $(\mathcal{P}, \mu)$ be an n-dimensional complex Poisson algebra such that $\mathfrak{g}_{\mathcal{P}}$ is a solvable rigid Lie algebra of rank r. If the roots are non-zero, then $(\mathcal{P}, \mu)$ is rigid and $\mathcal{A}^2_{\mathcal{P}} = \{0\}$.*

Remarks.

- We show how a rigid Lie algebra with $H^2_C(\mathfrak{g}_{\mathcal{P}}, \mathfrak{g}_{\mathcal{P}}) \neq 0$ leads to a rigid Poisson algebra with the same property. Consider a Poisson-admissible algebra satisfying the hypothesis of Proposition 22. Thus $\mu = \{\ ,\ \}_\mu$ and if $\varphi \in Z^2(\mathcal{P}, \mathcal{P})$ is the first term of a deformation of μ, then φ is a skew-symmetric map and $\delta\varphi(X, Y, Z) = (2/3)\delta_C\varphi(X, Y, Z)$. In particular, if $\mathfrak{g}_{\mathcal{P}}$ is rigid with $H^2_C(\mathfrak{g}_{\mathcal{P}}, \mathfrak{g}_{\mathcal{P}}) \neq 0$ then $\mathcal{P}$ is rigid with $H^2(\mathcal{P}, \mathcal{P}) \neq 0$. This gives examples of rigid Poisson algebras with non-trivial cohomology based on the constructions of [52].
- It may happen that a Poisson algebra $\mathcal{P}$ is rigid although $\mathfrak{g}_{\mathcal{P}}$ is not. An example is the Poisson algebra $\mathcal{P}_{3,6}$.
- We can consider deformations of $\mathcal{P}$ which leave the associated product of $\mathcal{A}_{\mathcal{P}}$ unchanged. This means that φ is a skew-bilinear map and cocycles of the Poisson cohomology are also cocycles of the Lichnerowicz-Poisson cohomology. In this case $H^2(\mathcal{P}, \mathcal{P}) = H^2_C(\mathfrak{g}_{\mathcal{P}}, \mathfrak{g}_{\mathcal{P}})$.

2.7.3 The Poisson algebra $S(\mathfrak{g})$

Let $\mathfrak{g}$ be a finite dimensional complex Lie algebra. We denote by $S(\mathfrak{g})$ the symmetric algebra on the vector space $\mathfrak{g}$. It is an associative commutative algebra. Let $\{e_1, \cdots, e_n\}$ be a fixed basis of $\mathfrak{g}$ and $\{e_i, e_j\} = \sum_{i,j}^k C_{ij}^k e_k$ its structure constants. We define on $S(\mathfrak{g})$ a structure of Lie algebra by

$$P_0(p, q) = \sum_{i,j,k=1}^{n} C_{ij}^k e_k \left(\frac{\partial p}{\partial e_i} \frac{\partial q}{\partial e_j} - \frac{\partial p}{\partial e_j} \frac{\partial q}{\partial e_i} \right),$$

where $p = p(e_1, \cdots, e_n)$ and $q = q(e_1, \cdots, e_n) \in S(\mathfrak{g}) = \mathbb{C}[e_1, \cdots, e_n]$. Let $p \bullet q$ be the ordinary associative product of the polynomials p and q. The Lie bracket satisfies the Leibniz rule with respect to this product. If

$$\tilde{P}_0(p, q) = P_0(p, q) + p \bullet q$$

then $(S(\mathfrak{g}), \tilde{P}_0)$ is a Poisson algebra. This structure is usually called the linear Poisson structure on $S(\mathfrak{g})$.

In this subsection we investigate deformations $\tilde{P}$ of $\tilde{P}_0$ on $S(\mathfrak{g})$ which leave the associated structure $(\mathcal{A}_{S(\mathfrak{g})}, \bullet)$ unchanged. We call such deformations Lie deformations of the Poisson algebra $(S(\mathfrak{g}), \tilde{P}_0)$. Any deformation of the bracket P_0 can be expanded into

$$P = P_0 + t\phi_1 + \cdots + t^k \phi_k + \cdots$$

and the corresponding Lie deformation of $\tilde{P}_0$ is

$$\tilde{P} = \tilde{P}_0 + t\phi_1 + \cdots + t^k \phi_k + \cdots .$$

Then $\phi_1 \in Z^2_{L,P}((S(\mathfrak{g}), \tilde{P}_0), (S(\mathfrak{g}), \tilde{P}_0))$.

Suppose now that $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{n}$ is a complex solvable rigid Lie algebra.

Proposition 23 *If $\mathfrak{g}$ is a complex solvable rigid Lie algebra with $\dim \mathfrak{t} \geq 2$, then the Lie algebra $(S(\mathfrak{g}), P_0)$ is not rigid.*

Proof. Let $\phi : S(\mathfrak{g}) \times S(\mathfrak{g}) \longrightarrow S(\mathfrak{g})$ be a skew-bilinear map given by $\phi(X_1, X_2) = \alpha_{12}.1$ when $X_1, X_2 \in \mathfrak{t}$ and $\phi(Y_1, Y_2) = 0$ when $Y_1, Y_2 \in \mathfrak{g}$ but Y_1 or Y_2 is not in $\mathfrak{t}$. By the assumption, ϕ is a derivation in each argument, so ϕ can be extended onto $S(\mathfrak{g})$. It is easy to see that $\phi \in Z^2_C(S(\mathfrak{g}), S(\mathfrak{g}))$. Since $P_0 + t\phi$ is not isomorphic to P_0, we have obtained a non-trivial deformation.

Corollary 23 *[103] If $\mathfrak{g}$ is a complex solvable rigid Lie algebra with* $\dim \mathfrak{t} \geq 2$, *then the Poisson algebra* $(S(\mathfrak{g}), \tilde{P}_0)$ *is not rigid.*

Now we consider the case $\dim \mathfrak{t} = 1$.

Lemma 5 *The maximal exterior torus $\mathfrak{t}$ is a Cartan subalgebra of $(S(\mathfrak{g}), P_0)$.*

Proof. We denote by $\{X, Y_1, \cdots, Y_{n-1}\}$ a basis of $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{n}$ adapted to this decomposition. By definition of $\mathfrak{t}$ we have $\{X, Y_i\} = \lambda_i Y_i$. Then

$$\begin{cases} P_0(X^i, X^j) = 0 \ \text{ for any } i, j, \\ P_0(X^i, Y_j) = i\lambda_j X^{i-1} Y_j, \\ P_0(X, XY_j) = \lambda_j XY_j, \\ P_0(X, Y_i Y_j) = (\lambda_i + \lambda_j) Y_i Y_j, \end{cases}$$

so that $ad_{P_0} X$ is a diagonal derivation of $S(\mathfrak{g})$.

We conclude that the Lie algebra $(S(\mathfrak{g}), P_0)$ is graded by the eigenvalues of $ad_{P_0} X$. In [52] families of rigid Lie algebras of rank 1 were classified. This classification can be used to study $S(\mathfrak{g})$ for a general rigid Lie algebra. We illustrate it on the case where the eigenvalues of $ad_{\mathfrak{g}} X$ are

$$1, 2, \cdots, n-1.$$

It follows from [4] that,

- If $3 \leq n \leq 6$ or $9 \leq n \leq 12$ then $\mathfrak{g}$ is not rigid.
- In the remaining cases, $\mathfrak{g}$ is rigid.

We consider a deformation of $\tilde{P}_0$ given as $\tilde{P} = \tilde{P}_0 + t\phi_1 + \cdots$ with $\phi_1 \in Z^2_{L,P}((S(\mathfrak{g}, \tilde{P}_0), (S(\mathfrak{g}, \tilde{P}_0))$. It is clear that if $\phi_1(Y, Z) = 0$ for every $Y, Z \in \mathfrak{g}$ then $\phi_1 = 0$. Let I_p be the Lie ideal of $S(\mathfrak{g})$ whose elements are polynomials of degree greater than or equal to p. If we denote by $S_p(\mathfrak{g})$ the quotient Lie algebra $S(\mathfrak{g})/I_{p+1}$, then $S_p(\mathfrak{g}) = \mathbb{C}\{1\} \oplus K_p(\mathfrak{g})$ where $K_p(\mathfrak{g})$ is generated by polynomials of degree greater than or equal to 1. Since $(\tilde{\mathcal{P}}, \cdot)$ is a Lie deformation it preserves this decomposition. Thus we need to study the Lie algebra $K_p(\mathfrak{g})$. The Lie subalgebra generated by $\{X\}$ is a maximal exterior torus of $K_p(\mathfrak{g})$. The vector X is, in the terminology of [4], a regular vector. The eigenvalues of $ad_{K_p(\mathfrak{g})} X$ are $(1, 2, \cdots, n-1, n, \cdots, p(n-1))$. Let $(S(X))$ be the corresponding root system [4]. It is easy to see that its rank is equal to $\dim(\mathfrak{n}) - 2$. This proves that $K_p(\mathfrak{g})$ is not rigid. But since we suppose that ϕ_1 is a derivation in each argument, this implies that $\phi_1(X, X^2) = 0$ and the rank of $(S(X))$ is $\dim(\mathfrak{n}) - 1$. The grading of $K_p(\mathfrak{g})$ by the roots of $ad_{K_p(\mathfrak{g})} X$ is preserved by such a deformation.

The cocycle ϕ_1 leaves each of the eigenspaces of adX invariant. Let $k, \ k \leq n-1$, be the smallest index such that ϕ_1 restricted to the eigenspace associated to the eigenvalue k of adX is non-zero. Then $H_k(\mathfrak{g})$ is a non-rigid Lie algebra such that ϕ_1 is a cocycle determined by a deformation. Conversely, let ϕ_1 be a 2-cocycle of the Lie algebra $K_p(\mathfrak{g})$ which is a derivation in each argument such that there exists i with $\phi_1(Y_i, Y_{p-i}) \neq 0$. Then we can extend ϕ_1 to $S(\mathfrak{g})$ to obtain a deformation of $S(\mathfrak{g})$.

Examples.

1. Let us suppose that $\mathfrak{g}$ is the two dimensional non-abelian rigid solvable Lie algebra with the bracket defined by $[X, Y] = Y$. Let $(S(\mathfrak{g}), P_0)$ be the corresponding Poisson algebra. Then $P_0(X, Y) = Y$. If P is a deformation of P_0, since $\dim(\mathfrak{n}) = 1$, $P = P_0$ and $(S(\mathfrak{g}), P_0)$ is rigid.

2. Let us suppose that $\mathfrak{g}$ is the decomposable 3-dimensional solvable Lie algebra whose brackets are in the basis $\{X, Y_1, Y_2\}$ given by:

$$[X, Y_i] = iY_i, \ i = 1, 2.$$

This Lie algebra is not rigid but, as we argued in Section 2.2, there exists only one Poisson algebra structure whose corresponding Lie algebra is $\mathfrak{g}$. This Poisson algebra is $\mathcal{P}_{3,7}(2)$ and it can be deformed into $\mathcal{P}_{3,7}(2+t)$.

The corresponding cocycle of deformation is given by $\phi(X, Y_2) = Y_2$. It defines a deformation of $(S(\mathfrak{g}), P_0)$. The cases $n = 4, 5$ can be discussed in the same manner.

3. If $n = 6$, then $\mathfrak{g}$ is rigid. Its structure constants are given by

$$\begin{cases} [X, Y_i] = iY_i, \quad i = 1, \cdots, 5, \\ [Y_1, Y_i] = Y_{i+1}, \quad i = 2, 3, 4, \\ [Y_2, Y_3] = Y_5. \end{cases}$$

The Lie algebra $K_2(\mathfrak{g})$ can be deformed using the cocycle $\phi_1(Y_1, Y_3) = Y_2^2$. Then $(S(\mathfrak{g}), P_0)$ is not rigid.

More generally, if we suppose $n > 12$, then $\mathfrak{g}$ is rigid. In this case, the deformations of $(S(\mathfrak{g}), P_0)$ have been studied in [48].

Chapter 3

Non-Coassociative coalgebras, Lie-admissible coalgebras

The theory of quantum groups is based on the deformation of the enveloping algebra of a rigid Lie algebra such as $sl(2,\mathbb{C})$. In this case, we do not deform the associative structure of the enveloping algebra, but the structure of Hopf algebra. Recall that a Hopf algebra is a vector space equipped simultaneously with an associative algebra and coassociative coalgebra structures, compatibility conditions between these two structures and existence of an antipode. The coassociative product is a linear map between the algebra and this algebra tensorised by itself. The enveloping algebra of a Lie algebra, tensor and symmetric algebras are also naturally provided with a Hopf structure. The aim of this chapter is to generalize the notion of coassociative comultiplication to noncoassociative comultiplication and define a similar concept of G-coassociative coalgebras. We prove that the usual properties, that we have for the associative case, extends to the G-associative case. For example, the dual vector space of a finite dimensional G-associative algebra has a G-coassociative coalgebra structure, existence of a convolution product provides $Hom(M,\mathcal{A})$ (where M is a G-associative coalgebra and $\mathcal{A}$ is a G-associative algebra) with a G-associative structure.

We define the category of $G^!$-algebras by the property that the tensor product of a G-algebra with a $G^!$-algebra is also a G-algebra. In fact, $G^!$-algebras are related to the dual of the operad for G-algebras. In Chapter 6, we generalize this construction of tensor product and study current algebras and operad.

We begin to explore bialgebras in the context of Lie-admissible algebras.

3.1 Definitions and examples

Let M be a $\mathbb{K}$-vector space and Δ a comultiplication in M, that is, a $\mathbb{K}$-linear map:

$$\Delta : M \to M \otimes M.$$

The coassociator of Δ is denoted by

$$\tilde{A}_{\Delta} \quad = \quad (\Delta \otimes Id) \circ \Delta - (Id \otimes \Delta) \circ \Delta \tag{3.1}$$

and the flip $\tau : M^{\otimes 2} \to M^{\otimes 2}$ is the linear map defined by $\tau(x \otimes y) = y \otimes x$.

For every σ in the symmetric group Σ_3, we define a linear map on $M^{\otimes 3}$

$$\Phi_{\sigma}^{M} : M^{\otimes 3} \to M^{\otimes 3}$$

by

$$\Phi_{\sigma}^{M}(x_1 \otimes x_2 \otimes x_3) = x_{\sigma^{-1}(1)} \otimes x_{\sigma^{-1}(2)} \otimes x_{\sigma^{-1}(3)}.$$

Definition 24 *The pair (M, Δ) is a Lie-admissible coalgebra if the linear map*

$$\Delta_L : M \to M \otimes M$$

defined by $\Delta_L = \Delta - \tau \circ \Delta$ is a Lie coalgebra comultiplication, that is, if Δ_L satisfies

$$\begin{cases} \tau \circ \Delta_L = -\Delta_L, \\ \tilde{A}_{\Delta_L} + \Phi^M_c \circ \tilde{A}_{\Delta_L} + \Phi^M_{c^2} \circ \tilde{A}_{\Delta_L} = 0. \end{cases}$$

We have the following characterization of a Lie-admissible comultiplication.

Proposition 24 *A comultiplication Δ on M is a Lie-admissible comultiplication if and only if Δ satisfies*

$$\sum_{\sigma \in \Sigma_3} (-1)^{\varepsilon(\sigma)} \Phi^M_\sigma \circ \tilde{A}_\Delta = 0 \tag{3.2}$$

where $(-1)^\varepsilon(\sigma)$ denotes the sign of the permutation σ.

Proof. It is a direct consequence of Equation (3.1) because

$$\begin{array}{rcl} \tilde{A}_{\tau\circ\Delta} & = & ((\tau \circ \Delta) \otimes Id) \circ \tau \circ \Delta - (Id \otimes (\tau \circ \Delta)) \circ (\tau \circ \Delta) \\ & = & -\Phi^M_{\tau_{13}} \circ \tilde{A}_\Delta. \end{array}$$

□

Examples.

- Every coassociative coalgebra is a Lie-admissible coalgebra.
- The comultiplication of a pre-Lie coalgebra (M, Δ) satisfies

$$\tilde{A}_\Delta - \Phi^M_{\tau_{23}} \circ \tilde{A}_\Delta = 0. \tag{3.3}$$

Since the composition of (3.3) by Φ^M_c and $\Phi^M_{c^2}$ gives respectively $\Phi^M_c \circ \tilde{A}_\Delta - \Phi^M_{\tau_{13}} \circ \tilde{A}_\Delta = 0$ and $\Phi^M_{c^2} \circ \tilde{A}_\Delta - \Phi^M_{\tau_{12}} \circ \tilde{A}_\Delta = 0$, we obtain Identity (3.2) by summation of (3.3) with these two equations and every pre-Lie coalgebra is Lie-admissible.

In the following sections we generalize these examples.

3.2 G-coalgebras

An interesting class of Lie-admissible coalgebras is obtained by dualizing the G-associative algebras. These Lie-admissible algebras, introduced in [106] and developed in [55], have been defined in Chapter 1.

3.2.1 Definition of G-coalgebras

Definition 25 *A G-coalgebra is a $\mathbb{K}$-vector space M provided with a comultiplication Δ satisfying*

$$\sum_{\sigma \in G} (-1)^{\varepsilon(\sigma)} \Phi^M_\sigma \circ \tilde{A}_\Delta = 0.$$

Remark. We can present an equivalent and axiomatic definition of the notion of G-associative algebra. A G-associative algebra is $(\mathcal{A}, \mu, \eta, G)$ where $\mathcal{A}$ is a vector space, G a subgroup of Σ_3, $\mu : \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A}$ and $\eta : \mathbb{K} \to \mathcal{A}$ are linear maps satisfying the following axioms:

1. (G-ass): The square

$$\begin{array}{ccc}
\mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A} & \xrightarrow{(\mu\otimes Id)_G} & \mathcal{A}\otimes\mathcal{A} \\
{\scriptstyle (Id\otimes\mu)_G}\downarrow & & \downarrow{\scriptstyle \mu} \\
\mathcal{A}\otimes\mathcal{A} & \xrightarrow{\mu} & \mathcal{A}
\end{array}$$

commutes, where $(Id\otimes\mu)_G$ is the linear map defined by:

$$(Id\otimes\mu)_G = \sum_{\sigma\in G}(-1)^{\varepsilon(\sigma)}(Id\otimes\mu)\circ\Phi_\sigma^{\mathcal{A}}.$$

If we impose that the algebra is unitary we have to add the following axiom:

2. (Un) The following diagram is commutative :

$$\begin{tikzcd}
\mathbb{K}\otimes\mathcal{A} \arrow[r, "\eta\otimes id"] \arrow[dr] & \mathcal{A}\otimes\mathcal{A} \arrow[d, "\mu"] & \mathcal{A}\otimes\mathbb{K} \arrow[l, "id\otimes\eta"'] \arrow[dl] \\
& \mathcal{A} &
\end{tikzcd}$$

The axiom (G-ass) expresses that the multiplication μ is G-associative whereas the axiom (Un) means that the element $\eta(1)$ of $\mathcal{A}$ is a left and right unit for μ. We want to dualize the previous diagrams to obtain the notions of corresponding coalgebras. Let Δ be a comultiplication on a vector space M:

$$\Delta : M \longrightarrow M\otimes M.$$

We define the bilinear map

$$G\circ(\Delta\otimes Id) : M^{\otimes 2} \longrightarrow M^{\otimes 3}$$

by

$$G\circ(\Delta\otimes Id) = \sum_{\sigma\in G}(-1)^{\varepsilon(\sigma)}\Phi_\sigma^M\circ(\Delta\otimes Id).$$

A G-coalgebra is a vector space M provided with a comultiplication $\Delta : M \longrightarrow M\otimes M$ and a counit $\epsilon : M\to\mathbb{K}$ such that :

1. (G-ass co) the following square is commutative:

$$\begin{array}{ccc}
M & \xrightarrow{\Delta} & M\otimes M \\
{\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle G\circ(Id\otimes\Delta)} \\
M\otimes M & \xrightarrow{G\circ(\Delta\otimes Id)} & M\otimes M\otimes M.
\end{array}$$

If we suppose moreover that the coalgebra is counitary we have to add the following axiom:

2. (Coun) the following diagram is commutative:

$$\begin{tikzcd}
\mathbb{K}\otimes M & M\otimes M \arrow[l, "\epsilon\otimes id"'] \arrow[r, "id\otimes\epsilon"] & M\otimes\mathbb{K} \\
& M \arrow[ul] \arrow[u, "\Delta"] \arrow[ur] &
\end{tikzcd}$$

A morphism of G-coalgebras

$$f : (M,\Delta,\epsilon,G) \to (M',\Delta',\epsilon',G)$$

is a linear map from M to M' such that

$$(f\otimes f)\circ\Delta = \Delta'\circ f \quad \text{and} \quad \epsilon = \epsilon'\circ f.$$

Proposition 25 *Every G-coalgebra is a Lie-admissible coalgebra.*

Proof. The Lie-admissible coalgebras are given by the relation:

$$\sum_{\sigma\in\Sigma_3}(-1)^{\varepsilon(\sigma)}\Phi_\sigma^M\circ\tilde{A}_\Delta=\Phi_V^M\circ\tilde{A}_\Delta=0.$$

Since for every $v_i=\sum_{\sigma\in G}(-1)^{\varepsilon(\sigma)}\sigma$ we have $V\in F_{v_i}$, we deduce the proposition. □

3.2.2 The dual space of a G-coalgebra

For any natural number n and any $\mathbb{K}$-vector spaces E and F, we denote by

$$\lambda_n : Hom(E,F)^{\otimes n}\longrightarrow Hom(E^{\otimes n},F^{\otimes n})$$

the natural embedding

$$\lambda_n(f_1\otimes\cdots\otimes f_n)(x_1\otimes\cdots\otimes x_n)=f_1(x_1)\otimes\cdots\otimes f_n(x_n).$$

Proposition 26 *The dual space of a G-coalgebra is provided with a structure of G-associative algebra.*

Proof. Let (M,Δ) be a G-coalgebra. We consider the multiplication on the dual vector space M^* of M defined by :

$$\mu=\Delta^*\circ\lambda_2.$$

It provides M^* with a G-associative algebra structure. In fact, we have

$$\mu(f_1\otimes f_2)=\mu_{\mathbb{K}}\circ\lambda_2(f_1\otimes f_2)\circ\Delta \tag{3.4}$$

for all $f_1,f_2\in M^*$ where $\mu_{\mathbb{K}}$ is the multiplication of $\mathbb{K}$. Equation (7.4) becomes :

$$\begin{aligned}\mu\circ(\mu\otimes Id)(f_1\otimes f_2\otimes f_3) &= \mu_{\mathbb{K}}\circ(\lambda_2(\mu(f_1\otimes f_2)\otimes f_3))\circ\Delta\\ &= \mu_{\mathbb{K}}\circ\lambda_2((\mu_{\mathbb{K}}\circ\lambda_2(f_1\otimes f_2)\circ\Delta)\otimes f_3)\circ\Delta\\ &= \mu_{\mathbb{K}}\circ(\mu_{\mathbb{K}}\otimes Id)\circ\lambda_3(f_1\otimes f_2\otimes f_3)\circ(\Delta\otimes Id)\circ\Delta.\end{aligned}$$

The associator A_μ satisfies :

$$\begin{aligned}A_\mu = \ &\mu_{\mathbb{K}}\circ(\mu_{\mathbb{K}}\otimes Id)\circ\lambda_3(f_1\otimes f_2\otimes f_3)\circ(\Delta\otimes Id)\circ\Delta\\ &-\mu_{\mathbb{K}}\circ(Id\otimes\mu_{\mathbb{K}})\circ\lambda_3(f_1\otimes f_2\otimes f_3)\circ(Id\otimes\Delta)\circ\Delta.\end{aligned}$$

and using associativity and commutativity of the multiplication in $\mathbb{K}$, we obtain

$$A_\mu=\mu_{\mathbb{K}}\circ(\mu_{\mathbb{K}}\otimes Id)\circ\lambda_3(f_1\otimes f_2\otimes f_3)\circ((\Delta\otimes Id)\circ\Delta-(Id\otimes\Delta)\circ\Delta).$$

Thus

$$\begin{aligned}&\textstyle\sum_{\sigma\in G}(-1)^{\varepsilon(\sigma)}A_\mu\circ\Phi_\sigma^{M^*}\\ &=\mu_{\mathbb{K}}\circ(\mu_{\mathbb{K}}\otimes Id)\circ\lambda_3(f_1\otimes f_2\otimes f_3)\circ(G\circ(\Delta\otimes Id)\circ\Delta-G\circ(Id\otimes\Delta)\circ\Delta)\\ &=0,\end{aligned}$$

and (M^*,μ) is a G-associative algebra. □

Proposition 27 *The dual vector space of a finite dimensional G-associative algebra has a G-coalgebra structure.*

Proof. Let $\mathcal{A}$ be a finite dimensional G-associative algebra and let $\{e_i, i = 1, \cdots, n\}$ be a basis of $\mathcal{A}$. If $\{f_i\}$ is the dual basis then $\{f_i \otimes f_j\}$ is a basis of $\mathcal{A}^* \otimes \mathcal{A}^*$. The coproduct Δ on $\mathcal{A}^*$ is defined by

$$\Delta(f) = \sum_{i,j} f(\mu(e_i \otimes e_j)) f_i \otimes f_j.$$

In particular

$$\Delta(f_k) = \sum_{i,j} C_{ij}^k f_i \otimes f_j,$$

where C_{ij}^k are the structure constants of μ related to the basis $\{e_i\}_{1 \leq i \leq n}$. Then Δ is the comultiplication of a G-associative coalgebra. $\square$

3.3 The convolution product

It is known that if $(\mathcal{A}, \mu)$ is associative $\mathbb{K}$-algebra and (M, Δ) a coassociative $\mathbb{K}$-coalgebra (i.e. a G_1-coalgebra) then the convolution product

$$f \star g = \mu \circ \lambda_2(f \otimes g) \circ \Delta$$

provides $Hom(M, \mathcal{A})$ with an associative algebra structure. We obtain similar results in the context of G-associative algebras

3.3.1 The $G^!$-algebras and coalgebras

Definition 26 *For $i \geq 2$, a $G^!$-algebra is an associative algebra satisfying :*

- *for $i = 2$: $x_1 \cdot x_2 \cdot x_3 = x_2 \cdot x_1 \cdot x_3$,*
- *for $i = 3$: $x_1 \cdot x_2 \cdot x_3 = x_1 \cdot x_3 \cdot x_2$,*
- *for $i = 4$: $x_1 \cdot x_2 \cdot x_3 = x_3 \cdot x_2 \cdot x_1$,*
- *for $i = 5$: $x_1 \cdot x_2 \cdot x_3 = x_2 \cdot x_3 \cdot x_1 = x_3 \cdot x_1 \cdot x_2$,*
- *for $i = 6$: $x_1 \cdot x_2 \cdot x_3 = x_{\sigma(1)} \cdot x_{\sigma(2)} \cdot x_{\sigma(3)}$ for all x_1, x_2, x_3 and $\sigma \in \Sigma_3$.*

The $G^!$-algebras are defined in the context of operads in Chapter 4 and [55]. We show that $G_i^!$-associative algebras are algebras on the operad $\mathcal{G}_i$-$ass^!$, the dual of the operad $\mathcal{G}_i$-ass.

Definition 27 *For $i \geq 2$, a $G^!$-coalgebra is a coassociative coalgebra satisfying :*

$$\Phi_\sigma^M \circ (Id \otimes \Delta) \circ \Delta = (Id \otimes \Delta) \circ \Delta,$$

for every $\sigma \in G$.

We will provide $Hom(M, \mathcal{A})$ with a structure of G-associative algebra.

Proposition 28 *Let $(\mathcal{A}, \mu)$ be a G-associative algebra and (M, Δ) a $G^!$-coalgebra. Then the algebra $(Hom(M, \mathcal{A}), \star)$ is a G-associative algebra where $\star$ is the convolution product :*

$$f \star g = \mu \circ \lambda_2(f \otimes g) \circ \Delta.$$

Proof. Let us compute the associator $A(\star)$ of the convolution product. Since

$$\begin{aligned}(f_1 \star f_2) \star f_3 &= \mu \circ \lambda_2((f_1 \star f_2) \otimes f_3) \circ \Delta \\ &= \mu \circ \lambda_2((\mu \circ \lambda_2(f_1 \otimes f_2) \circ \Delta) \otimes f_3) \circ \Delta \\ &= \mu \circ (\mu \otimes Id) \circ \lambda_3(f_1 \otimes f_2 \otimes f_3) \circ (\Delta \otimes Id) \circ \Delta,\end{aligned}$$

we have

$$\begin{aligned}A_\star(f_1 \otimes f_2 \otimes f_3) &= \mu \circ (\mu \otimes Id) \circ \lambda_3(f_1 \otimes f_2 \otimes f_3) \circ (\Delta \otimes Id) \circ \Delta \\ & -\mu \circ (Id \otimes \mu) \circ \lambda_3(f_1 \otimes f_2 \otimes f_3) \circ (Id \otimes \Delta) \circ \Delta.\end{aligned}$$

Therefore

$$\begin{aligned}&\textstyle\sum_{\sigma \in G} (-1)^{\varepsilon(\sigma)} A_\star \circ \Phi_\sigma^{Hom(M,A)}(f_1 \otimes f_2 \otimes f_3) \\ =\ & \mu \circ (\mu \otimes Id) \circ (\textstyle\sum_{\sigma \in G} \lambda_3(\Phi_\sigma^{Hom(M,A)}(f_1 \otimes f_2 \otimes f_3))) \circ (\Delta \otimes Id) \circ \Delta \\ & -\mu \circ (Id \otimes \mu) \circ (\textstyle\sum_{\sigma \in G} \lambda_3(\Phi_\sigma^{Hom(M,A)}(f_1 \otimes f_2 \otimes f_3))) \circ (Id \otimes \Delta) \circ \Delta.\end{aligned}$$

But

$$\lambda_3(\Phi_\sigma^{Hom(M,A)}(f_1 \otimes f_2 \otimes f_3)) = \Phi_\sigma^A \circ \lambda_3(f_1 \otimes f_2 \otimes f_3) \circ \Phi_{\sigma^{-1}}^M.$$

This gives

$$\begin{aligned}&\textstyle\sum_{\sigma \in G} (-1)^{\varepsilon(\sigma)} A_\star \circ \Phi_\sigma^{Hom(M,A)}(f_1 \otimes f_2 \otimes f_3) \\ =\ & \mu \circ (\mu \otimes Id) \circ (\textstyle\sum_{\sigma \in G} \Phi_\sigma^A \circ \lambda_3(f_1 \otimes f_2 \otimes f_3)) \circ \Phi_{\sigma^{-1}}^M \circ (\Delta \otimes Id) \circ \Delta \\ & -\mu \circ (Id \otimes \mu) \circ (\textstyle\sum_{\sigma \in G} \Phi_\sigma^A \circ \lambda_3(f_1 \otimes f_2 \otimes f_3)) \circ \Phi_{\sigma^{-1}}^M \circ (Id \otimes \Delta) \circ \Delta.\end{aligned}$$

Since Δ is coassociative

$$(\Delta \otimes Id) \circ \Delta = (Id \otimes \Delta) \circ \Delta,$$

the $G^!$-coalgebra structure implies

$$\Phi_{u_i}^M \circ (Id \otimes \Delta) \circ \Delta = \Phi_{u_i}^M \circ (\Delta \otimes Id) \circ \Delta = (\Delta \otimes Id) \circ \Delta.$$

Then

$$\begin{aligned}&\textstyle\sum_{\sigma \in G} (-1)^{\varepsilon(\sigma)} A_\star \circ \Phi_\sigma^{Hom(M,A)}(f_1 \otimes f_2 \otimes f_3) \\ =\ & \mu \circ (\mu \otimes Id) \circ (\textstyle\sum_{\sigma \in G} \Phi_\sigma^A \circ \lambda_3(f_1 \otimes f_2 \otimes f_3)) \circ (\Delta \otimes Id) \circ \Delta \\ & -\mu \circ (Id \otimes \mu) \circ (\textstyle\sum_{\sigma \in G} \Phi_\sigma^A \circ \lambda_3(f_1 \otimes f_2 \otimes f_3)) \circ (\Delta \otimes Id) \circ \Delta \\ =\ & \textstyle\sum_{\sigma \in G} A_\mu \circ \Phi_\sigma^A \circ \lambda_3(f_1 \otimes f_2 \otimes f_3) \circ (\Delta \otimes Id) \circ \Delta \\ =\ & 0.\end{aligned}$$

This proves the proposition. $\square$

3.3.2 Lie-admissible bialgebras.

Definition 28 *A Lie-admissible bialgebra is a triple $(\mathcal{A},\mu,\Delta)$ where $(\mathcal{A},\mu)$ is a Lie-admissible algebra and $(\mathcal{A},\Delta)$ a Lie-admissible coalgebra with a compatibility relation between Δ and μ:*

$$\Delta \circ A_\mu \circ \Phi^{\mathcal{A}}_{G_6} = 0.$$

Here we do not assume that the algebra and coalgebra are unitary and counitary. Among Lie-admissible bialgebras, we shall have the class of G-bialgebras. As example, a compatibility relation for pre-Lie bialgebras (that is G_3-bialgebras) is given by

$$\Delta \circ \mu = (Id \otimes \mu) \circ (\Delta \otimes Id) + (\mu \otimes Id) \circ \Phi^{\mathcal{A}}_{\tau_{23}} \circ (\Delta \otimes Id).$$

3.4 Tensor product of Lie-admissible algebras and coalgebras

3.4.1 Tensor product of G and $G^!$-algebras

We know that the tensor product of associative algebras can be provided with an associative algebra structure. In other words, the category of associative algebras is monoidal and closed for the tensor product. This is not true in general for other categories of $\mathbb{K}[\Sigma_3]$-associative algebras.

Proposition 29 *Let $(\mathcal{A},\mu_{\mathcal{A}})$ and $(\mathcal{B},\mu_{\mathcal{B}})$ be two $\mathbb{K}[\Sigma_3]$-associative algebras respectively defined by the relations $A_{\mu_{\mathcal{A}}} \circ \Phi^{\mathcal{A}}_v = 0$ and $A_{\mu_{\mathcal{B}}} \circ \Phi^{\mathcal{B}}_w = 0$. Then $(\mathcal{A} \otimes_{\mathbb{K}} \mathcal{B}, \mu_{\mathcal{A}} \otimes \mu_{\mathcal{B}})$ is a Σ_3-associative algebra if and only if $\mathcal{A}$ and $\mathcal{B}$ are associative algebras (i.e G_1-associative algebras).*

Proof. This is given by a simple computation.

Remark. In this proposition, the relations of $\mathbb{K}[\Sigma_3]$-associative algebras are considered to be strict. In fact, for example, if $\mathcal{A}$ is a Vinberg algebra and $\mathcal{B}$ a non strict Vinberg algebra satisfying $\mu_{\mathcal{B}}(x,\mu_{\mathcal{B}}(y,z)) = \mu_{\mathcal{B}}(y,\mu_{\mathcal{B}}(x,z))$ then the tensor product $\mathcal{A}\otimes\mathcal{B}$ is also a Vinberg algebra. This is also the case when $\mathcal{A}$ is an alternative algebra and $\mathcal{B}$ a commutative associative algebra.

Theorem 29 *If $\mathcal{A}$ is a G-associative algebra and $\mathcal{B}$ a $G^!$-algebra (with the same index) then $\mathcal{A}\otimes\mathcal{B}$ can be provided with a G-algebra structure for $i = 1,\cdots,6$.*

Proof. We give a sketch of proof. Let us consider on $\mathcal{A}\otimes\mathcal{B}$ the classical tensor product

$$\mu_{\mathcal{A}} \otimes \mu_{\mathcal{B}}((a_1\otimes b_1)\otimes(a_2\otimes b_2)) = \mu_{\mathcal{A}}(a_1\otimes a_2)\otimes\mu_{\mathcal{B}}(b_1\otimes b_2).$$

To simplify, we denote by μ the product $\mu_{\mathcal{A}}\otimes\mu_{\mathcal{B}}$. Since $\mathcal{B}$ is an associative algebra, the associator A_μ satisfies:

$$A_\mu((a_1\otimes b_1)\otimes(a_2\otimes b_2)\otimes(a_3\otimes b_3)) = A_{\mu_{\mathcal{A}}}(a_1\otimes a_2\otimes a_3)\otimes\mu_{\mathcal{B}}\circ(\mu_{\mathcal{B}}\otimes Id)(b_1\otimes b_2\otimes b_3).$$

Therefore

$$\begin{array}{l}\sum_{\sigma\in G}(-1)^{\varepsilon(\sigma)}A_\mu\circ\Phi^{\mathcal{A}\otimes\mathcal{B}}_\sigma((a_1\otimes b_1)\otimes(a_2\otimes b_2)\otimes(a_3\otimes b_3))\\ \quad = \sum_{\sigma\in G}(-1)^{\varepsilon(\sigma)}A_{\mu_{\mathcal{A}}}\circ\Phi^{\mathcal{A}}_\sigma(a_1\otimes a_2\otimes a_3)\otimes\mu_{\mathcal{B}}\circ(\mu_{\mathcal{B}}\otimes Id)\circ\Phi^{\mathcal{B}}_\sigma(b_1\otimes b_2\otimes b_3).\end{array}$$

But $\mathcal{B}$ a $G^!$-algebra. Then

$$\mu_{\mathcal{B}}\circ(\mu_{\mathcal{B}}\otimes Id)\circ\Phi^{\mathcal{B}}_\sigma(b_1\otimes b_2\otimes b_3) = \mu_{\mathcal{B}}\circ(\mu_{\mathcal{B}}\otimes Id)(b_1\otimes b_2\otimes b_3),$$

for any $\sigma\in G$. So we obtain

$$\begin{array}{l}\sum_{\sigma\in G}(-1)^{\varepsilon(\sigma)}A_\mu\circ\Phi^{\mathcal{A}\otimes\mathcal{B}}_\sigma((a_1\otimes b_1)\otimes(a_2\otimes b_2)\otimes(a_3\otimes b_3))\\ \quad = (\sum_{\sigma\in G}(-1)^{\varepsilon(\sigma)}A_{\mu_{\mathcal{A}}}\circ\Phi^{\mathcal{A}}_\sigma(a_1\otimes a_2\otimes a_3))\otimes\mu_{\mathcal{B}}\circ(\mu_{\mathcal{B}}\otimes Id)(b_1\otimes b_2\otimes b_3)\\ \quad = 0.\end{array}$$

□

3.4.2 Tensor product of G-coalgebras

Let (M_1, Δ_1) and (M_2, Δ_2) be two Lie-admissible coalgebras and Δ the composition

$$M_1 \otimes M_2 \xrightarrow{\Delta_1 \otimes \Delta_2} (M_1 \otimes M_1) \otimes (M_2 \otimes M_2) \xrightarrow{id_{M_1} \otimes \tau \otimes id_{M_2}} (M_1 \otimes M_2) \otimes (M_1 \otimes M_2)$$

If Δ_1 is a comultiplication of G-coalgebra, what should be the structure of (M_2, Δ_2) such that Δ is a comultiplication of G-coalgebra too?

Proposition 30 *Let (M_1, Δ_1) be a G-coalgebra and (M_2, Δ_2) a $G^!$-coalgebra. Then $(M_1 \otimes M_2, \Delta)$ is provided with a G-coalgebra structure.*

Proof. Using classical notations we have

$$\tilde{A}_\Delta(v \otimes w) = v_1^1 \otimes w_1^1 \otimes v_1^2 \otimes w_1^2 \otimes v_2 \otimes w_2 - v_1 \otimes w_1 \otimes v_2^1 \otimes w_2^1 \otimes v_2^2 \otimes w_2^2.$$

Let $\chi : (M_1 \otimes M_2)^{\otimes^3} \rightarrow M_1^{\otimes^3} \otimes M_2^{\otimes^3}$ be the isomorphism given by

$$\chi(v_1 \otimes w_1 \otimes v_2 \otimes w_2 \otimes v_3 \otimes w_3) = v_1 \otimes v_2 \otimes v_3 \otimes w_1 \otimes w_2 \otimes w_3.$$

Thus we obtain, from the hypothesis on Δ_2

$$\chi \circ \Phi_G^{M_1 \otimes M_2} \circ \tilde{A}_\Delta = \Phi_G^{M_1} \circ \tilde{A}_{\Delta_1} \otimes (\Delta_2 \otimes Id) \circ \Delta_2,$$

which is zero because Δ_1 is a G-comultiplication. As χ is an isomorphism, we deduce the proposition. $\square$

Remark. In Chapter 6 and [59] we have generalized, in the operadic framework, this study and defined a quadratic operad $\widetilde{\mathcal{P}}$ for any quadratic operad $\mathcal{P}$ so that the tensor product of a $\mathcal{P}$-algebra with a $\widetilde{\mathcal{P}}$-algebra is provided with a $\mathcal{P}$-algebra structure (see the following chapter). In the previous case we have always $\widetilde{\mathcal{P}} = \mathcal{P}^!$.

Chapter 4

Lie-Admissible operads, G-Associative operads

Operads were introduced by J.P. May in 1972 for the needs of homotopy theory. From an algebraic point of view, an operad is a system of data that formalizes properties of a collection of maps $X^n \to X$, for a certain set X, for each $n = 1, 2, ...$, that are closed under permutations of arguments of the maps and under all possible superpositions. Here, we are interested in $\mathbb{K}$-linear operad generated by binary (or later n-ary) multiplication and moreover in the case of quadratic operads. The whole structure of an operad is defined by a generating multiplication and a list of relations satisfied by it. We can give a finite list of relations when all additional relations follow from the initial data of operation(s) and relations. Thus, for a given multiplication (such as associative product, Lie product,...) the corresponding operad codes all the informations concerning all products computed starting from the generating product. In particular, operads permits to give some fundamental informations on the associated free algebras. The concept of quadratic operad is based on an analogy with the definition of quadratic algebra as the quotient of a free associative algebra by an ideal of quadratic relations. Then quadratic operads are defined as the quotient of a free operad by an operad ideal of quadratic relations. For quadratic operads we have also the concepts of duality and Koszulity which are also built on analogy with the similar concept for quadratic algebras. In this chapter, we begin to recall some classical and basic results on quadratic operads and, in particular, the notion of dual operad. We determine explicitly the operad $\mathcal{L}ieAdm$ associated with the Lie-admissible algebra. We prove that this operad is Koszul. This permits to describe naturally a cohomology with values in the algebra itself, also called operadic cohomology. In the case of Koszulity, this cohomology governs also the deformations in the Gerstenhaber's sense. We describe precisely the first spaces of this cohomology in terms of Gerstenhaber products (following the works of Nijenhuis). Since Lie-admissible algebras can be considered as G-associative algebras, we extend this study to other classes of G-associative algebras. This permits to produce some examples of nonKoszul quadratic linear operads.

4.1 Some classical results on quadratic operads

4.1.1 Quadratic operads

Recall the basic notions of operad and quadratic operad (see [46] or [91]). An *operad* $\mathcal{P}$ consists in a collection $\{\mathcal{P}(n)\}_{n\geq 1}$ of $\mathbb{K}$-vector spaces such that each $\mathcal{P}(n)$ is a Σ_n-module where Σ_n is the symmetric group on n elements, there is an element $1 \in \mathcal{P}(1)$ called the unit, linear maps

$$\circ_i : \mathcal{P}(n) \times \mathcal{P}(m) \to \mathcal{P}(n+m-1),$$

called comp-i operations satisfying associativity conditions: if $\lambda \in \mathcal{P}(l), \mu \in \mathcal{P}(m), \nu \in \mathcal{P}(n)$ then

$$(\lambda \circ_i \mu) \circ_j \nu = \left\{ \begin{array}{l} (\lambda \circ_j \nu) \circ_{i+n-1} \mu \text{ if } 1 \leq j \leq i-1, \\ \lambda \circ_i (\mu \circ_{j-i+1} \nu) \text{ if } i \leq j \leq m+1-1, \\ (\lambda \circ_{j-m+1} \nu) \circ_i \mu \text{ if } i+m \leq j, \end{array} \right.$$

and the comp-i operations are compatible with the action of the symmetric group.

A $\mathcal{P}$-algebra is a $\mathbb{K}$ vector space V equipped with a morphism of operads $f : \mathcal{P} \rightarrow \mathcal{E}_V$ where $\mathcal{E}_V$ is the operad of endomorphisms of V. Giving a structure of a $\mathcal{P}$-algebra on V is the same as giving a collection of linear maps

$$f_n : \mathcal{P}(n) \otimes V^{\otimes n} \rightarrow V$$

satisfying natural associativity, equivariance and unit conditions.

Let $E = \{E(n)\}_{n \geq 2}$ be a Σ-module, that is, $E(n)$ is a Σ_n-module for all n, $n \geq 2$. We assume that $E(a) = \{0\}$ for any $a \neq 2$. The free operad $\Gamma(E)$ generated by E is solution of the following universal problem: for any operad $\mathcal{Q} = \{\mathcal{Q}(n)\}_{n \geq 1}$ and any $\mathbb{K}[\Sigma_2]$-linear morphism $f : E \rightarrow \mathcal{Q}(2)$, there exists a unique operad morphism $\hat{f} : \Gamma(E) \rightarrow \mathcal{Q}$ which coincides with f on $E = \Gamma(E)(2)$. We have for example $\Gamma(E)(3) = (E \otimes E) \otimes_{\Sigma_2} \mathbb{K}[\Sigma_3]$. If R is a $\mathbb{K}[\Sigma_3]$-submodule of $\Gamma(E)(3)$, it generates an ideal $\mathcal{R} = (R)$ of $\Gamma(E)$. The quadratic operad generated by E with relations R is the operad $\mathcal{P} := \Gamma(E)/(R)$ with

$$\mathcal{P}(n) = \Gamma(E)(n)/(R)(n).$$

This notion of quadratic operad is related to binary algebras. We will see in Chapter 7 and 8 an extension of this definition adapted to n-ary algebras, that is, algebras whose multiplication is defined on n-arguments.

4.1.2 The dual operad

Let $E^\vee = \{E^\vee(n)\}_{n \geq 2}$ be a Σ-module with

$$E^\vee(n) := \left\{ \begin{array}{ll} \mathrm{sgn}_n \otimes E(n)^\#, & \text{if } n = 2 \text{ and} \\ 0, & \text{otherwise} \end{array} \right.$$

where sgn_n is the signum representation of the symmetric group Σ_n and $\#$ the linear dual of a graded vector space with the induced representation. Recall that $V^\# := \mathit{Hom}(V, \mathbb{K})$, so $(V^\#)_d = (V_{-d})^\#$. There is a non-degenerate, Σ_3-equivariant pairing

$$\langle - | - \rangle : \Gamma(E^\vee)(3) \otimes \Gamma(E)(3) \rightarrow \mathbb{K} \tag{4.1}$$

determined by requiring that

$$\langle e' \circ_i f' \mid e'' \circ_j f'' \rangle := \delta_{ij} (-1)^{(i+1)} e'(e'') f'(f'') \in \mathbb{K},$$

for arbitrary $e', f' \in E(2)^\#$, $e'', f'' \in E(2)$. Then $R^\perp \subset \Gamma(E^\vee)(3)$ is the annihilator of $R \subset \Gamma(E)(3)$ in the defined pairing (4.1),

Definition 30 *The* Koszul *or* quadratic *dual of the quadratic operad $\mathcal{P} = \Gamma(E)/(R)$ as above is the quotient*

$$\mathcal{P}^! := \Gamma(E^\vee)/(R^\perp),$$

where $R^\perp \subset \Gamma(E^\vee)(3)$ is the annihilator of $R \subset \Gamma(E)(3)$ in the pairing (8.1), and $(R^\perp)$ the operadic ideal generated by $R^\perp$.

When $\Gamma(E)$ is the free operad generated by the regular representation of Σ_2 the pairing can be reinterpreted as follows: a basis of $\Gamma(E)(n)$, as a vector space, is given by parenthesized products on n variables indexed by $\{1, 2, \cdots, n\}$. Then a basis of $\Gamma(E)(2)$ is given by $\{(x_1 \cdot x_2), (x_2 \cdot x_1)\}$, and a basis of $\Gamma(E)(3)$

by $\{(x_i \cdot x_j) \cdot x_k, x_i \cdot (x_j \cdot x_k), \{i,j,k\} = \{1,2,3\}\}$. Thus, as $\Gamma(E^\vee)(3) = \Gamma(E)(3)^\vee$, considering the scalar product on $\Gamma(E)(3)$ defined by

$$\left\{ \begin{array}{l} < (x_i \cdot x_j) \cdot x_k, (x_{i'} \cdot x_{j'}) \cdot x_{k'} >= 0, \text{ if } \{i,j,k\} \neq \{i',j',k'\}, \\ < (x_i \cdot x_j) \cdot x_k, (x_i \cdot x_j) \cdot x_k >= (-1)^{\varepsilon(\sigma)}, \\ \qquad \qquad \text{with } \sigma = \begin{pmatrix} i & j & k \\ i' & j' & k' \end{pmatrix} \text{ if } \{i,j,k\} = \{i',j',k'\}, \\ < x_i \cdot (x_j \cdot x_k), x_{i'} \cdot (x_{j'} \cdot x_{k'}) >= 0, \text{ if } \{i,j,k\} \neq \{i',j',k'\}, \\ < x_i \cdot (x_j \cdot x_k), x_i \cdot (x_j \cdot x_k) >= -(-1)^{\varepsilon(\sigma)}, \\ \qquad \qquad \text{with } \sigma = \begin{pmatrix} i & j & k \\ i' & j' & k' \end{pmatrix} \text{ if } (i,j,k) \neq (i',j',k'), \\ < (x_i \cdot x_j) \cdot x_k, x_{i'} \cdot (x_{j'} \cdot x_{k'}) >= 0. \end{array} \right. , \tag{4.2}$$

we obtain that $\mathcal{P} = \Gamma(E)/R^\perp$ where $R^\perp$ is the annihilator of R with respect to this scalar product.

4.2 The Lie-Admissible operad

4.2.1 Definition of $\mathcal{L}ieAdm$

Let $\Gamma(E)$ be the free operad generated by $E = \mathbb{K}[\Sigma_2]$. Consider R the $\mathbb{K}[\Sigma_3]$-submodule generated by the vector

$$\begin{array}{rcl} u & = & x_1 \cdot (x_2 \cdot x_3) + x_2 \cdot (x_3 \cdot x_1) + x_3 \cdot (x_1 \cdot x_2) - x_2 \cdot (x_1 \cdot x_3) - x_3 \cdot (x_2 \cdot x_1) \\ & & -x_1 \cdot (x_3 \cdot x_2) - (x_1 \cdot x_2) \cdot x_3 - (x_2 \cdot x_3) \cdot x_1 - (x_3 \cdot x_1) \cdot x_2 + (x_2 \cdot x_1) \cdot x_3 \\ & & +(x_3 \cdot x_2) \cdot x_1 + (x_1 \cdot x_3) \cdot x_2. \end{array}$$

The Lie-Admissible operad, noted $\mathcal{L}ieAdm$ is the binary quadratic operad defined by

$$\mathcal{L}ieAdm = \Gamma(E)/(R).$$

4.2.2 The dual operad of $\mathcal{L}ieAdm$

Before studying the dual operad of $\mathcal{L}ieAdm$, let us introduce some classes of associative algebras.

Definition 31 *An associative algebra A is called 3-order abelian if we have*

$$X_1 X_2 X_3 = X_{\sigma(1)} X_{\sigma(2)} X_{\sigma(3)},$$

for all $\sigma \in \Sigma_3$ and for all $X_i \in A$.

The unitary 3-order abelian algebras are the commutative algebras. But there exists non commutative 3-order abelian algebras. Let us consider, for example, the five dimensional associative algebra defined by

$$e_1^2 = e_2, \quad e_1^3 = e_3, \quad e_1 e_4 = e_5, \quad e_4 e_1 = e_3 + e_5, \quad e_4^2 = e_3.$$

If A is a 3-order abelian algebra, then the subalgebra $\mathcal{D}(A)$ generated by the product xy is abelian. Then A is an extension

$$0 \to V \to A \to A_1 \to 0,$$

where A_1 is abelian and V satisfying $vx = xv$ for all $x \in A_1$ and $v \in V$. In this case, the corresponding Lie algebra is 2-step nilpotent. In fact, as we have $abc = bac$, then $[a,b]c = 0$. We have also $c[a,b] = 0$ thus $[[a,b],c] = 0$.

Let R be the $\mathbb{K}[\Sigma_3]$-submodule which determines the $\mathcal{L}ieAdm$ operad. The annihilator $R^{\perp}$ is of dimension 11. Let R' be the $\mathbb{K}[\Sigma_3]$-submodule of $\Gamma(E)(3)$ generated by the relations

$$\begin{array}{c}(x_{\sigma(1)}x_{\sigma(2)})x_{\sigma(3)} - x_{\sigma(1)}(x_{\sigma(2)}x_{\sigma(3)}),\\ (x_{\sigma(1)}x_{\sigma(2)})x_{\sigma(3)} - (x_{\sigma(1)}x_{\sigma(3)})x_{\sigma(2)},\\ (x_{\sigma(1)}x_{\sigma(2)})x_{\sigma(3)} - (x_{\sigma(2)}x_{\sigma(1)})x_{\sigma(3)}.\end{array}$$

Then $\dim R' = 11$ and $< u, v >= 0$ for all $v \in R'$ where u is the vector which generates R. This implies $R' \simeq R^{\perp}$ and $(\Gamma(E)/(R))^!$ is by definition the binary quadratic operad $\Gamma(E)/(R)^{\perp}$.

Proposition 31 *The dual operad of $\mathcal{L}ieAdm$ is the binary quadratic operad whose corresponding algebras are associative and satisfying*

$$abc = acb = bac,$$

that is, there are 3-order abelian.

4.3 The operads $\mathcal{G}$-*ass*

Since we have distinguished 6 types of Lie-admissible algebras, the G-associative algebras (see Definition 12), we can define the operad $\mathcal{G}$-*ass* for each type of G-algebras. We obtain the following binary quadratic operads:

1. $\mathcal{A}ss = \mathcal{G}_1$-*ass* corresponding to the associative algebras.

2. $\mathcal{V}inb = \mathcal{G}_2$-*ass* corresponding to the Vinberg algebras. Here the $\mathbb{K}[\Sigma_3]$-submodule R defining the quadratic operad is generated by the vectors

$$x_1 \cdot (x_2 \cdot x_3) - (x_1 \cdot x_2) \cdot x_3 - x_2 \cdot (x_1 \cdot x_3) + (x_2 \cdot x_1) \cdot x_3.$$

3. $\mathcal{P}reLie = \mathcal{G}_3$-*ass* corresponding to the pre-Lie algebras (G_3-associative). The vectors which generate the ideal R are:

$$x_1 \cdot (x_2 \cdot x_3) - (x_1 \cdot x_2) \cdot x_3 - x_3 \cdot (x_2 \cdot x_1) + (x_3 \cdot x_2) \cdot x_1.$$

4. $\mathcal{G}_4$-*ass* corresponding to the G_4-associative algebras and R is generated by

$$x_1. \cdot (x_2 \cdot x_3) - (x_1 \cdot x_2) \cdot x_3 - x_3 \cdot (x_2 \cdot x_1) + (x_3 \cdot x_2) \cdot x_1.$$

5. $\mathcal{G}_5$-*ass* corresponding to the G_5-associative algebras and R is generated by

$$x_1 \cdot (x_2 \cdot x_3) - (x_1 \cdot x_2) \cdot x_3 + x_2 \cdot (x_3 \cdot x_1) - (x_2 \cdot x_3) \cdot x_1 + x_3 \cdot (x_1 \cdot x_2) - (x_3 \cdot x_1) \cdot x_2.$$

6. $\mathcal{L}ieAdm = \mathcal{G}_6$-*ass*.

4.3.1 The dual operads $\mathcal{G}$-$ass^!$

One knows that $\mathcal{A}ss^! = \mathcal{A}ss$ ([46]) and $\mathcal{P}reLie^! = \mathcal{P}erm$ ([21]). Recall that this last corresponds to the associative product with the identities:

$$abc = acb$$

Proposition 32 *The dual operads of $\mathcal{V}inb$, $\mathcal{G}_4$-ass, $\mathcal{G}_5$-ass are the quadratic operads for associative algebras satisfying respectively:*

- *for $\mathcal{V}inb^!$: $abc = bac$,*
- *for $\mathcal{G}_4$-$ass^!$: $abc = cba$,*
- *for $\mathcal{G}_5$-$ass^!$: $abc = bca = cab$.*

Sketch of proof. $R_2^{\perp}$ is the $\mathbb{K}[\Sigma_3]$-sub-module of $\Gamma(E)(3)$ generated by the vectors

$$\begin{aligned}&(x_1\cdot(x_2\cdot x_3)-(x_1\cdot x_2)\cdot x_3)\,,\,(x_1\cdot(x_2\cdot x_3)-x_2\cdot(x_1\cdot x_3))\,,\\&(x_1\cdot(x_2\cdot x_3)-(x_2\cdot x_1)\cdot x_3)\end{aligned}$$

for all $x_1, x_2, x_3 \in E$.

Likewise

$$\begin{aligned}R_4^{\perp} = \;& \langle (x_1\cdot(x_2\cdot x_3)-(x_1\cdot x_2)\cdot x_3),(x_1\cdot(x_2\cdot x_3)-x_3\cdot(x_2\cdot x_1)),x_1\cdot(x_2\cdot x_3)-(x_3\cdot x_2)\cdot x_1)\rangle\\ R_5^{\perp} = \;& \langle (x_1\cdot(x_2\cdot x_3)-(x_1\cdot x_2)\cdot x_3),(x_1\cdot(x_2\cdot x_3)-x_2\cdot(x_3\cdot x_1)),(x_1\cdot(x_2\cdot x_3)-(x_2\cdot x_3)\cdot x_1),\\ & (x_1\cdot(x_2\cdot x_3)-x_3\cdot(x_1\cdot x_2)),(x_1\cdot(x_2\cdot x_3)-(x_3\cdot x_1)\cdot x_2)\rangle.\end{aligned}$$

So $\dim R_4^{\perp} = 9$, $\dim R_5^{\perp} = 10$. Then it is sufficient to note that $(\Gamma(E)/(R))^!$ is by definition the binary quadratic operad $\Gamma(E)/(R)^{\perp}$.

4.4 On the (non)Koszulness of $\mathcal{G}$-*ass*

Recall that a quadratic operad $\mathcal{P}$ is called Koszul operad if for every $\mathcal{P}$-free algebra $F_{\mathcal{P}}(V)$ one has $H_i^{\mathcal{P}}(F_{\mathcal{P}}(V)) = 0, \quad i > 0$. We prove the Non-koszulness of an operad $\mathcal{P}$ using its Poincaré series also called generating function. This series is defined for a general operad but reduce for an operad $\mathcal{P}$ associated to a type of binary algebras with operation in degree 0 to

$$g_{\mathcal{P}}(x) := \sum_{i=1}^{\infty} \dim \mathcal{P}(n) \frac{x^n}{n!},$$

The Ginzburg-Kapranov criterium([46]) precises that the Poincaré series of a Koszul operad and its dual operad are connected by the functional equation

$$g_{\mathcal{P}}(-g_{\mathcal{P}^!}(-x)) = x.$$

Thus, if this relation is not satisfied, the operad $\mathcal{P}$ is not Koszul.

4.4.1 Koszulness of the operads $\mathcal{G}_i$-*ass*, $i = 1, 2, 3$

Proposition 33 *The operads $\mathcal{A}ss$, $\mathcal{V}inb$, $\mathcal{P}reLie$ are Koszul operads.*

Proof. In fact, from [46] and [21] the operads $\mathcal{A}ss$, $\mathcal{P}reLie$ are Koszul operads. Considering the relations between $\mathcal{P}reLie$ and $\mathcal{V}inb$, this last satisfies the same property.

4.4.2 NonKoszulness of $\mathcal{G}_4$-*ass*

Proposition 34 *The operad $\mathcal{G}_4$-ass is not a Koszul operad.*

Proof. We have

$$g_{\mathcal{G}_4\text{-}ass}(x) = x + x^2 + \frac{3}{2}x^3 + \frac{71}{4!}x^4 + \cdots \quad , \quad g_{\mathcal{G}_4\text{-}ass^!}(x) = x + x^2 + \frac{1}{2}x^3 + \frac{1}{4!}x^4 + \cdots$$

These series do not satisfy the functional equation. This proves the proposition.

4.4.3 Koszulness of $\mathcal{L}ieAdm$

The Koszulness of the operad $\mathcal{G}_6$-*ass* can be proved as follows. In Example 2.1.2 (Chapter 2), we observed that G_6-associative algebras consist of a commutative multiplication and a Lie bracket, with no relation between these two operations. Therefore $\mathcal{G}_6$-*ass* is the free product

$$\mathcal{G}_6\text{-}ass \cong \mathcal{L}ie * \Gamma(\mathbb{1}_2)$$

of the operad $\mathcal{L}ie$ for Lie algebras and the free operad $\Gamma(\mathbb{1}_2)$ generated by one commutative operation. The Koszulity of the operad $\mathcal{G}_6$-*ass* now follows from the obvious fact that the free product of two quadratic Koszul operads is again quadratic Koszul.

4.5 Cohomology of Lie-admissible algebras

4.5.1 Cohomology of $\mathcal{L}ieAdm$-algebras

Let $\mathcal{P}$ be a binary quadratic operad. A $\mathcal{P}$-algebra is given by a $\mathbb{K}$-vector space V and an operad morphism

$$\varphi : \mathcal{P} \longrightarrow \mathcal{E}nd(V)$$

where $\mathcal{E}nd(V)$ is the operad of endomorphisms of V defined by

$$\mathcal{E}nd(V)(n) = Hom_{\mathbb{K}}(V^{\otimes^n}, V).$$

Let A be a $\mathcal{L}ieAdm$-algebra. The cochain complex of the cohomology $H^*_{\mathcal{L}ieAdm}(A, A)$ is given by

$$C^n_{\mathcal{L}ieAdm}(A) = Hom_{\mathbb{K}}((\mathcal{L}ieAdm^!)^\vee(n) \otimes_{\Sigma_n} A^{\otimes^n}, A)$$

where $V^\vee = V^* \otimes_{\sum_n}(sgn)$ for V a $\mathbb{K}$-vector space provided with an action of the group Σ_n, $V^* = Hom(V, \mathbb{K})$ and sgn the signature representation.
Since every $\mathcal{L}ieAdm^!$-algebra is an associative 3-order commutative algebra, then the complex $C^*_{\mathcal{L}ieAdm}(A)$ is

$$A \longrightarrow Hom(A \otimes A, A) \longrightarrow Hom(\wedge^3(A), A) \longrightarrow Hom(\wedge^4(A), A) \longrightarrow \cdots$$

The differential operator is defined by the composition map of the operad $\mathcal{L}ieAdm$. To determine its expression, let us recall the definition of Nijenhuis-Gerstenhaber products.

Let f and g be n-, respectively m-, linear maps on a vector space V. We define the $(n+m-1)$-linear map $f \odot_6 g$ by

$$\begin{aligned} f \odot_6 g(X_1, \cdots, X_{n+m-1}) = \;& \textstyle\sum_{i=1,\cdots,n} \sum_{\sigma \in \sum_{n+m-1}} (-1)^{\varepsilon(\sigma)} (-1)^{(i-1)(m-1)} f(X_{\sigma(1)}, \cdots, \\ & X_{\sigma(i-1)}, g(X_{\sigma(i)}, \cdots, X_{\sigma(i+m-1)}), X_{\sigma(i+m)}, \cdots, X_{\sigma(n+m-1)}). \end{aligned}$$

This product corresponds to the composition product in the operad $\mathcal{L}ieAdm$.
For example if $f = g = \mu$ and $n = m = 2$ then

$$\mu \odot_6 \mu(X_1, X_2, X_3) = \sum_{\sigma \in \sum_3} (-1)^{\varepsilon(\sigma)} \{\mu(\mu(X_{\sigma(1)}, X_{\sigma(2)}), X_{\sigma(3)}) - \mu(X_{\sigma(1)}, \mu(X_{\sigma(2)}, X_{\sigma(3)})\}$$

and μ is a Lie-admissible multiplication if and only if $\mu \odot_6 \mu = 0$. Similarly if μ is a bilinear map and f an endomorphism of V, then

$$\mu \odot_6 f(X_1, X_2) = \mu(f(X_1), X_2) - \mu(f(X_2), X_1) + \mu(X_1, f(X_2)) - \mu(X_2, f(X_1))$$

and

$$f \odot_6 \mu(X_1, X_2) = f(\mu(X_1, X_2)) - f(\mu(X_2, X_1)) = f([X_1, X_2]_\mu)$$

as soon as $\mu \odot_6 \mu = 0$.

In [101], Nijenhuis defines the following product (denoted by $f\bar{o}g$):

$$\begin{array}{rl} f \odot_1 g(X_1, \cdots, X_{n+m-1}) = & \sum_{i=1,\cdots,n} (-1)^{(i-1)(m-1)} f(X_1, \cdots, X_{i-1}, g(X_i, \cdots, X_{i+m-1}), \\ & X_{i+m}, \cdots, X_{n+m-1}). \end{array}$$

We have

$$f \odot_6 g = \sum_{\sigma \in \sum_{n+m-1}} (-1)^{\varepsilon(\sigma)} g \odot_1 f \circ \sigma = \sum_{\sigma \in \sum_{n+m-1}} (-1)^{\varepsilon(\sigma)} f\bar{o}g \circ \sigma.$$

Let P be the antisymmetric operator. It is defined by

$$P(f)(X_1, \cdots, X_n) = \sum_{\sigma \in \sum_n} (-1)^{\varepsilon(\sigma)} f(X_{\sigma(1)}, X_{\sigma(2)}, \cdots, X_{\sigma(n)}).$$

It is clear that $f \odot_6 g = P(f \odot_1 g)$.

Lemma 6 *We have the following identities :*

$$P(P(f) \odot_1 g) = (n+m-1)! P(f \odot_1 g) = P(f \odot_1 P(g)).$$

This can be prove directly.
We deduce that the following bracket

$$[f, g]^{\odot_6} = f \odot_6 g - (-1)^{(n-1)(m-1)} g \odot_6 f.$$

satisfies :

1. $[g, f]^{\odot_6} = (-1)^{(n-1)(m-1)+1} [f, g]^{\odot_6}$
2. $(-1)^{(n-1)(p-1)} [[f, g]^{\odot_6}, h]^{\odot_6} + (-1)^{(m-1)(n-1)} [[g, h]^{\odot_6}, f]^{\odot_6} + (-1)^{(p-1)(m-1)} [[h, f]^{\odot_6}, g]^{\odot_6} = 0$

where h is a p-linear map on V.

Consequence. $[\ ,\ ]^{\odot_6}$ is a bracket of a graded Lie algebra (on the space of multilinear maps).

Definition 32 *Let ϕ be a in $C^n_{\mathcal{L}ieAdm}(A)$, where A is a $\mathcal{L}ieAdm$-algebra with multiplication μ. The differential operator*

$$\delta_\mu : C^n_{\mathcal{L}ieAdm}(A) \longrightarrow C^{n+1}_{\mathcal{L}ieAdm}(A)$$

is defined by

$$\delta_\mu \phi = -[\mu, \phi]^{\odot_6}.$$

This operator satisfies

$$\delta_\mu \circ \delta_\mu = 0$$

and $(C^*_{\mathcal{L}ieAdm}(A), \delta_\mu)$ is a complex defining a cohomology for Lie-admissible algebras.

4.5.2 Particular cases

Let us describe explicitly this coboundary operator for some n. i) $n = 2$. Let φ be a bilinear map. Then

$$\begin{array}{rl} -\delta_\mu \varphi(X_1, X_2, X_3) = & \mu(\varphi(X_1, X_2), X_3) - \mu(X_1, \varphi(X_2, X_3)) + \mu(\varphi(X_2, X_3), X_1) \\ & -\mu(X_2, \varphi(X_3, X_1)) - \mu(X_3, \varphi(X_1, X_2)) + \mu(\varphi(X_3, X_1), X_2) \\ & -\mu(\varphi(X_2, X_1), X_3) + \mu(X_2, \varphi(X_1, X_3)) - \mu(\varphi(X_3, X_2), X_1) \\ & +\mu(X_3, \varphi(X_2, X_1)) + \mu(X_1, \varphi(X_3, X_2)) - \mu(\varphi(X_1, X_3), X_2) \\ & +\varphi(\mu(X_1, X_2), X_3) - \varphi(X_1, \mu(X_2, X_3)) + \varphi(\mu(X_2, X_3), X_1) \\ & -\varphi(X_2, \mu(X_3, X_1)) - \varphi(X_3, \mu(X_1, X_2)) + \varphi(\mu(X_3, X_1), X_2) \\ & -\varphi(\mu(X_2, X_1), X_3) + \varphi(X_2, \mu(X_1, X_3)) - \varphi(\mu(X_3, X_2), X_1) \\ & +\varphi(X_3, \mu(X_2, X_1)) + \varphi(X_1, \mu(X_3, X_2)) - \varphi(\mu(X_1, X_3), X_2). \end{array}$$

ii) $n = 1$. Let f be in $\mathcal{C}^1$. Then

$$\begin{aligned}\delta_\mu f\left(X_1, X_2\right) &= -\mu\left(f(X_1), X_2\right) - \mu\left(X_1, f(X_2)\right) + f\left(\mu\left(X_1, X_2\right)\right)\\ &\quad +\mu\left(f(X_2), X_1\right) + \mu\left(X_2, f(X_1)\right) - f\left(\mu\left(X_2, X_1\right)\right).\end{aligned}$$

iii) $n = 0$. Here we can define directly the definition of $\delta_\mu(X)$. Let X be in A . Consider the map

$$h_X : Y \mapsto \mu\left(X, Y\right) - \mu\left(Y, X\right).$$

Then

$$\begin{aligned}-\delta_\mu h_X\left(X_1, X_2\right) &= \mu\left(h_X(X_1), X_2\right) + \mu\left(X_1, h_X(X_2)\right) - h_X\left(\mu\left(X_1, X_2\right)\right)\\ &\quad -\mu\left(h_X(X_2), X_1\right) - \mu\left(X_2, h_X(X_1)\right) + h_X\left(\mu\left(X_2, X_1\right)\right)\\ &= \mu\left(\mu\left(X, X_1\right), X_2\right) - \mu\left(\mu\left(X_1, X\right), X_2\right) + \mu\left(X_1, \mu\left(X, X_2\right)\right)\\ &\quad -\mu\left(X_1, \mu\left(X_2, X\right)\right) - h_X\left(\mu\left(X_1, X_2\right)\right) + h_X\left(\mu\left(X_2, X_1\right)\right)\\ &\quad -\mu\left(\mu\left(X, X_2\right), X_1\right) + \mu\left(\mu\left(X_2, X\right), X_1\right) - \mu\left(X_2, \mu\left(X, X_1\right)\right)\\ &\quad +\mu\left(X_2, \mu\left(X_1, X\right)\right),\end{aligned}$$

and

$$\begin{aligned}-\delta_\mu h_X\left(X_2, X_1\right) &= \mu\left(\mu\left(X, X_2\right), X_1\right) - \mu\left(\mu\left(X_2, X\right), X_1\right) + \mu\left(X_2, \mu\left(X, X_1\right)\right)\\ &\quad -\mu\left(X_2, \mu\left(X_1, X\right)\right) - h_X\left(\mu\left(X_2, X_1\right)\right) + h_X\left(\mu\left(X_1, X_2\right)\right)\\ &\quad -\mu\left(\mu\left(X, X_1\right), X_2\right) + \mu\left(\mu\left(X_1, X\right), X_2\right) - \mu\left(X_1, \mu\left(X, X_2\right)\right)\\ &\quad +\mu\left(X_1, \mu\left(X_2, X\right)\right).\end{aligned}$$

So

$$\delta_\mu h_X\left(X_2, X_1\right) - \delta_\mu h_X\left(X_1, X_2\right) = \mu \odot_6 \mu\left(X, X_1, X_2\right) = 0$$

and

$$\delta_\mu h_X\left(X_1, X_2\right) = \delta_\mu h_X\left(X_2, X_1\right)$$

Let us consider

$$\mathcal{C}^0 = \{X \in \mathcal{A} \quad / \quad P\left(\delta_\mu h_X\right) = \delta_\mu h_X\}$$

For $X \in \mathcal{C}^0$, $\quad \delta_\mu h_X = 0$. Then we can define $B^1\left(\mathcal{A}, \mathcal{A}\right)$ putting

$$\delta\left(X\right) = h_X$$

and $Z^1\left(\mathcal{A}, \mathcal{A}\right) = \left\{f \in \mathcal{C}^1 / \delta f = 0\right\}$. Then $H^0\left(\mathcal{A}, \mathcal{A}\right)$ is well defined.

Remark. In the same way, we can define the cohomology $H^*_{\mathcal{G}_i\text{-}ass}(A, A)$ for a $\mathcal{G}_i$-*ass*-algebra. We denote by $f \odot_i g$ the corresponding Nijenhuis product. We have already given the expression of this product for $i = 1$ and $i = 6$. In other cases we put :

$$\begin{aligned}f \odot_2 g(X_1, \cdots, X_{n+m-1}) = &\textstyle\sum_{\sigma\in\sum_{n+m-2}}(-1)^{\varepsilon(\sigma)}\{\sum_{i=1,\cdots,n-1}(-1)^{(i-1)(m-1)}f(X_{\sigma(1)}, \cdots,\\ &X_{\sigma(i-1)}, g(X_{\sigma(i)}, \cdots, X_{\sigma(m+i-1)}), X_{\sigma(m+i)}, \cdots, X_{\sigma(n+m-2)}, X_{n+m-1})\\ &+(-1)^{(n-1)}f(X_{\sigma(1)}, \cdots, X_{\sigma(n-1)}, g(X_{\sigma(n)}, \cdots, X_{\sigma(m+n-2)}, X_{n+m-1})\}.\end{aligned}$$

$$\begin{aligned}f \odot_3 g(X_1, \cdots, X_{n+m-1}) = &\textstyle\sum_{\sigma\in\sum_{n+m-2}}(-1)^{\varepsilon(\sigma)}\{f(g(X_1, X_{\sigma(2)}, \cdots, X_{\sigma(m)}), X_{\sigma(m+1)}, \cdots, X_{\sigma(m+n-1)})\\ &+\textstyle\sum_{i=2,\cdots,n-1}(-1)^{(i-1)(m-1)}f(X_1, X_{\sigma(2)}, \cdots, X_{\sigma(i)}, g(X_{\sigma(i+1)},\\ &\cdots, X_{\sigma(m+i)}), \cdots, X_{\sigma(n+m-1)})\}.\end{aligned}$$

$$\begin{aligned}f \odot_4 g(X_1, \cdots, X_{n+m-1}) = &\textstyle\sum_{\sigma\in\sum_{n+m-2}}\sum_{i=0,\cdots,n-1}(-1)^{\varepsilon(\sigma)+i(m-1)}f(X_{\sigma(1)}, X_2, X_{\sigma(3)}, \cdots, X_{\sigma(i)},\\ &g(X_{\sigma(i+1)}, \cdots, X_{\sigma(m+i)}), \cdots, X_{\sigma(n+m-1)}).\end{aligned}$$

$$\begin{aligned}f \odot_5 g(X_1, \cdots, X_{n+m-1}) = &\textstyle\sum_{\sigma\in A_{n+m-1}}\sum_{i=0,\cdots,n-1}(-1)^{i(m-1)}f(X_{\sigma(1)}, X_{\sigma(2)}, X_{\sigma(3)}, \cdots, X_{\sigma(i)},\\ &g(X_{\sigma(i+1)}, \cdots, X_{\sigma(m+i)}), \cdots, X_{\sigma(n+m-1)}),\end{aligned}$$

where A_p refers to the alternating group.

4.5.3 Lie-admissible cohomology of Lie algebras

If $\mathfrak{g}$ is a Lie algebra with product μ, it is also a Lie-admissible algebra. Then it is possible to consider the following cohomologies

1. $H^*_{\mathcal{L}ieAdm}(\mathfrak{g},\mathfrak{g})$
2. $H^*_{\mathcal{L}ie}(\mathfrak{g},\mathfrak{g})$

In these two cases, the spaces of cochains are the same, excepted for $n=2$. Recall also that the second cohomology in nothing other that the Chevalley cohomology.

Definition 33 *Let $\mathfrak{g}$ be a Lie algebra. We call Lie-admissible cohomology of $\mathfrak{g}$ with value in $\mathfrak{g}$ the cohomology $H^*_{\mathcal{L}ieAdm}(\mathfrak{g},\mathfrak{g})$.*

Remark. A 2-cochain corresponding to the Chevalley's cohomology is alternated and satisfies

$$\begin{aligned}\delta^c\varphi(X_1,X_2,X_3) &= \varphi(X_1,X_2)\cdot X_3+\varphi(X_2,X_3)\cdot X_1+\varphi(X_3,X_1)\cdot X_2\\ &\quad +\varphi(X_1\cdot X_2,X_3)+\varphi(X_2\cdot X_3,X_1)+\varphi(X_3\cdot X_1,X_2).\end{aligned}$$

where δ^c is the Chevalley operator. If we compute $\delta\varphi\in B^2_{\mathcal{L}ieAdm}(\mathfrak{g},\mathfrak{g})$, as we have $\varphi(X,Y)=-\varphi(Y,X)$, then

$$\begin{aligned}\delta\varphi(X_1,X_2,X_3) &= 2(\varphi(X_1,X_2)\cdot X_3-X_3\cdot\varphi(X_1,X_2)+\varphi(X_2,X_3)\cdot X_1\\ &\quad -X_1\cdot\varphi(X_2,X_3)+\varphi(X_3,X_1)\cdot X_2-X_2\cdot\varphi(X_3,X_1)\\ &\quad +\varphi(X_1\cdot X_2,X_3)-\varphi(X_3,X_1\cdot X_2)+\varphi(X_2\cdot X_3,X_1)\\ &\quad -\varphi(X_1,X_2\cdot X_3)+\varphi(X_3\cdot X_1,X_2)-\varphi(X_2,X_3\cdot X_1))\end{aligned}$$

and

$$\delta\varphi=4\delta^c\varphi.$$

4.6 Lie-admissible modules on a Lie algebra

4.6.1 Modules on a Lie-admissible algebra

Let $\mathcal{A}=(A,\mu)$ be a Lie-admissible algebra and M a vector space on $\mathbb{K}$.

Definition 34 *A group M is an $\mathcal{A}$-module if there are bilinear maps*

$$\begin{aligned}\lambda &: A\otimes M\to M,\\ \rho &: M\otimes A\to M,\end{aligned}$$

satisfying :
$\lambda(X,\lambda(Y,v))-\lambda(Y,\lambda(X,v))-\lambda([X,Y]_\mu,v)-\lambda(X,\rho(v,Y))+\rho(\lambda(X,v),Y)+\rho(v,[X,Y]_\mu)-\rho(\rho(v,X),Y)-\rho(\lambda(Y,v),X)+\lambda(Y,\rho(v,X))+\rho(\rho(v,Y),X)=0,$
for all $X,Y\in A$ and $v\in M$.

For example, the vector space A is an $\mathcal{A}$-module.

Proposition 35 *Let $\mathcal{A}=(A,\mu)$ be a Lie-admissible algebra and M an $\mathcal{A}$-module defined by the map λ and ρ. Then the bilinear map*

$$\widehat{\lambda}:A\otimes M\to M$$

defined by

$$\widehat{\lambda}(X,v)=\lambda(X,v)-\rho(v,X)$$

provides the vector space M with an $\mathcal{A}_L$-module structure where $\mathcal{A}_L$ is the Lie algebra $(A,[,]_\mu)$.

4.6.2 Modules on G-associative algebras

If $\mathcal{A}_V=(A,\mu_V)$ is a Vinberg algebra, an $\mathcal{A}_V$-module M is given by the maps λ and ρ satisfying the two conditions

$$\left\{ \begin{array}{l} \lambda(X,\lambda(Y,v)-\lambda(Y,\lambda(X,v))-\lambda([X,Y]_\mu,v)=0, \\ \lambda(X,\rho(v,Y)-\rho(\lambda(X,v),Y))-\rho(v,\mu(X,Y))+\rho(\rho(v,X),Y)=0. \end{array} \right.$$

Considering $\mathcal{A}_V$ as a Lie-admissible algebra, then M is also a module on this Lie-admissible algebra.

The notion of $\mathcal{A}$-module is well known in the cases of Lie-admissible algebras of type 1 (associative). For type 4 and 5 we define it by the following conditions:

- type 4:

$$\left\{ \begin{array}{l} \lambda(\mu(X,Y),v)-\lambda(X,\lambda(Y,v))-\rho(\rho(v,Y),X))+\rho(v,\mu(Y,X))=0, \\ \rho(\lambda(X,v),Y)-\lambda(X,\rho(v,Y))-\rho(\lambda(Y,v),X)+\lambda(Y,\rho(v,X))=0. \end{array} \right.$$

- type 5:

$$\lambda(\mu(X,Y),v))-\lambda(X,\lambda(Y,v))+\rho(\lambda(X,v),Y)-\lambda(X,\rho(v,Y))-\rho(v,\mu(X,Y))+\rho(\rho(v,X),Y)=0.$$

We can see that if μ is antisymmetric (i.e. a product of Lie algebra) then we find again the definition of module on Lie algebra considering $\rho(v,X)=-\lambda(X,v)$.

4.6.3 Lie-admissible modules on Lie algebras

Let $\mathcal{A}=(A,\mu)$ be a Lie algebra. Consider this algebra as a Lie-admissible algebra and we denote by $\mathcal{A}_{ad}=(A,\mu)$ this Lie-admissible algebra. It is clear that every module M on the Lie algebra $\mathcal{A}$ is also a module on the Lie-admissible algebra $\mathcal{A}_{ad}$. But the converse is false.

Definition 35 *We call Lie-admissible module on the Lie algebra $\mathcal{A}=(A,\mu)$ every module on the Lie-admissible algebra $\mathcal{A}_{ad}=(A,\mu)$.*

Let $\mathfrak{g}$ be the solvable non abelian 2-dimensional Lie algebra. There exists a basis $\{X_1,X_2\}$ such that $[X_1,X_2]=X_2$. Every one dimensional module on the Lie algebra $\mathfrak{g}$ is given by the map λ (here $\rho=-\lambda$) defined by

$$\left\{ \begin{array}{l} \lambda(X_1,v)=\alpha v \\ \lambda(X_2,v)=0. \end{array} \right.$$

On the other hand, a Lie-admissible module on $\mathfrak{g}$ is determined by the map λ and ρ given by

$$\left\{ \begin{array}{l} \lambda(X_1,v)=\alpha v, \\ \lambda(X_2,v)=\beta v, \end{array} \right. ; \quad \left\{ \begin{array}{l} \rho(v,X_1)=\gamma v, \\ \rho(v,X_2)=\beta v. \end{array} \right.$$

Suppose now that M is a n-dimensional Lie-admissible module on $\mathfrak{g}$. Then if A,B,C,D are the matrices on the linear operators $\lambda(X_1,.),\lambda(X_2,.),\rho(.,X_1),\rho(.,X_2)$ in a given basis of M, then these matrices satisfy

$$[(B-D),(C-A)]=B-D.$$

We describe in this way all the structures of Lie-admissible modules on $\mathfrak{g}$.

Now consider the Lie algebra $\mathfrak{g}=sl(2,\mathbb{C})$. By a similar computation we can see that every n-dimensional Lie-admissible module on $sl(2,\mathbb{C})$ is described by the following matrix representations:

$$\left\{ \begin{array}{l} [A_1-B_1,A_2-B_2]=4(A_2-B_2), \\ [A_1-B_1,A_3-B_3]=-4(A_3-B_3), \\ [A_2-B_2,A_3-B_3]=2(A_1-B_1). \end{array} \right.$$

Such a representation also is completely reducible.

Chapter 5

Operads for flexible and alternative algebras

In Chapter 1 (see also [56]) we have classified, for binary algebras, relations of nonassociativity that are invariant with respect to an action of the symmetric group on three elements Σ_3 on the associator. In particular we have investigated two classes of nonassociative algebras, the first one corresponds to algebras whose associator A_μ satisfies

$$A_\mu - A_\mu \circ \tau_{12} - A_\mu \circ \tau_{23} - A_\mu \circ \tau_{13} + A_\mu \circ c + A_\mu \circ c^2 = 0, \tag{5.1}$$

and the second

$$A_\mu + A_\mu \circ \tau_{12} + A_\mu \circ \tau_{23} + A_\mu \circ \tau_{13} + A_\mu \circ c + A_\mu \circ c^2 = 0. \tag{5.2}$$

where τ_{ij} denotes the transposition exchanging i and j, c is the 3-cycle $(1,2,3)$.

These relations are in correspondence with the only two irreducible one-dimensional subspaces of $\mathbb{K}[\Sigma_3]$ with respect to the action of Σ_3, where $\mathbb{K}[\Sigma_3]$ is the group algebra of Σ_3. In the previous chapter, we have studied the operadic and deformations aspects of the first class which corresponds to Lie-admissible algebras. Now we investigate the second class, that is, nonassociative algebras satisfying (5.2). Such an algebra is 3-power associative (see Chapter 1 for the definition) and is called p^3-associative. Among these algebras we have the class of G-p^3-associative algebras where the p^3-associative identity is given by the vector $\Sigma_{\sigma\in G}\sigma$, where G is a subgroup of Σ_3. And among the class of G-p^3-associative algebras we have flexible algebras, left and right alternative algebras which play to day, an important role in physic (for example, the octonion algebra). We determine explicitly the corresponding quadratic operads and their duals. We prove, in particular, that the operad of left and right alternative algebras, flexible algebras and of p^3-associative algebras are not Koszul. We focus on the deformations of alternative algebras, computing the operadic cohomology and proving that this cohomology cannot parameterize deformations.

5.1 G-p^3-associative algebras

5.1.1 Definition

We associate to each subgroup G of Σ_3 the vector $w_G = \sum_{\sigma\in G}\sigma$ of $\mathbb{K}[\Sigma_3]$.

We know that the one dimensional subspace $\mathbb{K}\{w_{\Sigma_3}\}$ of $\mathbb{K}[\Sigma_3]$ generated by

$$w_{G_6} = w_{\Sigma_3} = \sum_{\sigma\in\Sigma_3}\sigma$$

is an irreducible invariant subspace of $\mathbb{K}[\Sigma_3]$.

Definition 36 *A G-p^3-associative algebra is a $\mathbb{K}$-algebra $(\mathcal{A},\mu)$ whose associator*

$$A_\mu = \mu \circ (\mu \otimes Id - Id \otimes \mu)$$

satisfies

$$A_\mu \circ \Phi^{\mathcal{A}}_{w_{G_6}} = 0,$$

where $\Phi^{\mathcal{A}}_{w_G} : \mathcal{A}^{\otimes^3} \to \mathcal{A}^{\otimes^3}$ is the linear map

$$\Phi^{\mathcal{A}}_{w_G}(x_1 \otimes x_2 \otimes x_3) = \sum_{\sigma \in G} (x_{\sigma^{-1}(1)} \otimes x_{\sigma^{-1}(2)} \otimes x_{\sigma^{-1}(3)}).$$

Let $\mathcal{O}(w_G)$ be the orbit of w_G with respect to the natural action

$$\begin{array}{ccc} \Sigma_3 \times \mathbb{K}[\Sigma_3] & \rightarrow & \mathbb{K}[\Sigma_3] \\ (\sigma, \sum_i a_i \sigma_i) & \mapsto & \sum_i a_i \sigma^{-1} \circ \sigma_i. \end{array}$$

Then putting $F_{w_G} = \mathbb{K}(\mathcal{O}(w_G))$ we have

$$\left\{ \begin{array}{l} \dim F_{w_{G_1}} = 6, \\ \dim F_{w_{G_2}} = \dim F_{w_{G_3}} = \dim F_{w_{G_4}} = 3, \\ \dim F_{w_{G_5}} = 2, \\ \dim F_{w_{G_6}} = 1. \end{array} \right.$$

Proposition 36 *Every G-p^3-associative algebra is third power associative.*

Recall that a third power associative algebra is an algebra $(\mathcal{A},\mu)$ whose associator satisfies $A_\mu(x,x,x) = 0$. Linearizing this relation, we obtain

$$A_\mu \circ \Phi^{\mathcal{A}}_{w_{\Sigma_3}} = 0.$$

Since each of the invariant spaces F_G contains the vector w_{Σ_3}, we deduce the proposition.

Remark. An important class of third power associative algebras is the class of power associative algebras, that is, algebras such that any element generates an associative subalgebra.

5.1.2 What are G-p^3-associative algebras?

1. If $i = 1$, since $w_{G_1} = Id$, the class of G_1-p^3-associative algebras is the full class of associative algebras.
2. If $i = 2$, the associator of a G_2-p^3-associative algebra $\mathcal{A}$ satisfies

   $$A_\mu(x_1,x_2,x_3) + A_\mu(x_2,x_1,x_3) = 0$$

   and this is equivalent to

   $$A_\mu(x,x,y),$$

   for all $x, y \in \mathcal{A}$.
3. If $i = 3$, the associator of a G_3-p^3-associative algebra $\mathcal{A}$ satisfies

   $$A_\mu(x_1,x_2,x_3) + A_\mu(x_1,x_3,x_2) = 0,$$

   that is,

   $$A_\mu(x,y,y),$$

   for all $x, y \in \mathcal{A}$.

   Sometimes G_2-p^3-associative algebras are called left-alternative algebras, G_3-p^3-associative algebras are right-alternative algebras. An alternative algebra is an algebra which satisfies the G_2 and G_3-p^3-associativity.

4. If $i = 4$, we have $A_\mu(x, y, x)$, for all $x, y \in \mathcal{A}$, and the class of G_3-p^3-associative algebras is the class of flexible algebras.
5. If $i = 5$, the class of G_5-p^3-associative algebras corresponds to G_5-associative algebras ([55]).
6. If $i = 6$, the associator of a G_6-p^3-associative algebra satisfies

$$\begin{array}{l} A_\mu(x_1, x_2, x_3) + A_\mu(x_2, x_1, x_3) + A_\mu(x_3, x_2, x_1) \\ +A_\mu(x_1, x_3, x_2) + A_\mu(x_2, x_3, x_1)A_\mu(x_3, x_1, x_2) = 0. \end{array}$$

If we consider the symmetric product $x \star y = \mu(x, y) + \mu(y, x)$ and the skew-symmetric product $[x, y] = \mu(x, y) - \mu(y, x)$, then the G_6-p^3-associative identity is equivalent to

$$[x \star y, z] + [y \star z, x] + [z \star x, y] = 0.$$

Definition 37 *A* $([\ ,\ ], \star)$*-admissible-algebra is a* $\mathbb{K}$*-vector space* $\mathcal{A}$ *provided with two multiplication:*

(a) a symmetric multiplication $\star$*,*

(b) a skew-symmetric multiplication $[,]$ *satisfying the identity*

$$[x \star y, z] + [y \star z, x] + [z \star x, y] = 0$$

for any $x, y \in \mathcal{A}$*.*

Then a G_6-p^3-associative algebra can be defined as $([\ ,\ ], \star)$-admissible algebra.

5.2 The operads $\mathcal{G}_i$-p^3ass and their dual

For each $i \in \{1, \cdots, 6\}$, the operad for G-p^3-associative algebras will be denoted by $\mathcal{G}_i$-p^3ass. The operads $\{\mathcal{G}_i\text{-}p^3ass\}_{i=1,\cdots,6}$ are binary quadratic operads. Recall that a quadratic operad $\mathcal{P}$ is Koszul if the free $\mathcal{P}$-algebra based on a $\mathbb{K}$-vector space V is Koszul, for any vector space V. This property is conserved by duality and can be studied using generating functions of $\mathcal{P}$ and of $\mathcal{P}^!$ (see [46] or [93]).

Before studying the Koszulness of the operads G-$p^3\mathcal{A}ss$, we compute the homology of A_2 the two dimensional associative algebra A_2 given in a basis $\{e_1, e_2\}$ by $e_1e_1 = e_2, e_1e_2 = e_2e_1 = e_2e_2 = 0$. The Hochschild homology of an associative algebra is given by the complex $(C_n(\mathcal{A}, \mathcal{A}), d_n)$ where $C_n(\mathcal{A}, \mathcal{A}) = \mathcal{A} \otimes \mathcal{A}^{\otimes n}$ and the differentials $d_n : C_n(\mathcal{A}, \mathcal{A}) \to C_{n-1}(\mathcal{A}, \mathcal{A})$ are given by

$$\begin{array}{rl} d_n(a_0, a_1, \cdots, a_n) = & (a_0a_1, a_2, \cdots a_n) + \sum_{i=1}^{n-1}(-1)^i(a_0, a_1, \cdots, a_ia_{i+1}, \cdots, a_n) \\ & +(-1)^n(a_na_0, a_1, \cdots a_{n-1}). \end{array}$$

Concerning the algebra A_2, we have

$$d_1(e_i, e_j) = e_ie_j - e_je_i = 0,$$

for any $i, j = 1, 2$. Similarly we have

$$\left\{ \begin{array}{l} d_2(e_1, e_1, e_1) = 2(e_2, e_1) - (e_1, e_2), \\ d_2(e_1, e_1, e_2) = d_2(e_1, e_2, e_1) = -d_2(e_2, e_1, e_1) = (e_2, e_2) \end{array} \right.$$

and 0 in all the other cases. Then $\dim Imd_2 = 2$ and $\dim Kerd_1 = 4$. Then $H_1(A_2, A_2)$ is isomorphic to A_2. We have also

$$\left\{ \begin{array}{ll} d_3(e_1, e_1, e_1, e_1) & = -(e_1, e_2, e_1) + (e_1, e_1, e_2), \\ d_3(e_1, e_1, e_1, e_2) & = (e_2, e_1, e_2) - (e_1, e_2, e_2), \\ d_3(e_1, e_1, e_2, e_1) & = -d_2(e_2, e_1, e_1, e_1) = (e_2, e_2, e_1) - (e_2, e_1, e_2), \\ d_3(e_1, e_2, e_1, e_1) & = (e_1, e_2, e_2) - (e_2, e_2, e_1), \\ d_3(e_1, e_1, e_2, e_2) & = -d_3(e_1, e_2, e_2, e_1) = -d_3(e_2, e_1, e_1, e_2) = d_3(e_2, e_2, e_1, e_1) \\ & = (e_2, e_2, e_2) \end{array} \right.$$

and $d_3 = 0$ in all the other cases. Then $\dim Imd_3 = 4$ and $\dim Kerd_2 = 6$. Thus $H_2(A_2, A_2)$ is non trivial and A_2 is not a Koszul algebra.

Now we will study all the operads $\mathcal{G}_i$-p^3ass.

5.2.1 The operad $\mathcal{G}_1$-p^3ass

Since $\mathcal{G}_1$-$p^3ass = \mathcal{A}ss$, where $\mathcal{A}ss$ denotes the operad for associative algebras, and since the operad $\mathcal{A}ss$ is selfdual, we have

$$(\mathcal{G}_1\text{-}p^3ass)^! = \mathcal{A}ss^! = \mathcal{G}_1\text{-}p^3ass.$$

5.2.2 The operad $\mathcal{G}_2$-p^3ass

The operad $\mathcal{G}_2$-p^3ass is the operad for left-alternative algebras. It is the quadratic operad $\mathcal{P} = \Gamma(E)/(R)$, where the Σ_3-invariant subspace R of $\Gamma(E)(3)$ is generated by the vectors

$$(x_1 \cdot x_2) \cdot x_3 - x_1 \cdot (x_2 \cdot x_3) + (x_2 \cdot x_1) \cdot x_3 - x_2 \cdot (x_1 \cdot x_3).$$

The annihilator $R^\perp$ of R with respect to the pairing (4.2) is generated by the vectors

$$\begin{cases} (x_1 \cdot x_2) \cdot x_3 - x_1 \cdot (x_2 \cdot x_3), \\ (x_1 \cdot x_2) \cdot x_3 + (x_2 \cdot x_1) \cdot x_3. \end{cases} \tag{5.3}$$

We deduce from direct calculations that $\dim R^\perp = 9$ and

Proposition 37 *The $(\mathcal{G}_2$-$p^3ass)^!$-algebras are associative algebras satisfying*

$$abc = -bac.$$

Theorem 38 *The operad $(\mathcal{G}_2$-$p^3ass)^!$ is not Koszul.*

Proof. It is easy to describe $(\mathcal{G}_2$-$p^3ass)^!(n)$ for any n. In fact $(\mathcal{G}_2$-$p^3ass)^!(4)$ correspond to associative elements satisfying

$$x_1x_2x_3x_4 = -x_2x_1x_3x_4 = -x_2(x_1x_3)x_4 = x_1x_3x_2x_4 = -x_1x_2x_3x_4$$

and $(\mathcal{G}_2$-$p^3ass)^!(4) = \{0\}$. Let $\mathcal{P}$ be $\mathcal{G}_2$-p^3ass. The generating function of $\mathcal{P}^! = (\mathcal{G}_2$-$p^3ass)^!$ is

$$g_{\mathcal{P}^!}(x) = \sum_{a \geq 1} \frac{1}{a!} \dim(\mathcal{G}_2\text{-}p^3ass)^!(a)x^a = x + x^2 + \frac{x^3}{2}.$$

But the generating function of $\mathcal{P} = (\mathcal{G}_2$-$p^3ass)$ is

$$g_{\mathcal{P}}(x) = x + x^2 + \frac{3}{2}x^3 + \frac{5}{2}x^4 + O(x^5)$$

and if $\mathcal{G}_2$-p^3ass is Koszul, then the generating functions should be related by the functional equation

$$g_{\mathcal{P}}(-g_{\mathcal{P}^!}(-x)) = x$$

and it is not the case so both $\mathcal{G}_2$-p^3ass and $(\mathcal{G}_2$-$p^3ass)^!$ are not Koszul.

Remark 39 *Since the redaction of this work, Dzumadildaev and Zusmanovich have shown and published this result. For this reason we refer this theorem to these authors.*

By definition, a quadratic operad $\mathcal{P}$ is Koszul if any free $\mathcal{P}$-algebra on a vector space V is a Koszul algebra. Let us describe the free algebra $\mathcal{F}_{(\mathcal{G}_2\text{-}p^3ass)^!}(V)$ when $\dim V = 1$ and 2.

A $(\mathcal{G}_2\text{-}p^3ass)^!$-algebra $\mathcal{A}$ is an associative algebra satisfying

$$xyz = -yxz,$$

for any $x, y, z \in \mathcal{A}$. This implies $xyzt = 0$ for any $x, y, z \in \mathcal{A}$. In particular we have

$$\begin{cases} x^3 = 0, \\ x^2y = 0, \end{cases}$$

for any $x, y \in \mathcal{A}$. If $\dim V = 1$, $\mathcal{F}_{(\mathcal{G}_2\text{-}p^3ass)^!}(V)$ is of dimension 2 and given by

$$\begin{cases} e_1e_1 = e_2, \\ e_1e_2 = e_2e_1 = e_2e_2 = 0. \end{cases}$$

In fact if $V = \mathbb{K}\{e_1\}$ thus in $\mathcal{F}_{(\mathcal{G}_2\text{-}p^3ass)^!}(V)$ we have $e_1^3 = 0$. We deduce that $\mathcal{F}_{(\mathcal{G}_2\text{-}p^3ass)^!}(V) = A_2$ and $\mathcal{F}_{(\mathcal{G}_2\text{-}p^3ass)^!}(V)$ is not Koszul. It is easy to generalize this construction. If $\dim V = n$, then

$$\dim \mathcal{F}_{(\mathcal{G}_2\text{-}p^3ass)^!}(V) = \frac{n(n^2+n+2)}{2}$$

and if $\{e_1, \cdots, e_n\}$ is a basis of V then $\{e_i, e_i^2, e_ie_j, e_le_me_p\}$, for $i, j = 1, \cdots, n$ and $l, m, p = 1, \cdots, n$ with $m > l$, is a basis of $\mathcal{F}_{(\mathcal{G}_2\text{-}p^3ass)^!}(V)$. For example, if $n = 2$, the basis of $\mathcal{F}_{(\mathcal{G}_2\text{-}p^3ass)^!}(V)$ is

$$\{v_1 = e_1, v_2 = e_2, v_3 = e_1^2, v_4 = e_2^2, v_5 = e_1e_2, v_6 = e_2e_1, v_7 = e_1e_2e_1, v_8 = e_1e_2^2\}$$

and the multiplication table is

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
v_1	v_3	v_5	0	v_8	0	v_7	0	0
v_2	v_6	v_4	$-v_7$	0	$-v_8$	0	0	0
v_3	0	0	0	0	0	0	0	0
v_4	0	0	0	0	0	0	0	0
v_5	v_7	v_8	0	0	0	0	0	0
v_6	$-v_7$	$-v_8$	0	0	0	0	0	0
v_7	0	0	0	0	0	0	0	0
v_8	0	0	0	0	0	0	0	0

For this algebra we have

$$\begin{cases} d_1(v_1, v_2) = v_5 - v_6, \\ \frac{1}{2}d_1(v_1, v_6) = v_7 = -d_1(v_1, v_5) = d_1(v_2, v_3), \\ \frac{1}{2}d_1(v_2, v_5) = -v_8 = d_1(v_6, v_2) = -d_1(v_1, v_4), \end{cases}$$

and $Ker\, d_1$ is of dim 64. The space $Im\, d_2$ doesn't contain in particular the vectors (v_i, v_i) for $i = 1, 2$ because these vectors v_i are not in the derived subalgebra. Since these vectors are in $Ker\, d_1$ we deduce that the second space of homology is not trivial.

5.2.3 The operad $\mathcal{G}_3$-p^3ass

It is defined by the module of relations generated by the vector

$$(x_1x_2)x_3 - x_1(x_2x_3) + (x_1x_3)x_2 - x_1(x_3x_2),$$

and $R^\perp$ is the linear span of

$$\begin{cases} (x_1x_2)x_3 - x_1(x_2x_3), \\ (x_1x_2)x_3 + (x_1x_3)x_2. \end{cases}$$

Proposition 38 *A $(\mathcal{G}_3\text{-}p^3ass)^!$-algebra is an associative algebra $\mathcal{A}$ satisfying*

$$abc = -acb,$$

for any $a, b, c \in \mathcal{A}$.

Since $(\mathcal{G}_3\text{-}p^3ass)^!$ is basically isomorphic to $(\mathcal{G}_2\text{-}p^3ass)^!$ we deduce that $(\mathcal{G}_3\text{-}p^3ass)$ is not Koszul.

5.2.4 The operad $\mathcal{G}_4\text{-}p^3ass$

Remark that a $(\mathcal{G}_4\text{-}p^3ass)$-algebra is generally called flexible algebra. The relation

$$A_\mu(x_1, x_2, x_3) + A_\mu(x_3, x_2, x_1) = 0$$

is equivalent to $A_\mu(x, y, x) = 0$ and this denotes the flexibility of $(\mathcal{A}, \mu)$.

Proposition 39 *A $(\mathcal{G}_4\text{-}p^3ass)^!$-algebra is an associative algebra satisfying*

$$abc = -cba.$$

This implies that

$$\dim(\mathcal{G}_4\text{-}p^3ass)^!(3) = 3.$$

We have also $x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}x_{\sigma(4)} = (-1)^{\varepsilon(\sigma)}x_1x_2x_3x_4$ for any $\sigma \in \Sigma_4$. This gives $\dim(G_4\text{-}p^3\mathcal{A}ss)^!(4) = 1$. Similarly

$$\begin{array}{rl} x_1x_2(x_3x_4x_5) & = -x_3(x_4x_5x_2)x_1 = x_1(x_4x_5(x_2x_3)) = -x_1x_2(x_3x_5x_4) \\ & = (x_1x_2(x_4x_5))x_3 = -(x_4x_5)(x_2x_1)x_3 = (x_3x_2x_1)x_4x_5 \\ & = -x_1x_2x_3x_4x_5 \end{array}$$

(the algebra is associative so we put some parenthesis just to explain how we pass from one expression to an other). We deduce $(\mathcal{G}_4\text{-}p^3ass)^!(5) = \{0\}$ and more generally $(\mathcal{G}_4\text{-}p^3ass)^!(a) = \{0\}$ for $a \geq 5$.

The generating function of $(\mathcal{G}_4\text{-}p^3ass)^!$ is

$$f(x) = x + x^2 + \frac{x^3}{2} + \frac{x^4}{12}.$$

Let $\mathcal{F}_{(\mathcal{G}_4\text{-}p^3ass)^!}(V)$ be the free $(\mathcal{G}_4\text{-}p^3ass)^!$-algebra based on the vector space V. In this algebra we have the relations

$$\begin{cases} a^3 = 0, \\ aba = 0, \end{cases}$$

for any $a, b \in V$. Assume that $\dim V = 1$. If $\{e_1\}$ is a basis of V, then $e_1^3 = 0$ and $\mathcal{F}_{(\mathcal{G}_4\text{-}p^3ass)^!}(V) = \mathcal{F}_{(\mathcal{G}_2\text{-}p^3ass)^!}(V)$. We deduce that $\mathcal{F}_{(\mathcal{G}_4\text{-}p^3ass)^!}(V)$ is not a Koszul algebra.

Proposition 40 *The operad for flexible algebra is not Koszul.*

Let us note that, if $\dim V = 2$ and $\{e_1, e_2\}$ is a basis of V, then $\mathcal{F}_{(\mathcal{G}_4\text{-}p^3ass)^!}(V)$ is generated by $\{e_1, e_2, e_1^2, e_2^2, e_1e_2, e_2e_1, e_1e_2^2, e_1^2e_2, e_2e_1^2, e_2^2e_1, e_1^2e_2^2, e_2^2e_1^2\}$ and is of dimension 12.

5.2.5 The operad $\mathcal{G}_5\text{-}p^3ass$

It coincides with $\mathcal{G}_5$-*ass* and this last has been studied in [55].

5.2.6 The operad $\mathcal{G}_6$-p^3ass

A $\mathcal{G}_6$-p^3ass-algebra $(\mathcal{A},\mu)$ satisfies the relation

$$\begin{array}{c} A_\mu(x_1,x_2,x_3)+A_\mu(x_2,x_1,x_3)+A_\mu(x_3,x_2,x_1) \\ +A_\mu(x_1,x_3,x_2)+A_\mu(x_2,x_3,x_1)+A_\mu(x_3,x_1,x_2)=0. \end{array}$$

The dual operad $(\mathcal{G}_6\text{-}p^3ass)^!$ is generated by the relations

$$\begin{cases} (x_1x_2)x_3 = x_1(x_2x_3), \\ (x_1x_2)x_3 = (-1)^{\varepsilon(\sigma)}(x_{\sigma(1)}x_{\sigma(2)})x_{\sigma(3)}, \text{ for all } \sigma\in\Sigma_3. \end{cases}$$

We deduce

Proposition 41 *A $(\mathcal{G}_6\text{-}p^3ass)^!$-algebra is an associative algebra $\mathcal{A}$ which satisfies*

$$abc=-bac=-cba=-acb=bca=cab,$$

for any $a,b,c\in\mathcal{A}$. In particular

$$\begin{cases} a^3=0, \\ aba=aab=baa=0. \end{cases}$$

Lemma 7 *The operad $(\mathcal{G}_6\text{-}p^3ass)^!$ satisfies $(\mathcal{G}_6\text{-}p^3ass)^!(4)=\{0\}$.*

Proof. We have in $(\mathcal{G}_6\text{-}p^3ass)^!(4)$ that

$$x_1(x_2x_3)x_4 = x_2(x_3x_4)x_1 = -x_1(x_3x_4x_2) = x_1x_3x_2x_4 = -x_1x_2x_3x_4$$

so $x_1x_2x_3x_4=0$. We deduce that the generating function of $(\mathcal{G}_6\text{-}p^3ass)^!$ is

$$f^!(x)=x+x^2+\frac{x^3}{6}.$$

If this operad is Koszul the generating function of the operad $\mathcal{G}_6$-p^3ass should be of the form

$$f(x)=x+x^2+\frac{11}{6}x^3+\frac{25}{6}x^4+\frac{127}{12}x^5+\cdots$$

But if we look the free algebra generated by V with $\dim V=1$, it satisfies $a^3=0$ and coincides with $\mathcal{F}_{(\mathcal{G}_2\text{-}p^3ass)^!}(V)$. Then $\mathcal{G}_6$-p^3ass is not Koszul.

5.3 Cohomology and Deformations

Let $(\mathcal{A},\mu)$ be a $\mathbb{K}$-algebra defined by quadratic relations. It is attached to a quadratic linear operad $\mathcal{P}$. By deformations of $(\mathcal{A},\mu)$, we mean ([54])

- a $\mathbb{K}^*$ non archimedian extension field of $\mathbb{K}$, with a valuation v such that, if A is the ring of valuation and $\mathfrak{m}$ the unique ideal of A, then the residual field $A/\mathfrak{m}$ is isomorphic to $\mathbb{K}$.
- The $A/\mathfrak{m}$ vector space $\widetilde{\mathcal{A}}$ is $\mathbb{K}$-isomorphic to $\mathcal{A}$.
- For any $a,b\in\mathcal{A}$ we have that

  $$\tilde{\mu}(a,b)-\mu(a,b)$$

  belongs to the $\mathfrak{m}$-module $\widetilde{\mathcal{A}}$ (isomorphic to $\mathcal{A}\otimes\mathfrak{m}$) .

The most important example concerns the case where A is $\mathbb{K}[[t]]$, the ring of formal series. In this case $\mathfrak{m} = \left\{ \sum_{i \geq 1} a_i t^i, a_i \in \mathbb{K} \right\}$, $\mathbb{K}^* = \mathbb{K}((t))$ the field of rational fractions. This case corresponds to the classical Gerstenhaber deformations. Since A is a local ring, all the notions of valued deformations coincides ([30]).

We know ([82]) that there exists always a cohomology which parametrizes deformations. If the operad $\mathcal{P}$ is Koszul, this cohomology is the "standard"-cohomology called the operadic cohomology. If the operad $\mathcal{P}$ is not Koszul, the cohomology which governs deformations is based on the minimal model of $\mathcal{P}$ and the operadic cohomology and deformations cohomology differ.

In this section we focus on the case of left-alternative algebras, that is, on the operad $\mathcal{G}_2$-p^3ass and also by the classical alternative algebras.

5.3.1 Deformations and cohomology of left-alternative algebras

A $\mathbb{K}$-left-alternative algebra $(\mathcal{A}, \mu)$ is a $\mathbb{K}$-($\mathcal{G}_2$-p^3ass)-algebra. Then μ satisfies

$$A_\mu(x_1, x_2, x_3) + A_\mu(x_2, x_1, x_3) = 0.$$

A valued deformation can be viewed as a $\mathbb{K}[[t]]$-algebra $(A \otimes \mathbb{K}[[t]], \mu_t)$ whose product μ_t is given by

$$\mu_t = \mu + \sum_{i \geq 1} t^i \varphi_i.$$

a) *The operadic cohomology*

It is the standard cohomology $H^*_{\mathcal{G}_2\text{-}p^3ass}(\mathcal{A}, \mathcal{A})_{st}$ of the $\mathcal{G}_2$-p^3ass-algebra $(\mathcal{A}, \mu)$. It is associated to the cochains complex

$$\mathcal{C}^1_{\mathcal{P}}(\mathcal{A}, \mathcal{A})_{st} \xrightarrow{\delta^1_{st}} \mathcal{C}^2_{\mathcal{P}}(\mathcal{A}, \mathcal{A})_{st} \xrightarrow{\delta^2_{st}} \mathcal{C}^3_{\mathcal{P}}(\mathcal{A}, \mathcal{A})_{st} \xrightarrow{\delta^3_{st}} \cdots$$

where $\mathcal{P} = \mathcal{G}_2$-$p^3ass$ and

$$\mathcal{C}^p_{\mathcal{P}}(\mathcal{A}, \mathcal{A})_{st} = Hom(\mathcal{P}^!(p) \otimes_{\Sigma_p} \mathcal{A}^{\otimes p}, \mathcal{A}).$$

Since $(\mathcal{G}_2\text{-}p^3ass)^!(4) = 0$, we deduce that

$$H^p_{\mathcal{P}}(\mathcal{A}, \mathcal{A})_{st} = 0 \text{ for } p \geq 4,$$

because the cochains complex is a short sequence

$$\mathcal{C}^1_{\mathcal{P}}(\mathcal{A}, \mathcal{A})_{st} \xrightarrow{\delta^1_{st}} \mathcal{C}^2_{\mathcal{P}}(\mathcal{A}, \mathcal{A})_{st} \xrightarrow{\delta^2_{st}} \mathcal{C}^3_{\mathcal{P}}(\mathcal{A}, \mathcal{A})_{st} \xrightarrow{0} 0.$$

The coboundary operators are given by

$$\begin{cases} \delta^1 f(a,b) &= f(a)b + af(b) - f(ab), \\ \delta^2 \varphi(a,b,c) &= \varphi(ab,c) + \varphi(ba,c) - \varphi(a,bc) - \varphi(b,ac) \\ & \quad \varphi(a,b)c + \varphi(b,a)c - a\varphi(b,c) - b\varphi(a,c). \end{cases}$$

b) *The deformations cohomology*

The minimal model of $\mathcal{G}_2$-p^3ass is a homology isomorphism

$$(\mathcal{G}_2\text{-}p^3ass, 0) \xrightarrow{\rho} (\Gamma(E), \partial)$$

of dg-operads such that the image of ∂ consists of decomposable elements of the free operad $\Gamma(E)$. Since $\mathcal{G}_2$-$p^3ass(1) = \mathbb{K}$, this minimal model exists and it is unique. The deformations cohomology $H^*(\mathcal{A}, \mathcal{A})_{defo}$ of $\mathcal{A}$ is the cohomology of the complex ([87])

$$\mathcal{C}^1_{\mathcal{P}}(\mathcal{A}, \mathcal{A})_{defo} \xrightarrow{\delta^1} \mathcal{C}^2_{\mathcal{P}}(\mathcal{A}, \mathcal{A})_{defo} \xrightarrow{\delta^2} \mathcal{C}^3_{\mathcal{P}}(\mathcal{A}, \mathcal{A})_{defo} \xrightarrow{\delta^3} \cdots$$

where

$$\begin{cases} \mathcal{C}^1_{\mathcal{P}}(\mathcal{A},\mathcal{A})_{defo} = Hom(\mathcal{A},\mathcal{A}), \\ \mathcal{C}^k_{\mathcal{P}}(\mathcal{A},\mathcal{A})_{defo} = Hom(\oplus_{q\geq 2}E_{k-2}(q)\otimes_{\Sigma_q}\mathcal{A}^{\otimes q},\mathcal{A}). \end{cases}$$

The Euler characteristics of $E(q)$ can be read off from the inverse of the generating function of the operad $\mathcal{G}_2$-p^3ass

$$g_{\mathcal{G}_2\text{-}p^3ass}(t) = t + t^2 + \frac{3}{2}t^3 + \frac{5}{2}t^4 + \frac{53}{12}t^5$$

which is

$$g(t) = t - t^2 + \frac{t^3}{2} + \frac{13}{3}t^5 + O(t^6).$$

We obtain in particular

$$\chi(E(4)) = 0.$$

Each one of the modules $E(p)$ is a graded module $(E_*(p))$ and

$$\chi(E(p)) = \dim E_0(p) - \dim E_1(p) + \dim E_2(p) + \cdots$$

We deduce

- $E(2)$ is generated by two degree 0 bilinear operation $\mu_2 : V\cdot V \rightarrow V$,
- $E(3)$ is generated by three degree 1 trilinear operation $\mu_3 : V^{\otimes^3} \rightarrow V$,
- $E(4) = 0$.

Considering the action of Σ_n on $E(n)$ we deduce that $E(2)$ is generated by a binary operation of degree 0 whose differential satisfies

$$\partial(\mu_2) = 0,$$

$E(3)$ is generated by a trilinear operation of degree one such that

$$\partial(\mu_3) = \mu_2 \circ_1 \mu_2 - \mu_2 \circ_2 \mu_2 + \mu_2 \circ_1 (\mu_2\cdot\tau_{12}) - (\mu_2 \circ_2 \mu_2)\cdot\tau_{12}.$$

(we have $(\mu_2 \circ_2 \mu_2)\cdot\tau_{12}(a,b,c) = b(ac)$)

Since $E(4) = 0$ we deduce

Proposition 42 *The cohomology $H^*(\mathcal{A},\mathcal{A})_{defo}$ which governs deformations or left-alternative algebras is associated to the complex*

$$\mathcal{C}^1_{\mathcal{P}}(\mathcal{A},\mathcal{A})_{defo} \xrightarrow{\delta^1} \mathcal{C}^2_{\mathcal{P}}(\mathcal{A},\mathcal{A})_{defo} \xrightarrow{\delta^2} \mathcal{C}^3_{\mathcal{P}}(\mathcal{A},\mathcal{A})_{defo} \xrightarrow{\delta^3} \mathcal{C}^4_{\mathcal{P}}(\mathcal{A},\mathcal{A})_{defo} \rightarrow \cdots$$

with

$$\begin{array}{l} \mathcal{C}^1_{\mathcal{P}}(\mathcal{A},\mathcal{A})_{defo} = Hom(V^{\otimes 1},V), \\ \mathcal{C}^2_{\mathcal{P}}(\mathcal{A},\mathcal{A})_{defo} = Hom(V^{\otimes 2},V), \\ \mathcal{C}^3_{\mathcal{P}}(\mathcal{A},\mathcal{A})_{defo} = Hom(V^{\otimes 3},V), \\ \mathcal{C}^4_{\mathcal{P}}(\mathcal{A},\mathcal{A})_{defo} = Hom(V^{\otimes 5},V)\oplus\cdots\oplus Hom(V^{\otimes 5},V), \end{array}$$

In particular any 4-cochains consists of 5-linear maps.

5.3.2 Alternative algebras

Recall that an alternative algebra is given by the relation

$$A_\mu(x_1,x_2,x_3) = -A_\mu(x_2,x_1,x_3) = A_\mu(x_2,x_3,x_1).$$

Theorem 40 *An algebra $(\mathcal{A}, \mu)$ is alternative if and only if the associator satisfies*

$$A_\mu \circ \Phi_v^{\mathcal{A}} = 0,$$

with $v = 2Id + \tau_{12} + \tau_{13} + \tau_{23} + c_1$.

Proof. The associator satisfies $A_\mu \circ \Phi_{v_1}^{\mathcal{A}} = A_\mu \circ \Phi_{v_2}^{\mathcal{A}}$ with $v_1 = Id + \tau_{12}$ and $v_2 = Id + \tau_{23}$. The invariant subspace of $\mathbb{K}[\Sigma_3]$ generated by v_1 and v_2 is of dimension 5 and contains the vector $\sum_{\sigma \in \Sigma_3} \sigma$. From Chapter 1, the space is generated by the orbit of the vector v.

Proposition 43 *Let $\mathcal{A}lt$ be the operad for alternative algebras. Its dual is the operad for associative algebras satisfying*

$$abc + bac + cba + acb + bca + cab = 0.$$

In [28], one gives the generating functions of $\mathcal{P} = \mathcal{A}lt$ and $\mathcal{P}^! = \mathcal{A}lt^!$

$$\begin{array}{l} g_{\mathcal{P}}(x) = x + \frac{2}{2!}x^2 + \frac{7}{3!}x^3 + \frac{32}{4!}x^4 + \frac{175}{5!}x^5 + \frac{180}{6!}x^6 + O(x^7), \\ g_{\mathcal{P}^!}(x) = x + \frac{2}{2!}x^2 + \frac{5}{3!}x^3 + \frac{12}{4!}x^4 + \frac{15}{5!}x^5. \end{array}$$

and conclude to the non Koszulity of $\mathcal{A}lt$.

The operadic cohomology is the cohomology associated to the complex

$$(\mathcal{C}^p_{\mathcal{A}lt}(\mathcal{A}, \mathcal{A})_{st} = (Hom(\mathcal{A}lt^!(p) \otimes_{\Sigma_p} \mathcal{A}^{\otimes p}, \mathcal{A}), \delta_{st}).$$

Since $\mathcal{A}lt^!(p) = 0$ for $p \geq 6$ we deduce the short sequence

$$\mathcal{C}^1_{\mathcal{A}lt}(\mathcal{A}, \mathcal{A})_{st} \xrightarrow{\delta^1_{st}} \mathcal{C}^2_{\mathcal{A}lt}(\mathcal{A}, \mathcal{A})_{st} \to \cdots \to \mathcal{C}^5_{\mathcal{A}lt}(\mathcal{A}, \mathcal{A})_{st} \to 0.$$

But if we compute the formal inverse of the function $-g_{\mathcal{A}lt}(-x)$ we obtain

$$x + x^2 + \frac{5}{6}x^3 + \frac{1}{2}x^4 + \frac{1}{8}x^5 - \frac{11}{72}x^6 + O(x^7).$$

Because of the minus sign it can not be the generating function of the operad $\mathcal{P}^! = \mathcal{A}lt^!$. So this implies also that both operads are not Koszul. But it gives also some information on the deformation cohomology. In fact if $\Gamma(E)$ is the free operad associated to the minimal model, then

$$\left\{ \begin{array}{l} \dim \chi(E(2)) = 2, \\ \dim \chi(E(3)) = -5, \\ \dim \chi(E(4)) = 12, \\ \dim \chi(E(5)) = -15, \\ \dim \chi(E(6)) = -110. \end{array} \right.$$

Since $\chi(E(6)) = \sum_i (-1)^i \dim E_i(6)$, the graded space $E(6)$ is not concentred in even degree. Then the 6-cochains of the deformation cohomology are 6-linear maps of odd degree.

Chapter 6

Current operads, Dihedral, Cyclic and Hopf operads

In this chapter we investigate different properties of a quadratic operad $\mathcal{P}$ such as dihedrality, cyclicity and the property of being Hopf. We also define the current operad $\widetilde{\mathcal{P}}$ associated to $\mathcal{P}$ in the context of quadratic operad with only one binary generating operation (the algebras over this operad have only one binary operation). The dual operad of $\mathcal{P}$, denoted by $\mathcal{P}^!$ (see [46] for terminology), equals to $\mathcal{P}^! = hom(\mathcal{P}, \mathcal{L}ie)$, where $\mathcal{L}ie$ is the quadratic operad corresponding to Lie algebras. For any $\mathcal{P}$-algebra $\mathcal{A}$ and $\mathcal{P}^!$-algebra $\mathcal{B}$, the vector space $\mathcal{A} \otimes \mathcal{B}$ is naturally provided with a Lie algebra product [46]

$$\mu(a_1 \otimes b_1, a_2 \otimes b_2) = \mu_{\mathcal{A}}(a_1, a_2) \otimes \mu_{\mathcal{B}}(b_1, b_2) - \mu_{\mathcal{A}}(a_2, a_1) \otimes \mu_{\mathcal{B}}(b_2, b_1), \tag{6.1}$$

where $\mu_{\mathcal{A}}$ (resp. $\mu_{\mathcal{B}}$) is the product of $\mathcal{A}$ (resp. $\mathcal{B}$). So the "natural" tensor product $\mu_{\mathcal{A}\otimes\mathcal{B}} = \mu_{\mathcal{A}} \otimes \mu_{\mathcal{B}}$ provides $\mathcal{A} \otimes \mathcal{B}$ with a Lie-admissible algebra structure.

In 4 and [55] we introduced special classes of Lie-admissible algebras with defining axioms determined by the subgroups G_i of Σ_3 and their corresponding quadratic operads denoted $\mathcal{G}_i$*-ass*. Among these operads, we find operads for Lie-admissible, associative, Vinberg and pre-Lie algebras. For the operads $\mathcal{G}_i$*-ass* we proved in [55] that for every $\mathcal{P}$-algebra $\mathcal{A}$ and $\mathcal{P}^!$-algebra $\mathcal{B}$, the tensor product $\mathcal{A} \otimes \mathcal{B}$ is a $\mathcal{P}$-algebra. This is not true for general nonassociative algebras and, for example, if $\mathcal{P}$ is the operad for Leibniz algebras or for nonassociative algebras associated to Poisson algebras [59], the tensor product of a $\mathcal{P}$-algebra and a $\mathcal{P}^!$-algebra is not a $\mathcal{P}$-algebra. In this sense, G_i-associative algebras, which are algebras over $\mathcal{G}_i$*-ass*, are the most regular nonassociative algebras. So we introduce a quadratic operad, the *current* operad, denoted by $\widetilde{\mathcal{P}}$, such that the tensor product of a $\mathcal{P}$-algebra with a $\widetilde{\mathcal{P}}$-algebra is a $\mathcal{P}$-algebra and such that $\mathcal{P}$ is maximal with this property. Without requiring the maximality, one could take, for instance, $\mathcal{P} = \mathbb{1}$.

The name current refers to current Lie algebras, which are Lie algebras of the form $L \otimes A$, where L is a Lie algebra and A is a associative commutative algebra equipped with the bracket

$$[x \otimes a, y \otimes b]_{L\otimes A} = [x, y]_L \otimes ab.$$

Remark that in this case we have $\mathcal{P} = \mathcal{L}ie$ and that associative commutative algebras are algebras on $\mathcal{C}om = \mathcal{P}^!$.

6.1 Nonassociative algebras and operads

Let $\mathcal{P}$ be a quadratic operad with one binary generating operation. We consider the action of Σ_3 on $\Gamma(E)(3)$ given by:

$$\begin{array}{ll} \Sigma_3 \times \Gamma(E)(3) & \to \Gamma(E)(3) \\ (\sigma, X) & \mapsto \sigma(X) \end{array}$$

where

$$\begin{cases} \sigma(x_i \cdot (x_j \cdot x_k)) = x_{\sigma^{-1}(i)} \cdot (x_{\sigma^{-1}(j)} \cdot x_{\sigma^{-1}(k)}), \\ \sigma((x_i \cdot x_j) \cdot x_k) = (x_{\sigma^{-1}(i)} \cdot x_{\sigma^{-1}(j)}) \cdot x_{\sigma^{-1}(k)}). \end{cases}$$

For any $X \in \Gamma(E)(3)$, $\mathcal{O}(X)$ denotes the orbit of X and $\mathbb{K}(\mathcal{O}(X))$ the linear span of $\mathcal{O}(X)$ which is a Σ_3-invariant subspace of $\Gamma(E)(3)$. More generally, if $X_1, \cdots, X_k$ are vectors in $\Gamma(E)(3)$, we denote by $\mathbb{K}(\mathcal{O}(X_1, \cdots, X_k))$ the Σ_3-invariant subspace of $\Gamma(E)(3)$ generated by $\mathcal{O}(X_1) \cup \cdots \cup \mathcal{O}(X_k)$. It is clear that $\mathbb{K}(\mathcal{O}(X_1, \cdots, X_k))$ is a Σ_3-module of finite rank.

Definition 41 *If $\mathcal{P}$ is the quadratic operad*

$$\mathcal{P} = \Gamma(E)/(R),$$

the rank of $\mathcal{P}$ is the rank of the Σ_3-module $R \subset \Gamma(E)(3)$.

Using the action of Σ_3 on $\Gamma(E)(3)$ we define some linear maps on this module as follows. Let $\mathbb{K}[\Sigma_3]$ be the group algebra of Σ_3. Its elements are formal combinations $v = a_1\sigma_1 + a_2\sigma_2 + \cdots + a_6\sigma_6$ with $a_i \in \mathbb{K}$ and $\sigma_i \in \Sigma_3$. For $v = \sum_{i=1}^{6} a_i\sigma_i \in \mathbb{K}[\Sigma_3]$, we put

$$v(X) = \sum_{i=1}^{6} a_i\sigma_i(X), \ \forall X \in \Gamma(E)(3).$$

If F is an invariant subspace of $\Gamma(E)(3)$, we have $v(X) \in F$ for all $X \in F$.

Let $(\mathcal{A}, \mu)$ be a $\mathcal{P}$-algebra and let $A_\mu^L = \mu \circ (\mu \otimes Id)$ and $A_\mu^R = \mu \circ (Id \otimes \mu)$. Then the associator of μ is written $A_\mu = A_\mu^L - A_\mu^R$. For each vector $v = \sum_{i=1}^{6} a_i\sigma_i$ in $\mathbb{K}[\Sigma_3]$ we define the linear map $\Phi_v^{\mathcal{A}}$ on $\mathcal{A}^{\otimes 3}$ by

$$\begin{array}{cccc} \Phi_v^{\mathcal{A}}: & \mathcal{A}^{\otimes 3} & \rightarrow & \mathcal{A}^{\otimes 3} \\ & (X_1 \otimes X_2 \otimes X_3) & \mapsto & \sum_{i=1}^{6} a_i(X_{\sigma_i^{-1}(1)} \otimes X_{\sigma_i^{-1}(2)} \otimes X_{\sigma_i^{-1}(3)}). \end{array}$$

Any quadratic relation for μ is clearly of the form

$$A_\mu^L \circ \Phi_v^{\mathcal{A}} - A_\mu^R \circ \Phi_w^{\mathcal{A}} = 0, \tag{6.2}$$

for some $v = \sum_{i=1}^{6} a_i\sigma_i, w = \sum_{i=1}^{6} b_i\sigma_i \in \mathbb{K}[\Sigma_3]$. The module R is therefore spanned by the vectors

$$v((x_1 \cdot x_2) \cdot x_3)) - w(x_1 \cdot (x_2 \cdot x_3)).$$

By definition, if $\mathcal{P}$ is of rank 1, the product satisfies precisely one relation (6.2).

Proposition 44 *Let $\mathcal{P}$ be a quadratic operad with one generating operation such that the Σ_3-submodule R of relations is generated by the vectors:*

$$v_l((x_1 \cdot x_2) \cdot x_3 - x_1 \cdot (x_2 \cdot x_3)),$$

with $v_l \in \mathbb{K}[\Sigma_3]$, for some $l = 1, \cdots, k$. Then $\mathcal{P}$ is of rank 1.

Proof. The Σ_3-invariant subspace of $\Gamma(E)(3)$ generated by

$$((x_1 \cdot x_2) \cdot x_3 - x_1 \cdot (x_2 \cdot x_3))$$

is isomorphic to $\mathbb{K}[\Sigma_3]$. This isomorphism is given by:

$$v((x_1 \cdot x_2) \cdot x_3 - x_1 \cdot (x_2 \cdot x_3)) \longrightarrow v.$$

We saw in [56] that every Σ_3-submodule F of $\mathbb{K}[\Sigma_3]$ is of rank 1. Then there is a vector $v \in \mathbb{K}[\Sigma_3]$ such that $F = F_v = \mathbb{K}(\mathcal{O}(v))$ where $\mathcal{O}(v)$ is the orbit of v of the natural action of Σ_3 on $\mathbb{K}[\Sigma_3]$. We conclude that the rank of R, and then of $\mathcal{P}$, is 1.

6.2 The operad $\widetilde{\mathcal{P}}$ associated to a quadratic operad $\mathcal{P}$

In this section we define a quadratic operad $\widetilde{\mathcal{P}}$ whose fundamental property is that any $\mathcal{P}$-algebra tensored with a $\widetilde{\mathcal{P}}$-algebra is a $\mathcal{P}$-algebra.

Let $\mathcal{P}$ be a quadratic operad generated by $E \subset \mathbb{K}[\Sigma_2]$, R the module of relations and $(\mathcal{A}, \mu)$ a $\mathcal{P}$-algebra. The axioms of $\mathcal{A}$ determine the submodule R of $\Gamma(E)(3)$. If R is of rank k, then the product μ satisfies k relations

$$A^L_\mu \circ \Phi^{\mathcal{A}}_{v_p} - A^R_\mu \circ \Phi^{\mathcal{A}}_{w_p} = 0, \tag{6.3}$$

where $v_p, w_p \in \mathbb{K}[\Sigma_3]$ for $p \in I = \{1, \cdots, k\}$ and the vectors $(v_p \oplus w_p)_{p\in I}$ are linearly independent.

Relations (6.3) show that R can be described as

$$R = Span\,\{v_p((x_1 \cdot x_2) \cdot x_3) - w_p(x_1 \cdot (x_2 \cdot x_3)),\ 1 \leq p \leq k\}$$

with $v_p = \sum_{i=1}^{6} a^i_p \sigma_i$ and $w_p = \sum_{i=1}^{6} b^i_p \sigma_i$ for $1 \leq p \leq k$. Let $(A \mid B)$ be the $k \times 12$ matrix given by $A = (a^i_p)$ and $B = (b^i_p)$ for $i = 1, \cdots, 6$ and $p = 1, \cdots, k$. By hypothesis the matrix $(A \mid B)$ is of rank k.

Definition 42 *We say that the system of generators of R is reduced if there is an extracted matrix of $(A \mid B)$ of order k that equals to the identity matrix up permutations of lines. We say that such a matrix $(A \mid B)$ is reduced.*

It is clear that R can always be represented by a reduced system of generators. If fact if $(A \mid B)$ is of rank k, there is C invertible of order k, extracted from $(A \mid B)$ and

$$(A_1 \mid B_1) = C^{-1}(A \mid B) = (\alpha^i_p \mid \beta^i_p)$$

is reduced. Now we can consider the reduced system of generators of R associated to $(A_1 \mid B_1)$:

$$R = Span\left\{v'_p((x_1 \cdot x_2) \cdot x_3) - w'_p(x_1 \cdot (x_2 \cdot x_3)),\ 1 \leq p \leq k\right\} \tag{6.4}$$

with $v'_p = \sum_{i=1}^{6} \alpha^i_p \sigma_i$, $w'_p = \sum_{i=1}^{6} \beta^i_p \sigma_i$ for $1 \leq p \leq k$.

We denote by $\widetilde{R_C}$ the $\mathbb{K}[\Sigma_3]$-module generated by the vectors

$$\left\{ \begin{array}{l} \alpha^i_p \alpha^j_p (\sigma_i - \sigma_j)((x_1 \cdot x_2) \cdot x_3), \\ \beta^i_p \beta^j_p (\sigma_i - \sigma_j)(x_1 \cdot (x_2 \cdot x_3)), \\ \alpha^i_p \beta^j_p \left(\sigma_i((x_1 \cdot x_2) \cdot x_3) - \sigma_j(x_1 \cdot (x_2 \cdot x_3))\right), \end{array} \right. \tag{6.5}$$

for $1 \leq i, j \leq 6$ and $1 \leq p \leq k$. The subscript C means that the system of generators of $\widetilde{R_C}$ depends of the matrix C. With the system (6.5) we associate a matrix $k \times 12$

$$(E \mid F)$$

whose coefficients are 0 (resp. 1) if the corresponding coefficient in the matrix $(A \mid B)$ is 0 (resp. non zero). Then the lines of $(E \mid F)$ describe all the relations of type $\sigma_i((x_1 \cdot x_2) \cdot x_3) = \sigma_j((x_1 \cdot x_2) \cdot x_3)$, $\sigma_i((x_1 \cdot x_2) \cdot x_3) = \sigma_j(x_1 \cdot (x_2 \cdot x_3))$ or $\sigma_i(x_1 \cdot (x_2 \cdot x_3)) = \sigma_j(x_1 \cdot (x_2 \cdot x_3))$ which are given in (6.5).

Definition 43 *A path in the matrix $(E \mid F)$ is a maximal union of horizontal and vertical segments whose endpoints are* 1 *of the matrix.*

For example if $(E \mid F) = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}$ we have only one way

$$\begin{pmatrix} 1 & - & 1 & - & 1 \\ & & & & | \\ & 1 & - & 1- & 1 \end{pmatrix}$$

The matrix $(E \mid F)$ can be represented by a finite number $\chi(C)$ of such paths.

Definition 44 *A matrix C which reduces the initial system $(A \mid B)$ is called maximal if $\chi(C)$ is maximal among all systems associated to the initial one.*

Let C_0 be a maximal reduction matrix. We denote by

$$(A_0 \mid B_0) = C_0^{-1}(A \mid B) = (\rho^i_j | \lambda^i_j).$$

Let $\widetilde{E}$ be the sub-module of $\mathbb{K}[\Sigma_2]$ defined by

$$\widetilde{E} = \begin{cases} E \ \text{ if } E = \mathbb{K}[\Sigma_2] \simeq \mathbb{1} \oplus Sgn_2, \\ \mathcal{C}om(2) = \mathbb{1} \ \text{ if } \ E = \mathbb{1} \ \text{ or } \ E = Sgn_2, \end{cases}$$

where $\mathbb{1}$ is the one-dimensional trivial representation, Sgn_2 the one-dimensional signum representation and $\mathcal{C}om$ denotes the operad for associative commutative algebras. If $\widetilde{E} = \mathbb{K}[\Sigma_2]$, we denote by $\widetilde{R} = \widetilde{R_{C_0}}$ the $\mathbb{K}[\Sigma_3]$-module generated by the vectors

$$\begin{cases} \rho^i_p \rho^j_p (\sigma_i - \sigma_j)((x_1 \cdot x_2) \cdot x_3), \\ \lambda^i_p \lambda^j_p (\sigma_i - \sigma_j)(x_1 \cdot (x_2 \cdot x_3)), \\ \rho^i_p \lambda^j_p \left(\sigma_i((x_1 \cdot x_2) \cdot x_3) - \sigma_j(x_1 \cdot (x_2 \cdot x_3))\right), \end{cases}$$

for $1 \leq p \leq k$. If $\widetilde{E} = \mathcal{C}om(2)$, $\widetilde{R}$ is generated by the above vectors modulo the commutativity relations.

Definition 45 *Let $\mathcal{P}$ be a quadratic operad with one generating operation. We define the operad $\widetilde{\mathcal{P}}$ as the quadratic operad $\widetilde{\mathcal{P}} = \Gamma(\widetilde{E})/(\widetilde{R})$, where $\widetilde{E}$ and $\widetilde{R}$ are as above.*

This operad is called the current operad associated with $\mathcal{P}$. We have the main result:

Theorem 46 *Let $(\mathcal{A}, \mu_{\mathcal{A}})$ be a $\mathcal{P}$-algebra and $(\mathcal{B}, \mu_{\mathcal{B}})$ a $\widetilde{\mathcal{P}}$-algebra. Then $\mathcal{A} \otimes \mathcal{B}$ with the operation $\mu_{\mathcal{A} \otimes \mathcal{B}}$ is a $\mathcal{P}$-algebra. The operad $\widetilde{\mathcal{P}}$ is maximal among quadratic operads with one operation of the same arity having this property. The maximality means that each operad $\mathcal{Q}$ with the required property is a sub-operad of $\widetilde{\mathcal{P}}$.*

Proof. Let $(\mathcal{A}, \mu_{\mathcal{A}})$ be a $\mathcal{P}$-algebra. The following property holds for $\mu_{\mathcal{A}}$:

$$(A^L_{\mu_{\mathcal{A}}} \circ \Phi^{\mathcal{A}}_{v'_p} - A^R_{\mu_{\mathcal{A}}} \circ \Phi^{\mathcal{A}}_{w'_p})(x_1 \otimes x_2 \otimes x_3) = 0$$

for $p = 1, \cdots, k$ with $v'_p = \sum_{i=1}^6 \rho^i_p \sigma_i, w'_p = \sum_{i=1}^6 \lambda^i_p \sigma_i$ and $(\rho^i_p \mid \lambda^i_p)$ reduced. If $\mathcal{B}$ is a $\widetilde{\mathcal{P}}$-algebra, its product $\mu_{\mathcal{B}}$ satisfies

$$\begin{cases} A^L_{\mu_{\mathcal{B}}} \circ \Phi^{\mathcal{B}}_{\sigma_i - \sigma_j}(y_1 \otimes y_2 \otimes y_3) = 0, \ \text{if } \exists p, \ \rho^i_p \rho^j_p \neq 0, \\ \\ A^R_{\mu_{\mathcal{B}}} \circ \Phi^{\mathcal{B}}_{\sigma_i - \sigma_j}(y_1 \otimes y_2 \otimes y_3) = 0, \ \text{if } \exists p, \ \lambda^i_p \lambda^j_p \neq 0, \\ \\ A^L_{\mu_{\mathcal{B}}} \circ \Phi^{\mathcal{B}}_{\sigma_i} - A^R_{\mu_{\mathcal{B}}} \circ \Phi^{\mathcal{B}}_{\sigma_j}(y_1 \otimes y_2 \otimes y_3) = 0, \ \text{if } \exists p, \ \rho^i_p \lambda^j_p \neq 0. \end{cases}$$

The product $\mu_{\mathcal{A} \otimes \mathcal{B}}$ satisfies, for $(j, k) \in \{1, \cdots, 6\}^2$ such that $\rho^j_p \lambda^k_p \neq 0$,

$$
\begin{aligned}
&(A^L_{\mu_{\mathcal{A}\otimes\mathcal{B}}}\circ\Phi^{\mathcal{A}\otimes\mathcal{B}}_{v'_p}-A^R_{\mu_{\mathcal{A}\otimes\mathcal{B}}}\circ\Phi^{\mathcal{A}\otimes\mathcal{B}}_{w'_p})(x_1\otimes y_1\otimes x_2\otimes y_2\otimes x_3\otimes y_3)\\
&=\sum_{i=1}^{6}(\rho^i_pA^L_{\mu_{\mathcal{A}\otimes\mathcal{B}}}\circ\Phi^{\mathcal{A}\otimes\mathcal{B}}_{\sigma_i}-\lambda^i_pA^R_{\mu_{\mathcal{A}\otimes\mathcal{B}}}\circ\Phi^{\mathcal{A}\otimes\mathcal{B}}_{\sigma_i})(x_1\otimes y_1\otimes x_2\otimes y_2\otimes x_3\otimes y_3)\\
&=\sum_{i=1}^{6}\rho^i_p(A^L_{\mu_{\mathcal{A}}}\circ\Phi^{\mathcal{A}}_{\sigma_i}(x_1\otimes x_2\otimes x_3)\otimes A^L_{\mu_{\mathcal{B}}}\circ\Phi^{\mathcal{B}}_{\sigma_i}(y_1\otimes y_2\otimes y_3))\\
&\quad-\sum_{i=1}^{6}\lambda^i_p(A^R_{\mu_{\mathcal{A}}}\circ\Phi^{\mathcal{A}}_{\sigma_i}(x_1\otimes x_2\otimes x_3)\otimes A^R_{\mu_{\mathcal{B}}}\circ\Phi^{\mathcal{B}}_{\sigma_i}(y_1\otimes y_2\otimes y_3))\\
&=\sum_{i=1}^{6}\rho^i_p(A^L_{\mu_{\mathcal{A}}}\circ\Phi^{\mathcal{A}}_{\sigma_i}(x_1\otimes x_2\otimes x_3))\otimes A^L_{\mu_{\mathcal{B}}}\circ\Phi^{\mathcal{B}}_{\sigma_j}(y_1\otimes y_2\otimes y_3)\\
&\quad-\sum_{i=1}^{6}\lambda^i_p(A^R_{\mu_{\mathcal{A}}}\circ\Phi^{\mathcal{A}}_{\sigma_i}(x_1\otimes x_2\otimes x_3))\otimes A^R_{\mu_{\mathcal{B}}}\circ\Phi^{\mathcal{B}}_{\sigma_k}(y_1\otimes y_2\otimes y_3)\\
&=\sum_{i=1}^{6}\big(\rho^i_pA^L_{\mu_{\mathcal{A}}}\circ\Phi^{\mathcal{A}}_{\sigma_i}(x_1\otimes x_2\otimes x_3)-\lambda^i_pA^R_{\mu_{\mathcal{A}}}\circ\Phi^{\mathcal{A}}_{\sigma_i}(x_1\otimes x_2\otimes x_3)\big)\\
&\qquad\otimes A^L_{\mu_{\mathcal{B}}}\circ\Phi^{\mathcal{B}}_{\sigma_j}(y_1\otimes y_2\otimes y_3)\\
&=(A^L_{\mu_{\mathcal{A}}}\circ\Phi^{\mathcal{A}}_{v_p}-A^R_{\mu_{\mathcal{A}}}\circ\Phi^{\mathcal{A}}_{w_p})(x_1\otimes x_2\otimes x_3)\otimes A^L_{\mu_{\mathcal{B}}}\circ\Phi^{\mathcal{B}}_{\sigma_j}(y_1\otimes y_2\otimes y_3)\\
&=0.
\end{aligned}
$$

If, for a given p, all $\rho^j_p\lambda^k_p=0$, it means that we have

$$(A^L_{\mu_{\mathcal{A}}}\circ\Phi^{\mathcal{A}}_{v'_p})(x_1\otimes x_2\otimes x_3)=0$$

or

$$(A^R_{\mu_{\mathcal{A}}}\circ\Phi^{\mathcal{A}}_{w'_p})(x_1\otimes x_2\otimes x_3)=0$$

and the result will obviously follows from similar calculations.

Concerning the maximality, we consider the free algebra $L(V)$ on a $\mathbb{K}$-vector space V satisfying the axioms

$$\sum_{i=1}^{6}a^i_p(x_{\sigma_i(1)}x_{\sigma_i(2)})x_{\sigma_i(3)}-\sum_{i=1}^{6}b^j_px_{\sigma_j(1)}(x_{\sigma_j(2)}x_{\sigma_j(3)})=0, \tag{6.6}$$

for $p=1,\cdots,k$. It is the same to say that $L(V)$ is defined by the relations

$$\sum_{i=1}^{6}\rho^i_p(x_{\sigma_i(1)}x_{\sigma_i(2)})x_{\sigma_i(3)}-\sum_{i=1}^{6}\lambda^i_px_{\sigma_j(1)}(x_{\sigma_j(2)}x_{\sigma_j(3)})=0., \tag{6.7}$$

with matrix $(A_0\mid B_0)=(\rho^i_p\mid\lambda^i_p)$ related to this system admitting an extracted matrix of order k that equals to the identity after some permutations of the lines. Assume that

$$(\rho^{i_l}_p|\lambda^{j_m}_p)=Id$$

for $l=1,\cdots,k_1$ and $m=1,\cdots,k_2$ with $k_1+k_2=k$. We denote by $I_{k_1}=\{i_1,\cdots,i_{k_1}\}$ and $J_{k_2}=\{j_1,\cdots,j_{k_2}\}$. Then System (6.7) is equivalent to

$$\left\{\begin{array}{l}A^L_{\mu_{L(V)}}\circ\Phi^{L(V)}_{\sigma_{i_l}}=-A^L_{\mu_{L(V)}}\circ\sum_{i\notin I_{k_1}}\rho^i_l\Phi^{L(V)}_{\sigma_i}+A^R_{\mu_{L(V)}}\circ\sum_{j\notin J_{k_2}}\lambda^j_l\Phi^{L(V)}_{\sigma_j},\\ \qquad\qquad l=1,\cdots,k_1,\\ A^R_{\mu_{L(V)}}\circ\Phi^{L(V)}_{\sigma_{j_m}}=A^L_{\mu_{L(V)}}\circ\sum_{i\notin I_{k_1}}\rho^i_m\Phi^{L(V)}_{\sigma_i}-A^R_{\mu_{L(V)}}\circ\sum_{j\notin J_{k_2}}\lambda^j_m\Phi^{L(V)}_{\sigma_j},\\ \qquad\qquad m=k_1+1,\cdots,k.\end{array}\right. \tag{6.8}$$

Let B be a $\mathbb{K}$-algebra. Then $L(V)\otimes B$ satisfies (6.8) if and only if

$$\left\{\begin{array}{l}(A^L_{\mu_{L(V)}}\otimes A^L_{\mu_B})\circ(\Phi^{L(V)}_{\sigma_{i_l}}\otimes\Phi^B_{\sigma_{i_l}})=-(A^L_{\mu_{L(V)}}\otimes A^L_{\mu_B})\circ\sum_{i\notin I_{k_1}}\rho^i_l(\Phi^{L(V)}_{\sigma_i}\otimes\Phi^B_{\sigma_i})\\ \qquad\qquad\qquad\qquad\qquad\qquad+(A^R_{\mu_{L(V)}}\otimes A^R_{\mu_B})\circ\sum_{j\notin J_{k_2}}\lambda^j_l(\Phi^{L(V)}_{\sigma_j}\otimes\Phi^B_{\sigma_j}),\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad l=1,\cdots,k_1\\ (A^R_{\mu_{L(V)}}\otimes A^R_{\mu_B})\circ(\Phi^{L(V)}_{\sigma_{j_m}}\otimes\Phi^B_{\sigma_{j_m}})=(A^L_{\mu_{L(V)}}\otimes A^L_{\mu_B})\circ\sum_{i\notin I_{k_1}}\rho^i_m(\Phi^{L(V)}_{\sigma_i}\otimes\Phi^B_{\sigma_i})\\ \qquad\qquad\qquad\qquad\qquad\qquad-(A^R_{\mu_{L(V)}}\otimes A^R_{\mu_B})\circ\sum_{j\notin J_{k_2}}\lambda^j_m(\Phi^{L(V)}_{\sigma_j}\otimes\Phi^B_{\sigma_j}),\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad m=k_1+1,\cdots,k.\end{array}\right.$$

We deduce, for the first equation

$$\left\{\begin{array}{l}(A^L_{\mu_{L(V)}}\circ\sum_{i\notin I_{k_1}}\rho^i_l\Phi^{L(V)}_{\sigma_i})\otimes(A^L_{\mu_B}\circ\Phi^B_{\sigma_{i_l}})-(A^R_{\mu_{L(V)}}\circ\sum_{j\notin J_{k_2}}\lambda^j_l\Phi^{L(V)}_{\sigma_j})\otimes(A^L_{\mu_B}\circ\Phi^B_{\sigma_{i_l}})\\ =(A^L_{\mu_{L(V)}}\circ\sum_{i\notin I_{k_1}}\rho^i_l\Phi^{L(V)}_{\sigma_i})\otimes(A^L_{\mu_B}\circ\Phi^B_{\sigma_i})-(A^R_{\mu_{L(V)}}\circ\sum_{j\notin J_{k_2}}\lambda^j_l\Phi^{L(V)}_{\sigma_j})\otimes(A^R_{\mu_B}\circ\Phi^B_{\sigma_j}),\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad l=1,\cdots,k_1.\end{array}\right.$$

Similarly for the second equation, we get

$$\left\{\begin{array}{l}(A^L_{\mu_{L(V)}}\circ\sum_{i\notin I_{k_1}}\rho^i_m\Phi^{L(V)}_{\sigma_i})\otimes(A^R_{\mu_B}\circ\Phi^B_{\sigma_{j_m}})-(A^R_{\mu_{L(V)}}\circ\sum_{j\notin J_{k_2}}\lambda^j_m\Phi^{L(V)}_{\sigma_j})\otimes(A^R_{\mu_B}\circ\Phi^B_{\sigma_{j_m}})\\ =(A^L_{\mu_{L(V)}}\circ\sum_{i\notin I_{k_1}}\rho^i_m\Phi^{L(V)}_{\sigma_i})\otimes(A^L_{\mu_B}\circ\Phi^B_{\sigma_i})-(A^R_{\mu_{L(V)}}\circ\sum_{j\notin J_{k_2}}\lambda^j_m\Phi^{L(V)}_{\sigma_j})\otimes(A^R_{\mu_B}\circ\Phi^B_{\sigma_j}),\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad m=k_1+1,\cdots,k.\end{array}\right.$$

Since $L(V)$ is free, such equations are equivalent to

$$\left\{\begin{array}{l}\rho^i_l A^L_{\mu_B}\circ(\Phi^B_{\sigma_{i_l}}-\Phi^B_{\sigma_i})=0,\\ \lambda^j_l(A^L_{\mu_B}\circ\Phi^B_{\sigma_{i_l}}-A^R_{\mu_B}\circ\Phi^B_{\sigma_j})=0,\\ \rho^i_m(A^R_{\mu_B}\circ\Phi^B_{\sigma_{j_m}}-A^L_{\mu_B}\circ\Phi^B_{\sigma_i})=0,\\ \lambda^j_l A^R_{\mu_B}\circ(\Phi^B_{\sigma_{j_m}}-\Phi^B_{\sigma_j})=0,\\ \text{for } i\notin I_{k_1},\ j\notin J_{k_2},\ l\in\{1,\cdots,k_1\}\text{ and }\ m\in\{k_1+1,\cdots,k\}.\end{array}\right. \tag{6.9}$$

and using the property of the coefficients ρ^i_p and λ^i_p, we recognize the relations defining $\widetilde{R}$. The maximality of $\chi(C)$ implies the minimality of the system of relations.

6.3 Examples

6.3.1 $\mathcal{P}=\mathcal{L}ie$

If $\mathcal{P}=\mathcal{L}ie$ then $\widetilde{\mathcal{P}}=\mathcal{P}^!=\mathcal{C}om$ the operad for associative commutative algebras. Indeed, in this case, $k=1$ and $v_1=w_1=Id+c_1+c_2$. As $v_1=w_1$, a $\widetilde{\mathcal{P}}$-algebra is associative. Moreover $\widetilde{E}=\mathcal{C}om(2)$ because $\mathcal{L}ie=\Gamma(E)/(R)$ with $E=Sgn_2$. We conclude that $\widetilde{\mathcal{P}}=\mathcal{C}om$.

6.3.2 $\mathcal{P}=\mathcal{L}eib$

Let $\mathcal{P}=\mathcal{L}eib$ be the Leibniz operad. A Leibniz algebra satisfies the relation

$$x(yz)-(xy)z-(xz)y=0.$$

In this case, the associated elements of a $\widetilde{\mathcal{P}}$-algebra fulfill the relation

$$x(yz)=(xy)z$$

and

$$(xy)z=(xz)y.$$

Thus a $\widetilde{\mathcal{L}eib}$-algebra is an associative algebra satisfying

$$xyz = xzy.$$

This relation is equivalent to

$$x[y,z] = 0$$

with $[y,z] = yz - zy$. This last identity implies that the derived Lie subalgebra is abelian, and the Lie algebra is 2-step nilpotent. The dual operad, also denoted $\mathcal{Z}inb$, describes algebras satisfying

$$(xy)z - x(yz) - x(zy) = 0.$$

Thus if a $\widetilde{\mathcal{L}eib}$-algebra satisfies also the relation $x(yz) = (xy)z = 0$ (every product of 3 elements of the associative algebra is zero), it is a $\mathcal{Z}inb$-algebra (Zinbiel algebra) i.e. a $\mathcal{L}eib^!$-algebra ([16]). These algebras are nilalgebras $\mathcal{A}$ satisfying $\mathcal{A}^3 = 0$. For example, any associative commutative algebra is a $\widetilde{\mathcal{L}eib}$-algebra. Every $\widetilde{\mathcal{L}eib}$-algebra with unit is commutative. In dimension 3 the algebra defined by

$$e_1e_1 = e_2, \ e_1e_3 = e_3e_3 = e_2$$

is a noncommutative $\widetilde{\mathcal{L}eib}$-algebra.

6.3.3 $\mathcal{P} = \mathcal{P}oiss$

A Poisson algebra over $\mathbb{K}$ is a $\mathbb{K}$-vector space equipped with two bilinear products:

1) a Lie algebra product, denoted by $\{,\}$, called the Poisson bracket,

2) an associative commutative product, denoted by $\bullet$.

These two operations satisfy the Leibniz condition:

$$\{X \bullet Y, Z\} = X \bullet \{Y,Z\} + \{X,Z\} \bullet Y,$$

for all X, Y, Z. In [92], one proves that a Poisson algebra can be defined by only one nonassociative product, denoted by $X \cdot Y$, satisfying the following identity

$$3A.(X,Y,Z) = (X \cdot Z) \cdot Y + (Y \cdot Z) \cdot X - (Y \cdot X) \cdot Z - (Z \cdot X) \cdot Y, \tag{6.10}$$

where $A.(X,Y,Z) = (X \cdot Y) \cdot Z - X \cdot (Y \cdot Z)$ is the associator of the product $\cdot$. The corresponding quadratic operad has one generating operation and of rank 1. Let us denote by $\mathcal{P}oiss$ this operad. If

$$v_1((x_i \cdot x_j) \cdot x_k) - w_1(x_1 \cdot (x_2 \cdot x_3))$$

is the generator of the module of relations R of $\mathcal{P}oiss$, we have

$$v_1 = 3Id - \tau_{23} - c_1 + \tau_{12} + c_2$$

and

$$w_1 = 3Id.$$

So $\widetilde{\mathcal{P}oiss}$ is generated by

$$\begin{cases} (x_1 \cdot x_2) \cdot x_3 - (x_1 \cdot x_3) \cdot x_2, \\ (x_1 \cdot x_2) \cdot x_3 - (x_2 \cdot x_3) \cdot x_1, \\ (x_1 \cdot x_2) \cdot x_3 - (x_2 \cdot x_1) \cdot x_3, \\ (x_1 \cdot x_2) \cdot x_3 - (x_3 \cdot x_1) \cdot x_2, \\ (x_1 \cdot x_2) \cdot x_3 - x_1 \cdot (x_2 \cdot x_3), \end{cases}$$

and $\widetilde{\mathcal{P}oiss} = \mathcal{C}omm3$.

Remark. Here, we considered the standard product on the tensor product of algebras given by formula (6.1), but it is possible to consider also modified ones. For instance, let $(\mathcal{A}, \mu_{\mathcal{A}})$ and $(\mathcal{B}, \mu_{\mathcal{B}})$ be two Poisson algebras with a nonassociative product satisfying (6.10). Let τ be the twist map:

$$\tau(x \otimes y) = y \otimes x.$$

If we consider on $\mathcal{A} \otimes \mathcal{B}$ the following product

$$\mu_{\mathcal{A}} \otimes_\tau \mu_{\mathcal{B}} = 3\mu_{\mathcal{A}} \otimes \mu_{\mathcal{B}} - \mu_{\mathcal{A}} \otimes (\mu_{\mathcal{B}} \circ \tau) - (\mu_{\mathcal{A}} \circ \tau) \otimes \mu_{\mathcal{B}} + (\mu_{\mathcal{A}} \circ \tau) \otimes (\mu_{\mathcal{B}} \circ \tau)$$

then $(\mathcal{A} \otimes \mathcal{B}, \mu_{\mathcal{A}} \otimes_\tau \mu_{\mathcal{B}})$ is a Poisson algebra.

6.3.4 The operads $\mathcal{G}_i$-*ass*

Let $\mathcal{G}_i$-$ass^!$ be the quadratic dual operad of $\mathcal{G}_i$-*ass*. We denote by $(R_{V_i})^!$ the submodule of $\Gamma(E)(3)$ defining $(G_i\text{-}\mathcal{A}ss)^!$. We have

$$\begin{array}{rl}
(R_{V_1})^! = & R_{V_1}, \\
(R_{V_2})^! = & \mathbb{K}(\mathcal{O}((x_1 \cdot x_2) \cdot x_3 - x_1 \cdot (x_2 \cdot x_3); (x_1 \cdot x_2) \cdot x_3 - (x_2 \cdot x_1) \cdot x_3)), \\
(R_{V_3})^! = & \mathbb{K}(\mathcal{O}((x_1 \cdot x_2) \cdot x_3 - x_1 \cdot (x_2 \cdot x_3); (x_1 \cdot x_2) \cdot x_3) - (x_1 \cdot x_3) \cdot x_2)), \\
(R_{V_4})^! = & \mathbb{K}(\mathcal{O}((x_1 \cdot x_2) \cdot x_3 - x_1 \cdot (x_2 \cdot x_3); (x_1 \cdot x_2) \cdot x_3 - (x_3 \cdot x_2) \cdot x_1)), \\
(R_{V_5})^! = & \mathbb{K}(\mathcal{O}((x_1 \cdot x_2) \cdot x_3 - x_1 \cdot (x_2 \cdot x_3); (x_1 \cdot x_2) \cdot x_3 - (x_2 \cdot x_3) \cdot x_1)), \\
(R_{V_6})^! = & \mathbb{K}(\mathcal{O}((x_1 \cdot x_2) \cdot x_3 - x_1 \cdot (x_2 \cdot x_3); (x_1 \cdot x_2) \cdot x_3 - (x_2 \cdot x_1) \cdot x_3; \\
& \qquad (x_1 \cdot x_2) \cdot x_3 - (x_1 \cdot x_3) \cdot x_2)).
\end{array}$$

Proposition 45 *The operad $\mathcal{G}_1$-$ass^! = \mathcal{A}ss$ is of rank 1. For $2 \leq i \leq 6$, the operads $\mathcal{G}_i$-$ass^!$ are of rank 2.*

Proof. The case $i = 1$ is trivial (it is also a consequence of Proposition 2). Let us prove that the rank of $(R_{V_i})^!$ is 2 for $i = 2, \cdot, 5$. The case of $(R_{V_6})^!$ will be treated separately. We denote by v_i^j, $j = 1, 2$ the generators of $(R_{V_i})^!$. Then

$$\begin{cases} v_i^1 = (A^L_{\mu_{\mathcal{B}}} - A^R_{\mu_{\mathcal{B}}})(x_1 \otimes x_2 \otimes x_3), \\ v_i^2 = (A^L_{\mu_{\mathcal{B}}} - A^R_{\mu_{\mathcal{B}}} \circ \Phi_{\sigma_i})(x_1 \otimes x_2 \otimes x_3), \end{cases}$$

with

$$\begin{cases} \sigma_2 = \tau_{12}, \\ \sigma_3 = \tau_{23}, \\ \sigma_4 = \tau_{13}, \\ \sigma_5 = c_1 \ (\text{or} \ \ c_2). \end{cases}$$

For $i = 6$, the space $(R_{V_6})^!$ is generated by the vectors

$$(x_1 \cdot x_2) \cdot x_3 - x_1 \cdot (x_2 \cdot x_3), \ (x_1 \cdot x_2) \cdot x_3 - (x_2 \cdot x_1) \cdot x_3 \text{ and } \ (x_1 \cdot x_2) \cdot x_3 - (x_1 \cdot x_3) \cdot x_2.$$

But we can write

$$(x_1 \cdot x_2) \cdot x_3 - (x_2 \cdot x_1) \cdot x_3 = (Id - \tau_{12})((x_1 \cdot x_2) \cdot x_3)$$

and

$$(x_1 \cdot x_2) \cdot x_3 - (x_1 \cdot x_3) \cdot x_2 = (Id - \tau_{23})((x_1 \cdot x_2) \cdot x_3).$$

The Σ_3-invariant subspace of $\mathbb{K}[\Sigma_3]$ generated by the vectors $Id - \tau_{12}$ and $Id - \tau_{23}$ is of dimension 5, and from the classification [56], this space corresponds to $F_v = \mathbb{K}(\mathcal{O}(v))$ with

$$v = 2Id - \tau_{12} - \tau_{13} - \tau_{23} + c_1.$$

We conclude that this operad is of rank 2.

Proposition 46 *For $\mathcal{P} = \mathcal{G}_i$-ass, $1 \leq i \leq 6$, one has*

$$\mathcal{P}^! = \widetilde{\mathcal{P}}.$$

Proof. In this case $\widetilde{\mathcal{P}}$ is defined by the module of relations

$$\left\{\begin{array}{l}\rho_1^l\rho_1^m(\sigma_l-\sigma_m)((x_1\cdot x_2)\cdot x_3),\\ \\ \lambda_1^l\lambda_1^m(\sigma_l-\sigma_m)(x_1\cdot(x_2\cdot x_3)),\\ \\ \rho_1^l\lambda_1^m(\sigma_l((x_1\cdot x_2)\cdot x_3)-\sigma_m(x_1\cdot(x_2\cdot x_3))).\end{array}\right.$$

where σ_l and $\sigma_m \in G_i$ and

$$\rho_1^r=\lambda_1^r=\left\{\begin{array}{l}(-1)^{\varepsilon(\sigma_s)}\quad\text{if } \sigma_s\in G,\\ 0\quad\text{otherwise.}\end{array}\right.$$

This system reduces to

$$\left\{\begin{array}{l}((x_1\cdot x_2)\cdot x_3)-(x_1\cdot(x_2\cdot x_3)),\\ \rho_1^l\rho_1^m(\sigma_l-\sigma_m)((x_1\cdot x_2)\cdot x_3),\end{array}\right.$$

which defines the dual operad.

In particular, if $\mathcal{P}=\mathcal{L}ie\mathcal{A}dm$, then $\mathcal{P}^!=\widetilde{\mathcal{P}}$ is the quadratic operad $\mathcal{C}omm3$ defined by the submodule of relations

$$R=(R_{V_6})^!.$$

A $\mathcal{C}omm3$-algebra $\mathcal{A}$ is 3-commutative in the sense that it is associative and satisfies

$$x_i\cdot x_j\cdot x_k=x_{\sigma(i)}\cdot x_{\sigma(j)}\cdot x_{\sigma(k)}$$

for every $\sigma\in\Sigma_3$. If $\mathcal{A}$ is unitary this implies that $\mathcal{A}$ is commutative. In the general case $\mathcal{A}^2=\mathcal{A}\cdot\mathcal{A}$ is contained in the center of $\mathcal{A}$. The associated Lie algebra is two step nilpotent.

6.3.5 $\mathbb{K}[\Sigma_3]$-associative Lie-admissible algebras

In this section we describe the $\mathcal{P}$-algebras, $\mathcal{P}^!$-algebras and $\widetilde{\mathcal{P}}$-algebras, where $\mathcal{P}$ corresponds to the $\mathbb{K}[\Sigma_3]$-associative Lie admissible algebras. We use the classification of Chapter 1 (see also [56]). We denote by $(x,y,z)=(x\cdot y)\cdot z-x\cdot(y\cdot z)$.

$$(I)\left\{\begin{array}{l}\mathcal{P}=\mathcal{L}ie\mathcal{A}dm:(x,y,z)-(y,x,z)-(x,z,y)-(z,y,x)+(y,z,x)+(z,x,y)=0.\\ \mathcal{P}^!:(x,y,z)=0,\ x\cdot y\cdot z=y\cdot x\cdot z=x\cdot z\cdot y.\\ \widetilde{\mathcal{P}}=\mathcal{P}^!\end{array}\right.$$

$$(II)\left\{\begin{array}{l}\mathcal{P}=\mathcal{G}_5\text{-}ass:(x,y,z)+(y,z,x)+(z,x,y)=0.\\ \mathcal{P}^!:(x,y,z)=0,\ x\cdot y\cdot z=y\cdot z\cdot x=z\cdot x\cdot y.\\ \widetilde{\mathcal{P}}=\mathcal{P}^!\end{array}\right.$$

$$(III)\left\{\begin{array}{l}\mathcal{P}:\alpha(x,y,z)-\alpha(y,x,z)+(\alpha+\beta-3)(z,y,x)-\beta(x,z,y)+\beta(y,z,x)\\ \qquad+(3-\alpha-\beta)(z,x,y)=0,\ (\alpha,\beta)\neq(1,1),\\ \mathcal{P}^!:(x,y,z)=0,(\alpha-\beta)(x\cdot y\cdot z-y\cdot x\cdot z)+(\alpha+2\beta-3)(z\cdot y\cdot x-z\cdot x\cdot y)=0,\end{array}\right.$$

The computation of $\widetilde{\mathcal{P}}$ depends on the values of the parameters α and β. If $(\alpha,\beta)\notin\{(3,0),(0,3),(0,0)\}$, then:

$$\widetilde{\mathcal{P}}=\mathcal{L}ie\mathcal{A}dm^!$$

If $(\alpha,\beta)=(3,0)$ then $\mathcal{P}=\mathcal{G}_2$-*ass* and

$$\widetilde{\mathcal{P}}=\mathcal{P}^!=\mathcal{G}_2\text{-}ass^!.$$

If $(\alpha,\beta)=(0,3)$ then $\mathcal{P}=\mathcal{G}_4$-*ass* and

$$\widetilde{\mathcal{P}}=\mathcal{P}^!=\mathcal{G}_4\text{-}ass^!.$$

If $(\alpha,\beta)=(0,0)$, then $\mathcal{P}=\mathcal{G}_3$-*ass* and

$$\widetilde{\mathcal{P}}=\mathcal{P}^!=\mathcal{G}_3\text{-}ass^!.$$

$$(IV_1)\left\{\begin{array}{l}\mathcal{P}:(x,y,z)+(1+t)(y,x,z)+(z,y,x)+(y,z,x)+(1-t)(z,x,y)=0, t\neq 1,\\ \mathcal{P}^!:(x,y,z)=0,(t-1)x\cdot y\cdot z-(t-1)y\cdot x\cdot z-(t+2)z\cdot y\cdot x\\ \qquad +(1+2t)x\cdot z\cdot y-(1+2t)y\cdot z\cdot x+(t+2)z\cdot x\cdot y=0,\\ \widetilde{\mathcal{P}}=\mathcal{L}ie\mathcal{A}dm^!.\end{array}\right.$$

$$(IV_2)\left\{\begin{array}{l}\mathcal{P}:2(x,y,z)+(y,x,z)+(x,z,y)+(y,z,x)+(z,x,y)=0,\\ \mathcal{P}^!:(x,y,z)=0,x\cdot y\cdot z+y\cdot x\cdot z-z\cdot y\cdot x-z\cdot x\cdot y=0,\\ \widetilde{\mathcal{P}}=\mathcal{L}ie\mathcal{A}dm^!.\end{array}\right.$$

$$(V)\left\{\begin{array}{l}\mathcal{P}:2(x,y,z)-(y,x,z)-(z,y,x)-(x,z,y)+(y,z,x)=0,\\ \mathcal{P}^!:(x,y,z)=0,x\cdot y\cdot z-y\cdot x\cdot z-z\cdot y\cdot x-x\cdot z\cdot y+y\cdot z\cdot x+z\cdot x\cdot y=0,\\ \widetilde{\mathcal{P}}=\mathcal{L}ie\mathcal{A}dm^!.\end{array}\right.$$

$$(VI)\left\{\begin{array}{l}\mathcal{P}=\mathcal{A}ss:(x,y,z)=0,\\ \mathcal{P}^!=\mathcal{A}ss,\\ \widetilde{\mathcal{P}}=\mathcal{A}ss.\end{array}\right.$$

Proposition 47 *Let $\mathcal{P}$ be an operad corresponding to a class of $\mathbb{K}[\Sigma_3]$-associative Lie-admissible algebras. Then $\widetilde{\mathcal{P}}=\mathcal{P}^!$ if and only if $\mathcal{P}$ is the operad G_i-$\mathcal{A}ss$ for some i.*

6.3.6 3-power associative algebras

Every $\mathcal{G}_i$-p^3ass-algebra is 3-power associative (or third-power associative) algebra (see Chapter 5 or [56]). The corresponding dual operads are described by the following ideals of relations:

$$\begin{array}{rl}(R_{W_1})^! = & R_{W_1}=R_{V_1},\\ (R_{W_2})^! = & \mathbb{K}(\mathcal{O}((x_1\cdot x_2)\cdot x_3-x_1\cdot(x_2\cdot x_3);(x_1\cdot x_2)\cdot x_3+(x_2\cdot x_1)\cdot x_3)),\\ (R_{W_3})^! = & \mathbb{K}(\mathcal{O}((x_1\cdot x_2)\cdot x_3-x_1\cdot(x_2\cdot x_3);(x_1\cdot x_2)\cdot x_3)+(x_1\cdot x_3)\cdot x_2)),\\ (R_{W_4})^! = & \mathbb{K}(\mathcal{O}((x_1\cdot x_2)\cdot x_3-x_1\cdot(x_2\cdot x_3);(x_1\cdot x_2)\cdot x_3+(x_3\cdot x_2)\cdot x_1)),\\ (R_{W_5})^! = & R^!_{V_5},\\ (R_{W_6})^! = & \mathbb{K}(\mathcal{O}((x_1\cdot x_2)\cdot x_3-x_1\cdot(x_2\cdot x_3);(x_1\cdot x_2)\cdot x_3+(x_2\cdot x_1)\cdot x_3;\\ & \qquad (x_1\cdot x_2)\cdot x_3+(x_1\cdot x_3)\cdot x_2)).\end{array}$$

The proof is analogous to the Lie-admissible case. Note that these operads are also of rank 2 except for $i=1$ which is of rank 1.

Proposition 48 *If $\mathcal{P}=\mathcal{G}_i$-$p^3ass$, then $\widetilde{\mathcal{P}}=(\mathcal{G}_i\text{-}ass)^!=\widetilde{\mathcal{G}_i\text{-}ass}$.*

Proof. The computation is analogous to the proof of Proposition 46. Let us note that $\widetilde{\mathcal{P}}\neq\mathcal{P}^!$ except for $\mathcal{P}=(G_1\text{-}p^3\mathcal{A}ss)$ and $(G_5\text{-}p^3\mathcal{A}ss)$. For example,

- $$\widetilde{G_2\text{-}p^3\mathcal{A}ss}=\mathcal{P}erm.$$

- $$\widetilde{G_4\text{-}p^3\mathcal{A}ss}=(G_4\text{-}\mathcal{A}ss)^!.$$

  This means that a $\widetilde{G_4\text{-}p^3\mathcal{A}ss}$ is an associative algebra $\mathcal{A}$ satisfying

  $$abc=cba,$$

  for any $a,b,c\in\mathcal{A}$.

- $$\widetilde{G_6\text{-}p^3\mathcal{A}ss}=\mathcal{L}ie\mathcal{A}dm^!,$$

  that is, the binary quadratic operad whose corresponding algebras are associative and satisfying

  $$abc=acb=bac.$$

6.3.7 The operad $\mathcal{A}lt$

The current operad $\widetilde{\mathcal{A}lt}$ of $\mathcal{A}lt$ for alternative algebras is the operad for associative algebras satisfying

$$abc = bac = cba = acb = bca,$$

that is, 3-commutative associative algebras. Thus

$$\widetilde{\mathcal{A}lt} = \mathcal{L}ie\mathcal{A}dm^!.$$

6.4 Case where $\mathcal{P}$ has a generating operation with a symmetry

In all previous examples, except $\mathcal{P} = \mathcal{L}ie$, we considered generating operation without symmetry. We can also find the quadratic operads generated by a commutative operation (i.e $E = 1\!\!1$) or an anticommutative one (i.e $E = Sgn_2$) such that $\widetilde{\mathcal{P}} = \mathcal{P}^!$.

Proposition 49 *Let $\mathcal{P} = \Gamma(E)/(R)$ be a quadratic operad generated by an operation with a symmetry (i.e. $E = Sgn_2$ or $E = 1\!\!1$). If $\widetilde{\mathcal{P}} = \mathcal{P}^!$ then $E = Sgn_2$.*

We can easily verify that $\mathcal{P}_1 = \mathcal{L}ie$ or $\mathcal{P}_2 = \Gamma(Sgn_2)$ satisfy this property: $\widetilde{\mathcal{P}_1} = \mathcal{C}om = {\mathcal{P}_1}^!$ and $\widetilde{\mathcal{P}_2} = \Gamma(1\!\!1) = {\mathcal{P}_2}^!$, where $\Gamma(1\!\!1)$ denotes the free operad generated by the trivial representation.

Remark. We restricted our attention to quadratic operad with one binary generating operation $\mu : V^{\otimes 2} \to V$. But the concept of $\widetilde{\mathcal{P}}$ operad can by extended to an arbitrary quadratic operad $\mathcal{P}$, that is, generated by more than one n-ary operation.

6.5 Dihedral and cyclic operads

6.5.1 Monoidal structures

Consider the category $\mathcal{P}$-`alg` of algebras over a fixed operad $\mathcal{P}$. Following [44, 96] we say that $\mathcal{P}$ is a *Hopf operad*, if the category $\mathcal{P}$-`alg` admits a strict monoidal structure $\odot : \mathcal{P}\text{-}\mathtt{alg} \times \mathcal{P}\text{-}\mathtt{alg} \to \mathcal{P}\text{-}\mathtt{alg}$ such that the forgetful functor $\square : \mathcal{P}\text{-}\mathtt{alg} \to \mathtt{Vect}_{\mathbb{K}}$ to the category of $\mathbb{K}$-vector spaces with the standard tensor product, is a strict monoidal morphism, see [80, VII.1] for the terminology. This condition can be expressed solely in terms of $\mathcal{P}$ as in the following definition.

Definition 47 *An operad $\mathcal{P}$ is a Hopf operad if there exists an operadic map $\Delta : \mathcal{P} \to \mathcal{P} \otimes \mathcal{P}$ (the* diagonal*) which is coassociative in the sense that*

$$(\Delta \otimes 1\!\!1_{\mathcal{P}})\Delta = (1\!\!1_{\mathcal{P}} \otimes \Delta)\Delta, \tag{6.11}$$

where $1\!\!1_{\mathcal{P}} : \mathcal{P} \to \mathcal{P}$ denotes the identity. We also assume the existence of a counit *$e : \mathcal{P} \to \mathcal{C}om$, where $\mathcal{C}om$ is the operad for commutative associative algebras, satisfying*

$$(e \otimes 1\!\!1_{\mathcal{P}})\Delta = (1\!\!1_{\mathcal{P}} \otimes e)\Delta = 1\!\!1_{\mathcal{P}}. \tag{6.12}$$

The last equation uses the canonical identification $\mathcal{P} \cong \mathcal{C}om \otimes \mathcal{P} \cong \mathcal{P} \otimes \mathcal{C}om$. Our terminology slightly differs from the original one of [44] which did not assume the counit. Our assumption about the existence of the counit rules out trivial diagonals.

The diagonal $\Delta : \mathcal{P} \to \mathcal{P} \otimes \mathcal{P}$ induces a product $\odot : \mathcal{P}\text{-}\mathtt{alg} \times \mathcal{P}\text{-}\mathtt{alg} \to \mathcal{P}\text{-}\mathtt{alg}$ in a way described for example in [91, page 197]. Equation (6.11) is equivalent to the coassociativity of this product. To interpret (6.12), observe that, since $\mathcal{C}om$ is isomorphic to the endomorphism operad $\mathcal{E}nd_{\mathbb{K}}$ of the ground field, the counit e

equips $\mathbb{K}$ with a $\mathcal{P}$-algebra structure. Equation (6.12) then says that $\mathbb{K}$ with this structure is the unit object for the monoidal structure induced by Δ.

In the rest of this section we want to discuss the existence Hopf structures on quadratic operads $\mathcal{P} = \Gamma(E)/(R)$ with one operation. Let us look more closely at the map $e : \mathcal{P} \to \mathcal{C}om$ first. Since $\mathcal{C}om = \Gamma(1\!\!1_2)/(R_{\rm ass})$, with $1\!\!1_2$ the trivial representation of Σ_2 and $(R_{\rm ass})$ the ideal generated by the associativity, the counit e is determined by a Σ_2-equivariant map

$$e(2) : E \to 1\!\!1_2. \tag{6.13}$$

If $E = 1\!\!1_2$ (case (1) of the nomenclature of the introduction), such a map is the multiplication by a scalar α. If $E = \mathrm{sgn}_2$ (case (2)), the only equivariant $e(2)$ is the zero map. Finally, if $E = \mathbb{K}[\Sigma_2]$ (case (3)), $e(2)$ must be the projection $\mathbb{K}[\Sigma_2] \to 1\!\!1_2$ multiplied by some $\alpha \in \mathbb{K}$.

Equation (6.12) implies the non-triviality of $e(2)$. This excludes case (2) and implies that $\alpha \neq 0$ in cases (1) and (3). In these two cases we may moreover assume the *normalization* $\alpha = 1$, the general case can be brought to this form by rescaling $e \mapsto \alpha^{-1}e$, $\Delta \mapsto \alpha\Delta$.

Let us introduce the following useful pictorial language. Denote by $\square \in \mathcal{P}(2)$ the operadic generator for a type (3) operation (a multiplication with no symmetry). Similarly, we denote the generator for a commutative operation by $\bullet$ and for an anti-commutative one by $\circ$. The right action of the generator $\tau \in \Sigma_2$ on $\mathcal{P}(2)$ is, in this language, described by

$$\underset{1\ 2}{\square}\,\tau = \underset{2\ 1}{\square}\,,\quad \underset{1\ 2}{\bullet}\,\tau = \underset{2\ 1}{\bullet} = \underset{1\ 2}{\bullet} \quad \text{and} \quad \underset{1\ 2}{\circ}\,\tau = \underset{2\ 1}{\circ} = -\underset{1\ 2}{\circ}\,.$$

The polarization (2.1) is then given by

$$\underset{1\ 2}{\bullet} = \frac{1}{\sqrt{2}}\left(\underset{1\ 2}{\square} + \underset{2\ 1}{\square}\right) \quad \text{and} \quad \underset{1\ 2}{\circ} := \frac{1}{\sqrt{2}}\left(\underset{1\ 2}{\square} - \underset{2\ 1}{\square}\right),$$

and the depolarization (2.2) by

$$\underset{1\ 2}{\square} := \frac{1}{\sqrt{2}}\left(\underset{1\ 2}{\bullet} + \underset{1\ 2}{\circ}\right).$$

In the rest of this section we investigate the existence of diagonals for quadratic operads with one operation. Since the diagonal is, by assumption, an operadic homomorphism, it is uniquely determined by its value on a chosen generator of $\mathcal{P}(2)$. Let us see what can be concluded from this simple observation. As before, we distinguish three cases.

Case (1). In this case, the operad $\mathcal{P}$ is generated by one commutative bilinear operation $\bullet \in \mathcal{P}(2)$. The diagonal must necessarily satisfy

$$\Delta(\underset{1\ 2}{\bullet}) = A\left(\underset{1\ 2}{\bullet} \otimes \underset{1\ 2}{\bullet}\right), \text{ for some } A \in \mathbb{K}.$$

The coassociativity (6.11) is fulfilled automatically while (6.12) implies $A = 1$.

Case (2). Analyzing the counit, we already observed that operads with one anti-symmetric operation do not admit a (counital) diagonal. An easy argument shows that non-trivial diagonals for type (2) operads do not exists even if we do not demand the existence of a counit. Indeed, in case (2) we have $\mathcal{P}(2) \cong \mathrm{sgn}_2$ while $\mathcal{P}(2) \otimes \mathcal{P}(2) \cong \mathrm{sgn}_2 \otimes \mathrm{sgn}_2 \cong 1\!\!1_2$, therefore $\Delta(2) : \mathcal{P}(2) \to \mathcal{P}(2) \otimes \mathcal{P}(2)$ is trivial, as is any Σ_2 equivariant map $\mathrm{sgn}_2 \to 1\!\!1_2$. Let us formulate this observation as:

Theorem 48 *There is no non-trivial diagonal on a quadratic operad generated by an anti-symmetric product. In particular, the operad $\mathcal{L}ie$ for Lie algebras is not an Hopf operad.*

Case (3). As an operadic homomorphism, the diagonal (if exists) is uniquely determined by an element $\Delta(\underset{1\ 2}{\square}) \in \mathcal{P}(2) \otimes \mathcal{P}(2)$. The following proposition characterizes which choices of $\Delta(\underset{1\ 2}{\square})$ may lead to a coassociative counital diagonal.

Proposition 50 *Let $\mathcal{P}$ be a quadratic Hopf operad generated by a type (3) product $\pitchfork$. Then there exists $B \in \mathbb{K}$ such that the diagonal Δ is given by*

$$\Delta(\underset{1\;2}{\pitchfork}) = \underset{1\;2}{\pitchfork} \otimes \underset{1\;2}{\pitchfork} - B\left\{(\underset{1\;2}{\pitchfork} - \underset{2\;1}{\pitchfork}) \otimes (\underset{1\;2}{\pitchfork} - \underset{2\;1}{\pitchfork})\right\}. \tag{6.14}$$

The polarized version of this equation reads

$$\left\{\begin{array}{rcl} \Delta(\underset{1\;2}{\overset{\bullet}{\curlywedge}}) & = & \dfrac{1}{\sqrt{2}}\left\{\underset{1\;2}{\overset{\bullet}{\curlywedge}} \otimes \underset{1\;2}{\overset{\bullet}{\curlywedge}} + (1-4B)\left(\underset{1\;2}{\overset{\circ}{\curlywedge}} \otimes \underset{1\;2}{\overset{\circ}{\curlywedge}}\right)\right\}, \\ \Delta(\underset{1\;2}{\overset{\circ}{\curlywedge}}) & = & \dfrac{1}{\sqrt{2}}\left\{\underset{1\;2}{\overset{\circ}{\curlywedge}} \otimes \underset{1\;2}{\overset{\bullet}{\curlywedge}} + \underset{1\;2}{\overset{\bullet}{\curlywedge}} \otimes \underset{1\;2}{\overset{\circ}{\curlywedge}}\right\}. \end{array}\right. \tag{6.15}$$

Proof. A simple bookkeeping. The most general choice for $\Delta(2) : \mathcal{P}(2) \to \mathcal{P}(2) \otimes \mathcal{P}(2)$ is

$$\begin{aligned} \Delta(\underset{1\;2}{\pitchfork}) &= A(\underset{1\;2}{\pitchfork} \otimes \underset{1\;2}{\pitchfork}) + B(\underset{1\;2}{\pitchfork} \otimes \underset{2\;1}{\pitchfork}) \\ &\quad + C(\underset{2\;1}{\pitchfork} \otimes \underset{1\;2}{\pitchfork}) + D(\underset{2\;1}{\pitchfork} \otimes \underset{2\;1}{\pitchfork}) \end{aligned} \tag{6.16}$$

with some $A, B, C, D \in \mathbb{K}$. A straightforward calculation shows that the coassociativity $(\Delta \otimes 1\!\!1_{\mathcal{P}})\Delta = (1\!\!1_{\mathcal{P}} \otimes \Delta)\Delta$ for Δ defined by (6.16) has the following four families of solutions:

$$\begin{array}{ll} \text{(i) } D = C = 0,\ A = B, & \text{(ii) } B = C = -D,\ A \text{ arbitrary}, \\ \text{(iii) } D = B = 0,\ A = C, & \text{(iv) } B = C = D = A. \end{array}$$

The counit condition (6.12) leads to the system:

$$A + C = 1,\ B + D = 0,\ A + B = 1, \text{ and } C + D = 0.$$

We easily conclude that the only solution is a type (ii) one with $B = C = -D$ and $A = 1 - B$. This gives (6.14) whose polarization is (6.15).

Proposition 50 offers a useful tool to investigate the existence of Hopf structures for type (3) operads. It says that such an operad $\mathcal{P}$ is a Hopf operad if and only if there exists $B \in \mathbb{K}$ such that the diagonal defined by (6.14) (resp. (6.15) in the polarized form) extends to an operad map, i.e. preserves the relations R in the quadratic presentation $\Gamma(E)/(R)$ of $\mathcal{P}$.

Regarding the existence of diagonals in general, an operad might admit no Hopf structure at all (examples of this situation are provided by Theorem 48), it might admit exactly one Hopf structure (see Example 6.5.1 for operads with this property), or it might admit several different monoidal structures, as illustrated in Example 6.5.1.

Let us formulate another simple proposition whose proof we leave as an exercise. We say that $\mathcal{P}$ is a *set-operad*, if there exists an operad $\mathcal{S}$ in the monoidal category of sets such that, for any $n \geq 1$, $\mathcal{P}(n)$ is the $\mathbb{K}$-linear span of $\mathcal{S}(n)$, and that the operad structure of $\mathcal{P}$ is naturally induced from the operad structure of $\mathcal{S}$.

Proposition 51 *Every set-operad $\mathcal{P}$ admits an Hopf structure given by the formula $\Delta(p) := p \otimes p$, for any $p \in \mathcal{P}$.*

Examples

1. Let $\mathcal{LL}_q$ denote the operad for LL_q algebras. Then the one-parameter family $\{\mathcal{LL}_q\}_{q \neq \infty}$ is a family of Hopf operads, with the diagonal given by

$$\Delta(\underset{1\;2}{\pitchfork}) = \underset{1\;2}{\pitchfork} \otimes \underset{1\;2}{\pitchfork} - \frac{1-q}{4}\left\{(\underset{1\;2}{\pitchfork} - \underset{2\;1}{\pitchfork}) \otimes (\underset{1\;2}{\pitchfork} - \underset{2\;1}{\pitchfork})\right\}. \tag{6.17}$$

The polarized version of this equation reads

$$\left\{\begin{array}{lcl} \Delta(\underset{1\;2}{\bullet}) & = & \dfrac{1}{\sqrt{2}}\left\{(\underset{1\;2}{\bullet}\otimes\underset{1\;2}{\bullet}+q\left(\underset{1\;2}{\circ}\otimes\underset{1\;2}{\circ}\right)\right\}, \\ \Delta(\underset{1\;2}{\circ}) & = & \dfrac{1}{\sqrt{2}}\left\{\underset{1\;2}{\circ}\otimes\underset{1\;2}{\bullet}+\underset{1\;2}{\bullet}\otimes\underset{1\;2}{\circ}\right\}. \end{array}\right. \tag{6.18}$$

The above normalized diagonal is moreover unique for each $q \neq \infty$. Observe that the limit for $q \to \infty$ of formulas (6.17) (resp. (6.18)) does not make sense and, indeed, it can be easily shown that the operad $\mathcal{LL}_\infty$ is not Hopf.

2. We give an example of a category of algebras which admits several non-equivalent monoidal structures. Let us consider a type (3) product $x, y \mapsto x \cdot y$, with the axiom

$$(x \cdot y) \cdot z = z \cdot (y \cdot x).$$

Then

$$\Delta(\underset{1\;2}{\sqcap}) := \underset{1\;2}{\sqcap}\otimes\underset{1\;2}{\sqcap} - B\left[(\underset{1\;2}{\sqcap}-\underset{2\;1}{\sqcap})\otimes(\underset{1\;2}{\sqcap}-\underset{2\;1}{\sqcap})\right].$$

defines an Hopf structure for any $B \in \mathbb{K}$.

Above we saw an algebra admitting a one-parameter family of non-equivalent monoidal structures. It would be interesting to see a structure that admits a *discrete* family of non-equivalent Hopf structures.

3. It can be shown that the only G-admissible algebras that admit a monoidal structure are associative algebras. In particular, pre-Lie algebras do not form a monoidal category.

6.5.2 Cyclicity and dihedrality

In this section we study cyclicity [45] of operads mentioned in the previous sections. We then introduce the notion of *dihedrality* of operads and investigate this property. To complete the picture, we also list results concerning Koszulness [46] of some operads with one operation.

Let us recall first what is a cyclic operad. Let Σ_n^+ be the group of automorphisms of the set $\{0, \ldots, n\}$. This group is, of course, isomorphic to the symmetric group Σ_{n+1}, but the isomorphism is canonical only up to an identification $\{0, \ldots, n\} \cong \{1, \ldots, n+1\}$. We interpret Σ_n as the subgroup of Σ_n^+ consisting of permutations $\sigma \in \Sigma_n^+$ with $\sigma(0) = 0$. If $\gamma_n^+ \in \Sigma_n^+$ denotes the cycle $(0, \ldots, n)$, that is, the permutation with $\gamma_n^+(0) = 1,\ \gamma_n^+(1) = 2,\ \ldots, \gamma_n^+(n) = 0$, then γ_n^+ and Σ_n generate Σ_n^+.

By definition, each operad $\mathcal{P}$ has a natural right action of Σ_n on each piece $\mathcal{P}(n)$, $n \geq 1$. The operad $\mathcal{P}$ is *cyclic* if this action extends, for any $n \geq 1$, to a Σ_n^+-action in a way compatible with structure operations. See [91, Definition II.5.2] or the original paper [45] for a precise definition.

We already recalled in the introduction that an (ordinary) operad $\mathcal{P}$ is quadratic if it can be presented as $\mathcal{P} = \Gamma(E)/(R)$, where $E = \mathcal{P}(2)$ and $R \subset \Gamma(E)(3)$. The action of Σ_2 on E extends to an action of Σ_2^+, via the sign representation $\mathrm{sgn} : \Sigma_2^+ \to \{\pm 1\} \cong \Sigma_2$. It can be easily verified that this action induces a cyclic operad structure on the free operad $\Gamma(E)$. In particular, $\Gamma(E)(3)$ is a right Σ_3^+-module. An operad $\mathcal{P}$ as above is called *cyclic quadratic* if the space of relations R is invariant under the action of Σ_3^+. Since R is, by definition, Σ_3-invariant, $\mathcal{P}$ is cyclic quadratic if and only if R is preserved by the action of the generator γ_3^+.

Remarks. There are operads that are both quadratic and cyclic but not cyclic quadratic. The simplest example of this exotic phenomenon is provided by the free operad $\Gamma(V_{2,2})$ generated by the 2-dimensional irreducible representation $V_{2,2}$ of $\Sigma_3 \cong \Sigma_2^+$ placed in arity 2. In general, an operad $\mathcal{P}$ is cyclic quadratic if and only if it is both quadratic and cyclic and if the Σ_2^+-action on $\mathcal{P}(2)$ is induced from the operadic Σ_2-action on $\mathcal{P}(2)$ via the homomorphism $\mathrm{sgn} : \Sigma_2^+ \to \{\pm 1\} \cong \Sigma_2$.

Let us turn our attention to the cyclicity of operads for algebras with one operation. Since, as proved in [45, Proposition 3.6], each quadratic operad with one operation of type (1) or (2) is cyclic quadratic,

Operad	Type of algebras	Koszul	Cyclic	Dihedral	Hopf
$\mathcal{A}ss = \mathcal{LL}_1 = \mathcal{G}_1\text{-}ass$	associative	yes	yes	yes	yes
$\mathcal{P}oiss = \mathcal{LL}_0$	Poisson	yes	yes	yes	yes
$\mathcal{LL}_q,\ q \neq 0, \infty$	LL_q-algebras	yes	yes	yes	yes
$\mathcal{LL}_\infty$	LL_∞-algebras	yes	yes	yes	no
$\mathcal{V}inb = \mathcal{G}_2\text{-}ass$	Vinberg	yes	no	no	no
$pre - \mathcal{L}ie$	pre-Lie	yes	no	no	no
$\mathcal{G}_4\text{-}ass$	G_4-associative		yes	yes	no
$\mathcal{G}_5\text{-}ass$	G_5-associative	no	no	yes	no
$\mathcal{G}_6\text{-}ass$	Lie-admissible	yes	yes	yes	no

Figure 6.1: Koszulness, quadratic cyclicity, dihedrality and Hopfness of operads with one type (3) operation.

we shall focus on operads with a type (3) multiplication. The right action of the generator $\gamma_3^+ \in \Sigma_3^+$ on $\Gamma(\mathbb{K}[\Sigma_2])(3)$ is described in the following table:

$$\begin{array}{rclrcl}
((x\cdot y)\cdot z)\gamma_3^+ &=& x\cdot(y\cdot z), & (x\cdot(y\cdot z))\gamma_3^+ &=& (x\cdot y)\cdot z,\\
((y\cdot z)\cdot x)\gamma_3^+ &=& (y\cdot x)\cdot z, & (y\cdot(z\cdot x))\gamma_3^+ &=& y\cdot(x\cdot z),\\
((z\cdot x)\cdot y)\gamma_3^+ &=& y\cdot(z\cdot x), & (z\cdot(x\cdot y))\gamma_3^+ &=& (y\cdot z)\cdot x,\\
((y\cdot x)\cdot z)\gamma_3^+ &=& x\cdot(z\cdot y), & (y\cdot(x\cdot z))\gamma_3^+ &=& (x\cdot z)\cdot y,\\
((z\cdot y)\cdot x)\gamma_3^+ &=& z\cdot(y\cdot x), & (z\cdot(y\cdot x))\gamma_3^+ &=& (z\cdot y)\cdot x,\\
((x\cdot z)\cdot y)\gamma_3^+ &=& (z\cdot x)\cdot y, & (x\cdot(z\cdot y))\gamma_3^+ &=& z\cdot(x\cdot y).
\end{array}$$

Using this table, it is easy to investigate the cyclicity of operads with one operation, see Example 6.5.2 where the corresponding analysis was done for G_5-associative algebras. The results are summarized in Figure 6.1.

In Definition 49 below we single out a property of quadratic operads responsible for the existence of the dihedral cohomology [40, 78] of associated algebras. As far as we know, this property has never been considered before. Let $\mathcal{P} = \Gamma(E)/(R)$ be a quadratic operad. Let $\lambda \in \Sigma_2$ be the generator and define a left Σ_2-action on E using the operadic right Σ_2-action by $\lambda e := e\lambda$, for $e \in E$. It follows from the universal property of free operads that this action extends to a left Σ_2-action on $\Gamma(E)$.

Definition 49 *We say that a quadratic operad $\mathcal{P} = \Gamma(E)/(R)$ is* dihedral *if the left Σ_2-action on $\Gamma(E)$ induces a left Σ_2-action on $\mathcal{P}$. A quadratic operad is* cyclic dihedral*, if it is both cyclic and dihedral and if these two structures are compatible, by which we mean that*

$$(\lambda u)\sigma = \lambda(u\sigma),$$

for each $u \in \mathcal{P}(n)$, $\lambda \in \Sigma_2$, $\sigma \in \Sigma_n^+$ and $n \geq 1$. In other words, the cyclic and dihedral actions make each piece $\mathcal{P}(n)$ of a cyclic dihedral operad a left Σ_2- right Σ_n^+-bimodule.

Remarks. We emphasize that dihedrality is a property defined only for quadratic operads. We do not know how to extend this definition for a general operad. Observe that the left Σ_2-action on $\Gamma(E)$ induces an action on $\mathcal{P}$ as required in Definition 49 if and only if the space of relations $R \subset \Gamma(E)(3)$ is Σ_2-stable.

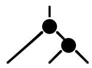
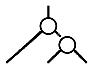
The operad $\Gamma(V_{2,2})$ considered in Remark 6.5.2 is quadratic, cyclic and dihedral, but not cyclic dihedral, because the left Σ_2-action on $V_{2,2}$ is clearly not compatible with the right Σ_2^+-action. On the other hand, each cyclic quadratic operad which is dihedral is cyclic dihedral.

We leave as an exercise to prove that all quadratic operads generated by one operation of type (1) or (2) are dihedral. Therefore again the only interesting case to investigate is a type (3) operation. The dihedrality is then easily understood if we write the axioms in the polarized form as follows. Let $E = \mathbb{K}[\Sigma_2]$ and decompose

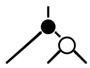
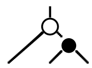
$$\Gamma(E)(3) = \Gamma_+(E)(3) \oplus \Gamma_-(E)(3), \tag{6.19}$$

where $\Gamma_+(E)(3)$ is the Σ_3-subspace of $\Gamma(E)(3)$ generated by compositions $x(yz)$ and $[x,[y,z]]$, and $\Gamma_-(E)(3)$ is the Σ_3-subspace of $\Gamma(E)(3)$ generated by compositions $x[y,z]$ and $[x,yz]$.

In the pictorial language of Section 6.5.1, $\Gamma_+(E)(3)$ is the Σ_3-invariant subspace generated by compositions of the following two types

 and 

while $\Gamma_-(E)(3)$ is the Σ_3-invariant subspace generated by

 and  .

Decomposition (6.19) is obviously Σ_3^+-invariant. It is almost evident that λ acts trivially on $\Gamma_+(E)(3)$ while on $\Gamma_-(E)(3)$ it acts as the multiplication by -1. We therefore get for free the following:

Proposition 52 *A quadratic operad $\mathcal{P} = \Gamma(E)/(R)$ generated by a type (3) multiplication is dihedral if and only if the space of relations R decomposes as*

$$R = R_+ \oplus R_-,$$

with $R_+ \subset \Gamma_+(E)(3)$ and $R_- \subset \Gamma_-(E)(3)$.

Example. In this example we show that the operad $\mathcal{G}_5$*-ass* for G_5-associative algebras is dihedral but not cyclic. Recall from Example 2.1.2 that the polarized form of the axioms for these algebras consists of the Jacobi identity

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$$

and equation (2.7)

$$[xy,z] + [yz,x] + [zx,y] = 0.$$

Since the left-hand side of the Jacobi identity belongs to $\Gamma_+(E)(3)$ and the right-hand side of (2.7) to $\Gamma_-(E)(3)$, the space of relations obviously decomposes as required by Proposition 52. Therefore $\mathcal{G}_5$*-ass* is dihedral.

Let us inspect the cyclicity. By definition, the unpolarized form of the axiom for G_5-associative algebras reads

$$A(x,y,z) + A(y,z,x) + A(z,x,y) = 0, \tag{6.20}$$

where A denotes, as usual, the associator (8.9). The action of γ_3^+ converts this equation to

$$-A(x,y,z) + A(y,x,z) - A(y,z,x) = 0.$$

The sum of the above equations gives

$$A(y,x,z) + A(z,x,y) = 0.$$

It is then a simple linear algebra to prove that this equation does not belong to the Σ_3-closure of (6.20). Therefore $\mathcal{G}_5$*-ass* is not cyclic.

Theorem 50 *Cyclic quadratic operads generated by one operation are dihedral.*

Proof. The claim is obvious when $\mathcal{P}$ is generated by one operation of type (1) or (2). Suppose $\mathcal{P}$ is a quadratic operad of the form

$$\mathcal{P} = \Gamma(E)/(R), \text{ where } E =: \mathbb{K}[\Sigma_2].$$

It was calculated in [45] that, as Σ_3^+-modules,

$$\Gamma_+(E)(3) = 1\!\!1_3 \oplus V_{2,2} \oplus \mathrm{sgn} \oplus V_{2,2} \ \text{ and } \ \Gamma_-(E)(3) = V_{3,1} \oplus V_{2,1,1}, \tag{6.21}$$

where the irreducible representations $1\!\!1, \mathrm{sgn}, V_{2,2}, V_{3,1}$ and $V_{2,1,1}$ are given by the following character table:

	I	(01)	(012)	(0123)	(01)(23)
$1\!\!1$	1	1	1	1	1
sgn	1	-1	1	-1	1
$V_{2,2}$	2	0	-1	0	2
$V_{3,1}$	3	1	0	-1	-1
$V_{2,1,1}$	3	-1	0	1	-1

Observe that there are no common factors in $\Gamma_+(E)(3)$ and $\Gamma_-(E)(3)$, therefore it follows from an elementary representation theory that each Σ_3^+-invariant subspace R of $\Gamma(E)(3)$ decomposes as $R = R_+ \oplus R_-$ with $R_+ \subset \Gamma_+(E)(3)$ and $R_- \subset \Gamma_-(E)(3)$. This means that $\mathcal{P}$ is dihedral, by Proposition 52.

Theorem 50 was a consequence of the fact that for operads generated by one operation, the Σ_3^+-spaces $\Gamma_+(E)(3)$ and $\Gamma_-(E)(3)$ do not contain a common irreducible factor. The following example shows that this is not longer true for general quadratic operads.

Example. Consider the quadratic operad $\mathcal{P} = \Gamma(E)/(R)$, where $E := \mathbb{K}[\Sigma_2] \oplus 1\!\!1_2$ and where (R) is the operadic ideal generated by the relations

$$\begin{aligned} r_1 &:= x \cdot (yz) + y \cdot (zx) + z \cdot (xy) = 0 \ \text{ and} \\ r_2 &:= (xy) \cdot z + (z \cdot x)y + x(z \cdot y) = 0. \end{aligned}$$

In the above display, $\cdot$ denotes the multiplication corresponding to a generator of $\mathbb{K}[\Sigma_2]$ and we, as usual, omit the symbol for the commutative multiplication corresponding to a generator of $1\!\!1_2$. Then $\mathcal{P}$ is cyclic but not dihedral.

Let us explain how this example was constructed. It can be calculated that in decomposition (6.19) of the 27-dimensional space $\Gamma(E)(3)$,

$$\begin{aligned} \Gamma_+(E)(3) &= 3 1\!\!1_2 \oplus \mathrm{sgn} \oplus 4V_{2,2} \oplus V_{3,1} \ \text{ and} \\ \Gamma_-(E)(3) &= 2V_{3,1} \oplus 2V_{2,1,1}. \end{aligned}$$

There is a common irreducible factor $V_{3,1}$ which occurs both in $\Gamma_+(E)(3)$ and in $\Gamma_-(E)(3)$. Therefore, to construct an operad which is cyclic but not dihedral, it is enough to choose a generator e_+ of $V_{3,1}$ in $\Gamma_+(E)$ and a generator e_- of $V_{3,1}$ in $\Gamma_-(E)$ and define R to be the Σ_3^+-subspace of $\Gamma(E)(3)$ generated by $e_+ + e_-$. Operad $\mathcal{P}$ above corresponds to one of these choices.

Chapter 7

n-ary algebras

The notion of n-ary algebra, that is, algebra whose product is given by a multilinear map, is not new. But, in the last years, the Nambu model for the statistic and classical mechanics has been developed to obtain, for example, a new formalization of Hamilton equations. If in the classical mechanic, the model corresponds to a Poisson algebra, in the Nambu model, we have to consider a new type of algebras called 3-Lie algebras, whose bracket concerns 3-vectors. This type of algebras has been studied, in a algebraic point of view, by Filippov. The notion of Filippov algebras corresponds exactly to Lie-Nambu algebras used in the Nambu mechanic. Thus, it is natural to consider n-ary algebras in a more general sense and to approach the Filippov algebras, for example, via associative algebras. But what is an n-ary associative algebra? Two notions can be considered as natural generalization of the binary associativity. They are called totally and partially associative algebras. At the end of the chapter we will introduce another associative notion to speak on ternary product of matrices. But at first, we consider totally and partially associative algebras. The n-ary associative algebras have been studied in details in the thesis of Gnedbaye [47]. But, in this work and in the following, the n-odd case has been treated in the same way that the n-even case, which cannot be the case. For example, in [47], the determination of the free n-ary partially associative algebra is true for even n but it is wrong for odd n. In this chapter we compute the free associative 3-ary algebra. We determine explicitly the dimensions and basis of the first homogeneous components of this free algebra. For this, we have used an original coding of ternary and, more generally, n-ary trees. The results of this chapter are in [62]. These results have conduced us to consider the notion of duality for ternary and more generally n-ary operads. But this is the subject of the following chapter. As a consequence, we prove that the operad for 3-ary partially associative algebras is not Koszul (this new result contradicts the works of [47] and others). So the operadic cohomology directly defined by the equations of the operad is different from the cohomology of the minimal model used, for example, to describe a deformation cohomology. All these results are developed in the next chapter. In the present chapter we develop a simpler cohomological theory for 3-ary partially associative algebras, where the coboundary operator is similar to the Hochschild operator for classical associative algebras. We end this chapter, considering a 3-product on the vector space of the non-square matrices. This product is extended to the vector spaces of tensors. Looking to the associativity property of this product, we prove that we have to consider a generalization of the classical partially-associativity with some permutations on the arguments appearing naturally. Thus, from this viewpoint, we find again some classical properties of Lie triple systems.The operadic part of this generalized associativity will be treated in the next chapter.

7.1 Associative n-ary algebras

Let $\mathbb{K}$ be a commutative field of characteristic zero and V a $\mathbb{K}$-vector space. An n-ary algebra structure on V is given by a linear map

$$\mu : V^{\otimes n} \to V.$$

We denote by (V, μ) such an algebra. In the following, the notation $I_0 \otimes \mu$ and $\mu \otimes I_0$ means μ.

Definition 51 *The n-ary algebra (V,μ) is*

- *partially associative if μ satisfies*

$$\sum_{p=0}^{n-1}(-1)^{p(n-1)}\mu\circ(I_p\otimes\mu\otimes I_{n-p-1})=0 \tag{7.1}$$

- *totally associative if μ satisfies*

$$\mu\circ(\mu\otimes I_{n-1})=\mu\circ(I_p\otimes\mu\otimes I_{n-p-1}), \tag{7.2}$$

for any $p=0,\cdots,n-1$.

Example. The Gerstenhaber products. Let $\mathcal{A}$ be a (binary) associative algebra and $H^*(\mathcal{A},\mathcal{A})$ its Hochschild cohomology. The space of k-cochains is $\mathcal{C}^k(\mathcal{A})=Hom_{\mathbb{K}}(\mathcal{A}^{\otimes k},\mathcal{A})$. Gerstenhaber defines a graded pre-Lie algebra $\oplus_k\mathcal{C}^k(\mathcal{A})$ considering the product

$$\bullet_{n,m}:\mathcal{C}^n(\mathcal{A})\times\mathcal{C}^m(\mathcal{A})\to\mathcal{C}^{n+m-1}(\mathcal{A})$$

given by

$$(f\bullet_{n,m}g)(X_1\otimes\cdots\otimes X_{n+m-1})=\sum_{i=1}^{m}(-1)^{(i-1)(m-1)}f(X_1\otimes\cdots\otimes g(X_i\otimes\cdots X_{i+m-1})\otimes\cdots\otimes X_{n+m-1}).$$

A k-cochain μ satisfying $\mu\bullet_{k,k}\mu=0$ provides $\mathcal{A}$ with a k-ary partially associative structure.

7.2 Partially Associative free n-ary algebras

The free algebras in the totally associative case, as well as in the partially associative case, are described in [47] when the n-product corresponds to n even. We prove in Chapter 8 that the description are correct in the even case. But when n is odd, we have noticed that the situation is completely different and we have to study the $(2k+1)$-ary partially associative free algebras more precisely. In particular we prove in [107] that the corresponding operad is not Koszul, and this result contradicts (in the odd case) some results presented until now. In this chapter we focus on the case $n=2k+1=3$. Using the Gerstenhaber notations we denote by $L(V,\bullet_{3,3})$ the free 3-ary partially associative algebra on the $\mathbb{K}$-vector space V. This algebra is graded

$$L(V,\bullet_{3,3})=\oplus_{p\geq 1}L^{2p+1}(V)$$

with $L^1(V)=V,\ L^3(V)=V^{\otimes 3}$. We denote by $F(V,\bullet)$ the free algebra on V corresponding to a 3-ary product $\bullet$ on V. This algebra is also graded

$$F(V,\bullet)=\oplus_{p\geq 1}F^{2p+1}(V)$$

with $F^1(V)=V,\ F^3(V)=V^{\otimes 3}$. Then, for $p>1$, $L^{2p+1}(V)$ is a quotient space of $F^{2p+1}(V)$. Note that $L^{2p+1}(V)$ is also a quotient space of

$$\oplus_{(a,b,c)\in D(2p+1,3)}L^a(V)\otimes L^b(V)\otimes L^c(V),$$

where $D(k,3)$ is the set of triples (a,b,c) of odd positif integers such that $a+b+c=k$. Now we have to investigate each one of the components $L^{2p+1}(V)$ and their dimension. For this, we process through several steps.

i) Coding a vector of $F^{2p+1}(V)$. We denote by $(v_1\cdot v_2\cdot v_3)$ the vector of $L^3(V)$ or $F^3(V)$ which is a 3-product of 3 vectors of V. An element of $F^{2p+1}(V)$ is a linear combination of vectors which are written as a sequence of p parenthesis inserted on a sequence $v_1\cdots v_{2p+1}$ of length $2p+1$, each parenthesis containing 3 vectors with the condition that a triple $(a\cdot b\cdot c)$ between parenthesis is considered as one vector of V.

For example $(v_1 \cdot (v_2 \cdot (v_3 \cdot v_4 \cdot v_5) \cdot v_6) \cdot v_7) \in F^7(V)$. If $\{e_1, \cdots, e_n\}$ is a basis of V, then $F^{2p+1}(V)$ is generated by all $(2p+1)$-uple of vectors of the basis on which we have inserted p parenthesis, each one containing 3 vectors. We can code the place of these parenthesis by the position of the left parenthesis. To simplify, we forget the first one which is always before the first vector (and the corresponding right parenthesis which is after the last vector). Thus in $F^5(V)$ the parenthesis are coded by $\{1\}, \{2\}, \{3\}$ and corresponds respectively to the vectors $((v_1 \cdot v_2 \cdot v_3) \cdot v_4 \cdot v_5), (v_1 \cdot (v_2 \cdot v_3 \cdot v_4) \cdot v_5), (v_1 \cdot v_2 \cdot (v_3 \cdot v_4 \cdot v_5))$. In $F^7(V)$, the parenthesis are coded by $\{11\}, \{12\}, \{13\}, \{14\}, \{15\}, \{22\}, \{23\}, \{24\}, \{25\}, \{33\}, \{34\}, \{35\}$. For example $(v_1 \cdot (v_2 \cdot (v_3 \cdot v_4 \cdot v_5) \cdot v_6) \cdot v_7)$ corresponds to $\{23\}$. Moreover starting with a coding $\{n_1 n_2 \cdots n_{p-1}\}$ and a vector $v_1 \otimes \cdots \otimes v_{2p+1}$ of V^{2p+1} we have only one vector of $F^{2p+1}(V)$ after putting brackets with the defined coding.

Lemma 8 *A $(p-1)$-sequence $\{n_1 \cdots n_{p-1}\}$ of positive integers is a coding of an element of $F^{2p+1}(V)$ or $L^{2p+1}(V)$ if $1 \leq n_1 \leq 3, n_1 \leq n_2 \leq 5, \cdots, n_{p-2} \leq n_{p-1} \leq 2p-1$. Such a sequence is called admissible.*

We denote by $\widetilde{C}_{p-1}$ the linear span of these sequences. Thus $F^{2p+1}(V) = \widetilde{C}_{p-1} \otimes V^{\otimes 2p+1}$. For example, if $p = 2$, then $\widetilde{C}_1 = \{\{1\}, \{2\}, \{3\}\}$ and $\dim F^5(V) = 3 \cdot \dim V^{\otimes 5}$.

ii) non necessarily independent and independent relations. Let R_{2p+1} be the subspace of $F^{2p+1}(V)$ generated by the relations defining $L^{2p+1}(V)$, that is, $L^{2p+1}(V) = F^{2p+1}(V)/R_{2p+1}$. We denote by C_{p-1} the corresponding independent relations of R_{2p+1} restricted to the elements of $\widetilde{C}_{p-1}$. Then $R_{2p+1} = C_{p-1} \otimes V^{\otimes 2p+1}$.

- For $p = 2$ we have $C_1 = \{\{1\} + \{2\} + \{3\}\}$. Thus $\dim R_5 = \dim V^{\otimes 5}$ and $\dim L^5(V) = 2 \dim V^{\otimes 5}$.
- For $p = 3$ we have

$$\begin{array}{rl} C_2 = & \{\{11\} + \{14\} + \{15\}, \{12\} + \{22\} + \{25\}, \{13\} + \{23\} + \{33\}, \\ & \{14\} + \{24\} + \{34\}, \{15\} + \{25\} + \{35\}, \{11\} + \{12\} + \{13\}, \\ & \{22\} + \{23\} + \{24\}, \{33\} + \{34\} + \{35\}\}. \end{array}$$

All these relations are independent. Then $\dim R_7 = 8 \dim V^{\otimes 7}$ and $\dim L^7(V) = 4 \dim V^{\otimes 7}$.

iii) Writing the relations of R_{2p+1}. Assume that C_{p-2} is known, that is, we have independent relations in degree $2p-1$. The following rules permits to determinate all the relations (non independent) in degree $2p+1$:

Consider an element of C_{p-2}. If $\{i_1, \cdots, i_{p-2}\}$ appears in this vector, then we obtain elements of $\widetilde{C}_{p-1}$ using the following two rules:

- Add the index i in front of each $(p-2)$ coded vector of this relation where i is successively equal to $1, 2, 3$ and we replace i_l by $i_l + (i-1)$ for all the elements $\{i_1, \cdots, i_{p-2}\}$ of the element of C_{p-2}.

- For $i = 1, 2, \cdots, 2p-1$, add the index i in front of any $(p-2)$-uple of element involved in the relation with the addition condition: if $\{i_1, \cdots, i_{p-2}\}$ is the relation, if $i_1 \leq i$, conserve i_1 otherwise replace i_1 by $i_1 + 2$, and apply the same rule for all further indices and at last rearrange the subscripts to get an admissible sequence (described in the previous lemma).

Consequence. Starting from an element of C_{p-2} we get $2p+2$ elements of $\widetilde{C}_{p-1}$.

Example. We consider $\{11\} + \{12\} + \{13\} \in C_2$. We obtain the following elements of $\widetilde{C}_3$ (and relations in $F^9(V)$):

$$\left\{ \begin{array}{ll} \{111\} + \{112\} + \{113\} & \text{we add 1,} \\ \{222\} + \{223\} + \{224\} & \text{we add 2 and change } i_l \text{ by } i_l + 1, \\ \{333\} + \{334\} + \{335\} & \text{we add 3 and change } i_l \text{ by } i_l + 2, \\ \{111\} + \{114\} + \{115\} & \text{we add 1 and change } i_l \text{ by } i_l + 2 \text{ if } i_l > 1, \\ \{112\} + \{122\} + \{125\} & \text{we add 2 and change } i_l \text{ by } i_l + 2 \text{ if } i_l > 2, \\ \{113\} + \{123\} + \{133\} & \text{we add 3 and change } i_l \text{ by } i_l + 2 \text{ if } i_l > 3, \\ \{114\} + \{124\} + \{134\} & \text{we add 4 and reorganize the sequence,} \\ \{115\} + \{125\} + \{135\} & \text{we add 5 and reorganize the sequence,} \\ \{116\} + \{126\} + \{136\} & \text{we add 6 and reorganize the sequence,} \\ \{117\} + \{127\} + \{137\} & \text{we add 7 and reorganize the sequence.} \end{array} \right.$$

In this case we obtain 80 relations on $\widetilde{C}_3$ starting from the 8 elements of C_2.

iv) Independent relations. Using Mathematica we reduce the previous system of $\widetilde{C}_{p-1}$ to obtain C_{p-1} .

Thus we obtain

Proposition 53 *If* $\dim V = n$*, the dimensions of homogeneous components of* $L(V, \bullet_{3,3})$ *are:*

$$\begin{array}{lll} \dim L^3(V) = n^3, & \dim L^5(V) = 2n^5, & \dim L^7(V) = 4n^7, \\ \dim L^9(V) = 5n^9, & \dim L^{11}(V) = 6n^{11}, & \dim L^{13}(V) = 7n^{13}. \end{array}$$

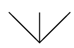
and more generally we conjecture that

$$\dim L^{2p+1}(V) = (p+1)n^{2p+1}.$$

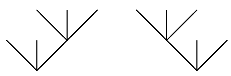
Proof. For $p = 1, \cdots, 6$, the computation have been made using Mathematica and verified directly (without computer). Let us note that for $p = 1, \cdots, 4$, the dimensions were obtained using a different coding by Bremner [13].

It remains to give a basis of the free algebra. We have already computed the dimensions of the first homogeneous components. Let us complete these results by describing a basis. For this we use a graphic representation by planar trees with 3-branching nods (three entries and one exit as the multiplication is 3-ary). We decorate each leave with a vector of a basis of V to obtain a free family of elements of the free algebra. Suppose V is n-dimensional. Then

- $\dim L^3(V) = n^3$. A basis is associated to the tree:

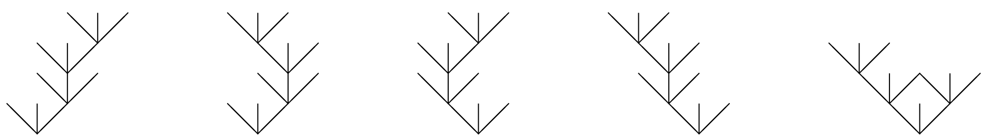


- $\dim L^5(V) = 2n^5$. A basis is associated to the trees

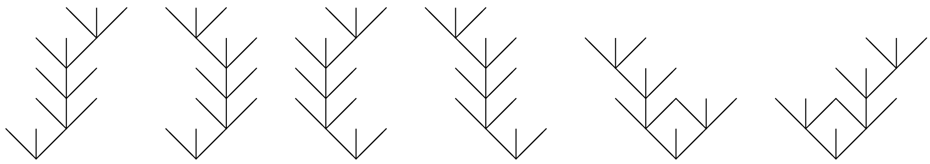


- $\dim L^7(V) = 4n^7$. A basis corresponds to the trees

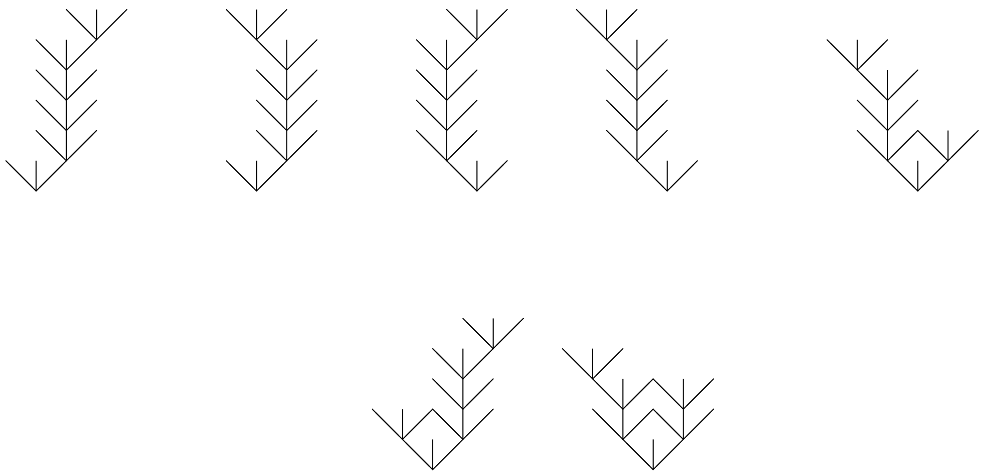
- $\dim L^9(V) = 5n^9$. A basis is given by



- $\dim L^{11}(V) = 6n^{11}$. A basis is given by



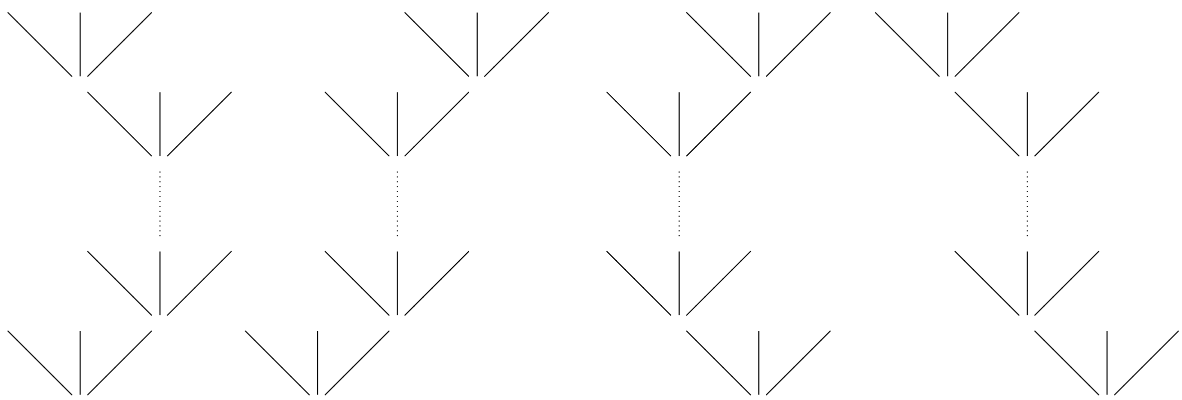
- $\dim L^{13}(V) = 7n^{13}$. A basis is given by



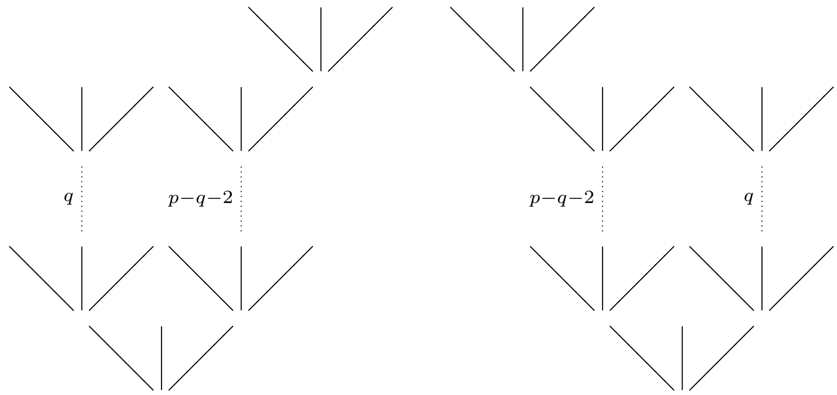
The choice of the basis is non canonical. But we choose them for symmetry reasons. The rules providing the relations of the subspace R_{2p+1} are easy to implement in order to solve the corresponding linear system. This gives the dimensions of the spaces $L^{2p+1}(V)$ (in fact, we find the dimensions of the modules of the associated operad) for $p = 2, 3, 4, 5$ and 6. Let us notice that we can however present basic vectors for the relations associated to the elements of

$$L^{2p-1} \otimes L^1 \otimes L^1 \oplus L^1 \otimes L^{2p-1} \otimes L^1 \oplus L^1 \otimes L^1 \otimes L^{2p-1}.$$

We conjecture that a free family of L^{2p+1} for $p \geq 4$ corresponds to the trees



The other are of the form



where $q = 1, \cdots, [\frac{p-2}{2}]$ and $[,]$ indicates the integer part of a rational number. If p is even, the last two trees are related. If p is odd, these trees are independent.

Remark. Recall that for any vector space V, the associated tensor algebra $T(V)$ is the unique solution, up to isomorphism, of the universal problem which determine from a linear application $f : M \to A$ in an associative algebra A, a morphism of associative algebra $T(V) \to A$. The construction of this algebra comes from the isomorphisms

$$\Phi_{n,m} : T^{\otimes n}(V) \otimes T^{\otimes m}(V) \to T^{\otimes (n+m)}(V)$$

defined by

$$\Phi_{n,m}((x_1 \otimes x_2 \otimes \cdots \otimes x_n) \otimes (y_1 \otimes y_2 \cdots \otimes y_m)) = x_1 \otimes x_2 \otimes \cdots x_n \otimes y_1 \otimes y_2 \otimes \cdots \otimes y_m.$$

In fact the multiplication μ of $T(V)$ is given by

$$\mu((x_1 \otimes x_2 \cdots \otimes x_n) \otimes (y_1 \otimes y_2 \cdots \otimes y_m)) = \Phi_{n,m}((x_1 \otimes x_2 \cdots \otimes x_n) \otimes (y_1 \otimes y_2 \cdots \otimes y_m))$$

and the associativity of the multiplication follows from

$$\Phi_{n+m,p} \bullet (\Phi_{n,m} \otimes Id_p) = \Phi_{n+m,p} \bullet (Id_n \otimes \Phi_{m,p}).$$

We can define an other isomorphism non longer adapted to the associative structure but adapted to the n-ary structure. For this we consider the family of vectorial isomorphisms

$$\Psi_{n,m,p} : T^{\otimes n}(V) \otimes T^{\otimes m}(V) \otimes T^{\otimes p}(V) \to T^{\otimes n+m+p}(V)$$

satisfying

$$\left\{ \begin{array}{ll} \Psi_{n,m+p+q,r} \bullet (Id_n \otimes \Psi_{m,p,q} \otimes Id_r) & = -2\Psi_{n,m+p+q,r} \bullet (Id_{n+m} \otimes \Psi_{p,q,r}) \\ & = -2\Psi_{n,m+p+q,r} \bullet (\Psi_{n,m,p} \otimes Id_{q+r}). \end{array} \right.$$

7.3 Cohomology of n-ary partially associative algebras: A μ-complex

In [47], an Hochschild cohomology is defined when n is even. In this case the operad $p\mathcal{A}ss_0^n$ for n-ary partially associative algebras is Koszul and this cohomology coincides with the operadic cohomology developed in [91] and governs deformations. For n odd, this is no longer true. If fact we prove (see [107] and Chapter 8) that the operad $pAss_0^{2k+1}$ is not Koszul, then the operadic cohomology can be different from the deformations cohomology. To define directly an Hochschild cohomology for $(2p+1)$-associative algebras (that is a cohomology with a coboundary operator defined by a relation similar to the associative binary case), we define a complex whose cochains satisfy some relations linked to the multiplication μ. Thus, this cohomology is different to the operadic cohomology. But, it corresponds to the approach of [47] in the even case.

Let V be a $\mathbb{K}$-vector space. For any $f \in \mathcal{C}^n(V)$ and $g \in \mathcal{C}^m(V)$ we can write the Gerstenhaber product $f \bullet_{n,m} g$ as

$$f \bullet_{n,m} g = \sum_{i=1}^{n} (-1)^{(i-1)(m-1)} f \circ_i g$$

with

$$(f \circ_i g)(X_1 \otimes \cdots \otimes X_{n+m-1}) = f(X_1 \otimes \cdots \otimes g(X_i \otimes \cdots X_{i+m-1}) \otimes \cdots \otimes X_{n+m-1}).$$

Recall that a map $\mu \in \mathcal{C}^n(V)$ such that $\mu \bullet_{n,n} \mu = 0$ is a n-ary partially associative product on V. If $f \in \mathcal{C}^n(V)$ and $g \in \mathcal{C}^m(V)$ we will write simply $f \bullet g$ instead of $f \bullet_{n,m} g$ as the subscripts of the Gerstenhaber products are clear.

Lemma 9 *Let μ be an n-ary partially associative product on V and $\varphi \in \mathcal{C}^k(V)$.*

1) If n is even, then $(\varphi \bullet \mu) \bullet \mu = 0$.

2) If n is odd, then

$$(\varphi \bullet \mu) \bullet \mu = \sum_{1 \leq p \leq q-n \leq k-1} (\varphi \circ_p \mu) \circ_q \mu.$$

Proof. We have the pre-Lie Identity ([42]):

$$(\varphi \bullet_{k,n} \mu) \bullet_{k+n-1,n} \mu - \varphi \bullet_{k,2n-1} (\mu \bullet_{n,n} \mu) = (-1)^{(n-1)^2} [\varphi \bullet_{k,n} \mu) \bullet_{k+n-1,n} \mu - \varphi \bullet_{k,2n-1} (\mu \bullet_{n,n} \mu)].$$

As $\mu \bullet_{n,n} \mu = 0$, the previous equation reduces to

$$(\varphi \bullet_{k,n} \mu) \bullet_{k+n-1,n} \mu = (-1)^{(n-1)^2} (\varphi \bullet_{k,n} \mu) \bullet_{k+n-1,n} \mu.$$

If n is even, it implies that $(\varphi \bullet_{k,n} \mu) \bullet_{k+n-1,n} \mu = 0$ but if n is odd, the previous identity is trivial. Computing directly $(\varphi \bullet_{k,n} \mu) \bullet_{k+n-1,n} \mu$, we obtain the identity 2 of Lemma 9.

Consequence. Assume that n is even. Let φ be in $\mathcal{C}^k(V)$ and consider for any $i \in \{0, \cdots, n-2\}$ the linear maps

$$\delta_i^k : \mathcal{C}^{i+k(n-1)}(V) \to \mathcal{C}^{i+(k+1)(n-1)}(V)$$

given by

$$\delta_i^k(\varphi) = (-1)^{k-1} \mu \bullet \varphi - \varphi \bullet \mu.$$

It is proven in [47] that

$$\delta_i^{k+1} \circ \delta_i^k = 0,$$

for any $i = 0, \cdots, n-2$. Then we have the following sequence

$$\begin{aligned} \mathcal{C}^i(V) \xrightarrow{\delta_i^0} \mathcal{C}^{n-1+i}(V) \xrightarrow{\delta_i^1} \mathcal{C}^{2(n-1)+i}(V) \to \\ \cdots \to \mathcal{C}^{k(n-1)+i}(V) \xrightarrow{\delta_i^k} \mathcal{C}^{k+1)(n-1)+i}(V) \to \cdots \end{aligned} \tag{7.3}$$

Assume now that n is odd. Lemma 9 shows that $\delta_i^{k+1} \circ \delta_i^k \neq 0$ and (9) is not a complex. To define a cohomology in this case, we will restrict the spaces of cochains. Let $\chi^k(V)$ be the subspace of $\mathcal{C}^k(V)$ given by

$$\chi^k(V) = \left\{ \varphi \in \mathcal{C}^k(V), (\varphi \bullet \mu) \bullet \mu = (\mu \bullet \varphi) \bullet \mu = \mu \bullet (\varphi \bullet \mu) = 0 \right\}.$$

Pre-Lie identity applied to the triple (μ, φ, μ) implies

$$(\mu \bullet \varphi) \bullet \mu = \mu \bullet (\varphi \bullet \mu) - \mu \bullet (\mu \bullet \varphi),$$

and if $\varphi \in \chi^k(V)$ we have also $\mu \bullet (\mu \bullet \varphi) = 0$.

Proposition 54 *Let $\partial^k : \chi^k(V) \to \mathcal{C}^{k+n-1}(V)$ be the linear map defined by*

$$\partial^k \varphi = (-1)^{k-1} \mu \bullet \varphi - \varphi \bullet \mu.$$

Then $Im\partial^k \subset \chi^{k+n-1}(V)$ and

$$\partial^{k+n-1} \circ \partial^k = 0.$$

Proof. Let φ be in $\chi^k(V)$. Let us prove that $\partial^k \varphi \in \chi^{k+n-1}(V)$. We have

$$(\partial^k \varphi \bullet \mu) \bullet \mu = (-1)^{k-1}((\mu \bullet \varphi) \bullet \mu) \bullet \mu - ((\varphi \bullet \mu) \bullet \mu) \bullet \mu = 0,$$

and

$$(\mu \bullet \partial^k \varphi) \bullet \mu = (-1)^{k-1}(\mu \bullet (\mu \bullet \varphi)) \bullet \mu - (\mu \bullet (\varphi \bullet \mu)) \bullet \mu = 0,$$

finally

$$\mu \bullet (\partial^k \varphi \bullet \mu) = (-1)^{k-1} \mu \bullet ((\mu \bullet \varphi) \bullet \mu) - \mu \bullet ((\varphi \bullet \mu) \bullet \mu) = 0.$$

Thus $\partial^k \varphi \in \chi^{k+n-1}(V)$. But

$$\begin{aligned}(\partial^{k+n-1} \circ \partial^k)\varphi &= \partial^{k+n-1}((-1)^{k-1}\mu \bullet \varphi - \varphi \bullet \mu) \\ &= \mu \bullet (\mu \bullet \varphi) + (-1)^k \mu \bullet (\varphi \bullet \mu) + (-1)^k (\mu \bullet \varphi) \bullet \mu + (\varphi \bullet \mu) \bullet \mu = 0,\end{aligned}$$

so

$$\partial^{k+n-1} \circ \partial^k = 0$$

which proves the result.

Consequences. Considering $\delta_i^j = \partial^{i+j(n-1)}$ we obtain the following complexes

$$(\chi^{k(n-1)+i}(V), \delta_i^k)_{k \geq 0}$$

which are attached to μ. We denote by $H_i^*(\mu, \mu)$ the corresponding cohomology.

7.4 n-ary-product of degree 1 and graded n-ary product

In this section, we introduce the notion of graded n-ary product. This notion will be fundamental, in the following chapter, to give a good definition of the notion of duality for operads associated with n-ary products, $n \geq 3$.

7.4.1 n-ary associative product of degree d.

To define the notion of product of degree d we recall the notion of comp-i operations of Gerstenhaber. If $\mu : V^{\otimes n} \to V$ is a linear map, then $\mu \circ_i \mu : V^{\otimes 2n-1} \to V$ is the linear map defined by

$$\mu \circ_i \mu(e_1 \otimes \cdots \otimes e_{2n-1}) = \mu(e_1 \otimes \cdots \otimes e_{i-1} \otimes \mu(e_i \otimes \cdots \otimes e_{n+i-1}) \otimes e_{n+i} \otimes \cdots \otimes e_{2n-1}).$$

Such a product is defined for $i = 1, \cdots, n$. Then

$$(\mu \bullet_{n,n} \mu) = \sum_{i=1}^{m} (-1)^{(i-1)(m-1)} \mu \circ_i \mu.$$

Similar calculations may be carried out leading to:

Definition 52 *An n-ary product is of degree d if we have for $1 \leq j \leq n$,*

$$(\mu \circ_j \mu) \circ_i \mu = \begin{cases} (-1)^{d^2} (\mu \circ_i \mu) \circ_{j+n-1} \mu \text{ if } 1 \leq i \leq j-1, \\ \mu_j \circ (\mu \circ_{i-j+1} \mu) \text{ if } j \leq i \leq n+j-1, \\ (-1)^{d^2} (\mu \circ_{i-n+1} \mu) \circ_j \mu \text{ if } i \geq j+n. \end{cases}$$

Definition 53 *A n-product μ of degree d is called totally associative if*

$$\mu \circ_1 \mu = \mu \circ_i \mu,$$

for $i = 1, \cdots, n$. It is partially associative if

$$\sum_{i=1}^{n} (-1)^{(i-1)(n-1)} \mu \circ_i \mu = 0.$$

Examples.

1. If $d = 0$, we find the classical totally and partially associative algebras again and μ is totally or partially associative.

2. If $n = 2$ and $d = 1$. Then if μ is associative, it satisfies

$$\begin{cases} \mu \circ_1 \mu = \mu \circ_2 \mu, \\ (\mu \circ_1 \mu) \circ_1 \mu = \mu \circ_1 (\mu \circ_1 \mu), \\ (\mu \circ_1 \mu) \circ_2 \mu = \mu \circ_1 (\mu \circ_2 \mu), \\ (\mu \circ_1 \mu) \circ_3 \mu = -(\mu \circ_2 \mu) \circ_1 \mu, \\ (\mu \circ_2 \mu) \circ_2 \mu = \mu \circ_2 (\mu \circ_1 \mu), \\ (\mu \circ_2 \mu) \circ_3 \mu = \mu \circ_2 (\mu \circ_2 \mu). \end{cases}$$

If we want to explain the minus sign which appears in the fourth equation, we consider $(\mu \circ_1 \mu) \circ_3 \mu$ and $(\mu \circ_2 \mu) \circ_1 \mu$ applied to $(v_1 \otimes \cdots \otimes v_4)$ with the convention degree$(v_i) = 0$ and the Koszul rules and we get

$$\begin{array}{ll} (\mu \circ_1 \mu) \circ_3 \mu(v_1 \otimes \cdots \otimes v_4) & = (\mu \circ_1 \mu)(v_1 \otimes v_2 \otimes \mu(v_3 \otimes v_4)) \\ & = \mu(\mu(v_1 \otimes v_2) \otimes \mu(v_3 \otimes v_4)), \\ (\mu \circ_2 \mu) \circ_1 \mu(v_1 \otimes \cdots \otimes v_4) & = (\mu \circ_2 \mu)(\mu(v_1 \otimes v_2) \otimes v_3 \otimes v_4) \\ & = -\mu(\mu(v_1 \otimes v_2) \otimes \mu(v_3 \otimes v_4)) \end{array}$$

because $\mu(v_1 \otimes v_2)$ is of degree 1.

3. If $n = 3, d = 1$ and μ is totally associative, we have

$$\mu \circ_1 \mu = \mu \circ_2 \mu = \mu \circ_3 \mu,$$

and the relations with some minus sign given by the degree are

$$\begin{cases} (\mu \circ_1 \mu) \circ_4 \mu = -(\mu \circ_2 \mu) \circ_1 \mu, \\ (\mu \circ_1 \mu) \circ_5 \mu = -(\mu \circ_3 \mu) \circ_1 \mu, \\ (\mu \circ_2 \mu) \circ_5 \mu = -(\mu \circ_3 \mu) \circ_2 \mu. \end{cases}$$

We denote by $(n)tot^1\mathcal{A}ss$ the operad for n-ary algebras with totally associative product of degree 1. We proved in [107] that $((n)pa\mathcal{A}ss)^! = (n)tot^1\mathcal{A}ss$ where $((n)pa\mathcal{A}ss)^!$ denotes the dual operad for the n-ary partially associative algebras. As a consequence we deduce that the free algebra $L(V)$ is not Koszul.

7.4.2 Graded n-ary algebras and n-ary super-algebras

Let μ be an n-ary product on V. We assume that V is a $\mathbb{Z}$-graded vector space, $V = \oplus_{n \in \mathbb{Z}} V_n$.

Definition 54 *The n-ary algebra $(\oplus_{n \in \mathbb{Z}} V_n, \mu)$ is a graded totally (resp. partially) associative n-ary algebra if μ is a totally (resp. partially) associative product satisfying*

$$\mu(V_i, V_j, V_k) \subset V_{i+j+k},$$

for any $i, j, k \in \mathbb{Z}$.

In particular V_0 is an n-ary totally (resp.) partially associative subalgebra.

Remark that as in the associative case the notion of super-algebra coincides with the notion of $\mathbb{Z}$-graded algebra. In fact we can define, for example for $n = 3$, the notion of super-algebra by:

Definition 55 *A 3-ary totally associative super-algebra is a pair (V, μ) where*

1) $V = \oplus_{i\in\mathbb{Z}_3} V_i = V_{-1} \oplus V_0 \oplus V_1$ i.e. V is $\mathbb{Z}_3$-graded,

2) $\mu(V_i, V_j, V_k) \in V_{i+j+k(mod3)},\ i, j, k \in \{-1, 0, 1\}$,

3) μ satisfies the super-identity

$$\mu \circ_1 \mu = \mu \circ_2 \mu = \mu \circ_3 \mu.$$

We see that the super-identity is just the classical identity but for τ_{13}-partially associative algebras, these notions are not similar.

Definition 56 *A 3-ary τ_{13}-totally associative super-algebra is a pair (V, μ) where*

1) V is $\mathbb{Z}_3$-graded,

2) $\mu(V_i, V_j, V_k) \in V_{i+j+k(mod3)},\ i, j, k \in \{-1, 0, 1\}$,

3) μ satisfies the super-identity

$$\mu(\mu(x_1, x_2, x_3), x_4, x_5) = (-1)^{|x_2|^2|x_3|^2|x_4|^2}\mu(x_1, \mu(x_4, x_3, x_2), x_5) = \mu(x_1, x_2, \mu(x_3, x_4, x_5))$$

where x_i are homogeneous vectors of V, that is, belonging to some V_i and $\mid x \mid$ denotes the degree of x, that is, $-1, 0$ or 1.

7.4.3 An Hochschild cohomology for partially associative algebras with operation in degree one

We suppose, to simplify, that $n = 3$ and let (V, μ) be a partially associative 3-ary algebra with operation in degree 1. In next chapter we will define a "natural" cohomology dicted by the structure of the corresponding operad. In this section we explore what type of complex is associated to an "Hochschild"-coboundary operator extended to an n-ary algebra.

Proposition 55 *For any $\varphi \in \mathcal{C}^k(V)$ we have $(\varphi \bullet \mu) \bullet \mu = 0$.*

Lemma 10 *(Graded pre-Lie identity)*

Let φ_1 be in $C^n(V)$, φ_2 in $C^m(A)$ and φ_3 in $C^p(A)$ of respective degree $|\varphi_1|, |\varphi_2|, \varphi_3|$. They satisfy

$$(\varphi_1 \bullet \varphi_2) \bullet \varphi_3 - \varphi_1 \bullet (\varphi_2 \bullet \varphi_3) = (-1)^{(m-1)(p-1)}(-1)^{|\varphi_2||\varphi_3|}((\varphi_1 \bullet \varphi_3) \bullet \varphi_2 - \varphi_1 \bullet (\varphi_3 \bullet \varphi_2))$$

We deduce, if $\varphi \in \mathcal{C}^k(V)$,

$$(\varphi \bullet \mu) \bullet \mu = (-1)((\varphi \bullet \mu) \bullet \mu)$$

and $(\varphi \bullet \mu) \bullet \mu = 0$.

Consequence. Let

$$\delta : C^n(V) \longrightarrow C^{n+2}(A)$$

be the 1 degree operation defined by

$$\delta\varphi = \mu \bullet \varphi - (-1)^{|\varphi|}\varphi \bullet \mu$$

where $|\varphi|$ is the degree of φ. The graded pre-Lie identity gives

$$(\mu \bullet \mu) \bullet \varphi - \mu \bullet (\mu \bullet \varphi) = (-1)^{|\varphi|}((\mu \bullet \varphi) \bullet \mu - \mu \bullet (\varphi \bullet \mu)).$$

This implies

$$\delta(\delta\varphi) = 0$$

Proposition 56 *The operator $\delta : C^n(V) \longrightarrow C^{n+2}(V)$ defined by*

$$\delta\varphi = \mu \bullet \varphi - (-1)^{|\varphi|}\varphi \bullet \mu$$

satisfies

$$\delta(\delta\varphi) = 0.$$

Then we obtain a complex

$$(C^{2k+1}(V), \delta)_{k\geq 1}$$

whose coboundary operator satisfies a Hochschild condition. Nevertheless, remark that the cochains spaces depend of μ and, if n is odd, we can not define "universal" spaces of cochains. Thus for a cohomological theory we have to consider another approach (see Chapter 8).

7.5 Extension of the notion of coassociative algebras for n-ary algebras

For $n = 2$, we have that 2-ary partially associative algebras are just associative algebras and we can define coassociative coalgebras with the well known relations between these two structures. In fact, the dual space of a coassociative algebra can be provided with a structure of associative algebra, the dual space of a finite dimensional associative algebra can be provided with a structure of coassociative coalgebra structure and also, if (A, μ) is an associative algebra and (M, Δ) a coassociative coalgebra, the space $Hom(M, A)$ can be provided with an associative algebra structure. All these notions can be extended to n-ary algebras.
An n-ary partially associative algebra has a product μ satisfying:

$$\sum_{p=0}^{n-1}(-1)^{p(n-1)}\mu \circ (Id_p \otimes \mu \otimes Id_{n-1-p}) = 0.$$

Then we get the definition of partially coassociative n-ary coalgebra.

Definition 57 *An n-ary comultiplication on a $\mathbb{K}$-vector space M is a map*

$$\Delta : M \to M^{\otimes n}.$$

An n-ary partially coassociative coalgebra is a $\mathbb{K}$-vector space M provided with an n-ary comultiplication Δ satisfying

$$\sum_{p=0}^{n-1}(-1)^{p(n-1)}(Id_p \otimes \Delta \otimes Id_{n-1-p}) \circ \Delta = 0.$$

An n-ary totally coassociative coalgebra is a $\mathbb{K}$-vector space M provided with an n-ary comultiplication Δ satisfying

$$(Id_p \otimes \Delta \otimes Id_{n-1-p}) \circ \Delta = (\Delta \otimes Id_{n-1}) \circ \Delta,$$

for any $p \in \{0, \cdots, n-1\}$.

If $(\mathcal{A}, \mu)$ is an n-ary algebra and (M, Δ) an n-ary coalgebra we denote by

$$A(\mu) = \sum_{p=0}^{n-1}(-1)^{p(n-1)}\mu \circ (Id_p \otimes \mu \otimes Id_{n-1-p}),$$

$$\tilde{A}(\Delta) = \sum_{p=0}^{n-1}(-1)^{p(n-1)}(Id_p \otimes \Delta \otimes Id_{n-1-p}) \circ \Delta.$$

Then an n-ary algebra $(\mathcal{A}, \mu)$ is partially associative if and only if $A(\mu) = 0$ and an n-ary coalgebra (M, Δ) is partially associative if and only if $\tilde{A}(\Delta) = 0$.

For any natural number k and any $\mathbb{K}$-vector spaces E and F, we denote by

$$\lambda_k : Hom(E,F)^{\otimes k} \longrightarrow Hom(E^{\otimes k}, F^{\otimes k})$$

the natural embedding

$$\lambda_k(f_1 \otimes \cdots \otimes f_k)(x_1 \otimes \cdots \otimes x_k) = f_1(x_1) \otimes \cdots \otimes f_k(x_k).$$

Proposition 57 *The dual space of an n-ary partially coassociative coalgebra is provided with a structure of n-ary partially associative algebra.*

Proof. Let (M, Δ) be an n-ary partially coassociative coalgebra. We consider the multiplication on the dual vector space M^* of M defined by :

$$\mu = \Delta^* \circ \lambda_n.$$

It provides M^* with an n-ary partially associative algebra structure. In fact we have

$$\mu(f_1 \otimes f_2 \otimes \cdots \otimes f_n) = \mu_{\mathbb{K}} \circ \lambda_n(f_1 \otimes f_2 \otimes \cdots \otimes f_n) \circ \Delta \tag{7.4}$$

for all $f_1, \cdots, f_n \in M^*$ and $\mu_{\mathbb{K}}$ is the multiplication in $\mathbb{K}$. Equation (7.4) becomes:

$$\begin{aligned} &\mu \circ (Id_p \otimes \mu \otimes Id_{n-1-p})(f_1 \otimes f_2 \otimes \cdots \otimes f_{2n-1}) \\ &\qquad = \mu_{\mathbb{K}} \circ (\lambda_n(f_1 \otimes \cdots \otimes f_p \otimes \mu(f_{p+1} \otimes \cdots \otimes f_{p+n}) \otimes f_{p+n+1} \otimes \cdots \otimes f_{2n-1})) \circ \Delta \\ &\qquad = \mu_{\mathbb{K}} \circ \lambda_n(f_1 \otimes \cdots \otimes f_p \otimes (\mu_{\mathbb{K}} \circ \lambda_n(f_{p+1} \otimes \cdots \otimes f_{p+n}) \circ \Delta) \otimes f_{p+n+1} \otimes \cdots \otimes f_{2n-1}) \circ \Delta \\ &\qquad = \mu_{\mathbb{K}} \circ (Id_p \otimes \mu_{\mathbb{K}} \otimes Id_{n-1-p}) \circ \lambda_{2n-1}(f_1 \otimes \cdots \otimes f_{2n-1}) \circ (Id_p \otimes \Delta \otimes Id_{n-1-p}) \circ \Delta. \end{aligned}$$

Using associativity and commutativity of the multiplication in $\mathbb{K}$, we obtain

$$\forall p \in \{0, \cdots, n-1\}, \quad \mu_{\mathbb{K}} \circ (Id_p \otimes \mu_{\mathbb{K}} \otimes Id_{n-1-p}) = \mu_{\mathbb{K}} \circ (\mu_{\mathbb{K}} \otimes Id_{n-1}),$$

so

$$\begin{aligned} &\sum_{p=0}^{n-1} (-1)^{p(n-1)} \mu \circ (Id_p \otimes \mu \otimes Id_{n-1-p}) \\ &\qquad = \mu_{\mathbb{K}} \circ (\mu_{\mathbb{K}} \otimes Id_{n-1}) \circ \lambda_{2n-1}(f_1 \otimes \cdots \otimes f_{2n-1}) \circ \sum_{p=0}^{n-1} (-1)^{p(n-1)} (Id_p \otimes \Delta \otimes Id_{n-1-p}) \circ \Delta = 0 \end{aligned}$$

and (M^*, μ) is an n-ary partially associative algebra. □

Proposition 58 *The dual vector space of a finite dimensional n-ary partially associative algebra has an n-ary partially associative coalgebra structure.*

Proof. Let $\mathcal{A}$ be a finite dimensional n-ary partially associative algebra and let $\{e_i\}_{i=1,\cdots,n}$ be a basis of $\mathcal{A}$. If $\{f_i\}$ is the dual basis then $\{f_{i_1} \otimes \cdots \otimes f_{i_n}\}$ is a basis of $(\mathcal{A}^*)^{\otimes n}$. We define a coproduct Δ on $\mathcal{A}^*$ by

$$\Delta(f) = \sum_{i_1,\cdots,i_n} f(\mu(e_{i_1} \otimes \cdots \otimes e_{i_n})) f_{i_1} \otimes \cdots \otimes f_{i_n}.$$

In particular

$$\Delta(f_k) = \sum_{i_1,\cdots,i_n} C^k_{i_1\cdots,i_n} f_{i_1} \otimes \cdots \otimes f_{i_n}$$

where $C^k_{i_1\cdots,i_n}$ are the structure constants of μ related to the basis $\{e_i\}$. Then Δ is a comultiplication of an n-ary partially coassociative coalgebra. □

Now we study the convolution product. Let us recall that if $(\mathcal{A},\mu)$ is associative $\mathbb{K}$-algebra and (M,Δ) a coassociative $\mathbb{K}$-coalgebra then the convolution product

$$f \star g = \mu \circ \lambda_2(f \otimes g) \circ \Delta$$

provides $Hom(M,\mathcal{A})$ with an associative algebra structure. This result can be extended to the n-ary partially associative algebras and partially coassociative coalgebras.

Proposition 59 *Let $(\mathcal{A},\mu)$ be an n-ary partially associative algebra and (M,Δ) an n-ary totally coalgebra. Then the algebra $(Hom(M,\mathcal{A}),\star)$ is an n-ary partially associative algebra where $\star$ is the convolution product:*

$$f_1 \star f_2 \star \cdots \star f_n = \mu \circ \lambda_n(f_1 \otimes f_2 \otimes \cdots \otimes f_n) \circ \Delta.$$

Proof. Let us compute the convolution product of functions of $Hom(M,\mathcal{A})$. We have for any $i = 1,\cdots,n$

$$\begin{array}{l} f_1 \star \cdots \star f_{i-1} \star (f_i \star f_{i+1} \star \cdots \star f_{i+n-1}) \star f_{i+n} \star \cdots \star f_{2n-1} \\ = \mu \circ \lambda_n(f_1 \otimes f_2 \otimes \cdots \otimes f_{i-1} \otimes (f_i \star \cdots \star f_{i+n-1}) \otimes f_{i+n} \otimes \cdots \otimes f_n) \circ \Delta \\ \\ = \mu \circ \lambda_n(f_1 \otimes \cdots f_{i-1} \otimes (\mu \circ \lambda_n(f_i \otimes \cdots \otimes f_{i+n-1}) \circ \Delta) \otimes f_{i+n} \otimes f_{2n-1}) \circ \Delta \\ \\ = \mu \circ (Id_{i-1} \otimes \mu \otimes Id_{n-i}) \circ \lambda_{2n-1}(f_1 \otimes f_2 \otimes \cdots \otimes f_{2n-1}) \circ (Id_{i-1} \otimes \Delta \otimes Id_{n-i}) \circ \Delta. \end{array}$$

Since Δ is an n-ary totally associative product, we have

$$\begin{array}{l} A(\star)(f_1 \otimes \cdots \otimes f_{2n-1}) \\ = \sum_{p=0}^{n-1} (-1)^{p(n-1)} \mu \circ (Id_p \otimes \mu \otimes Id_{n-1-p}) \circ \lambda_{2n-1}(f_1 \otimes \cdots \otimes f_{2n-1}) \circ (Id_p \otimes \Delta \otimes Id_{n-1-p}) \circ \Delta \\ \\ = \left(\sum_{p=0}^{n-1} (-1)^{p(n-1)} \mu \circ (Id_p \otimes \mu \otimes Id_{n-1-p}) \right) \circ \lambda_{2n-1}(f_1 \otimes \cdots \otimes f_{2n-1}) \circ (\Delta \otimes Id_{n-1}) \circ \Delta = 0, \end{array}$$

which proves the result.

7.6 Some examples of n-ary algebras

1. Let $\mathfrak{g}$ be a Lie algebra. The associator related to the Lie bracket is

$$A(X,Y,Z) = [[X,Y],Z] - [X,[Y,Z]] = [[X,Z],Y].$$

If $\mathfrak{g}$ is a 4-step nilpotent Lie algebra the multiplication $\mu(X,Y,Z) = A(X,Y,Z)$ is 3-ary of type $\bullet_{3,3}$.

2. Let μ an n-ary multiplication of type $\bullet_{n,n}$ on a vector space V. This multiplication is commutative if, for any $v_i \in V$,

$$\sum_{\sigma \in S_n} \varepsilon(\sigma)\mu(v_{\sigma(1)},\ldots,v_{\sigma(n)}) = 0,$$

where S_n is the symmetric group on n-elements and $\varepsilon(\sigma)$ is the signature of the element σ of S_n. The 3-ary algebras of the previous examples are commutative. A non-commutative version is based on the Roby algebras. A Roby algebra is constructed in the following way: let V be a vector space and $T(V)$ its associated tensor algebra. For any integer k, we consider the ideal $I(V,k)$ of $T(V)$ generated by the products of symmetric tensors of length k. The exterior algebra of order k, or Roby algebra of order k, is by definition

$$\Lambda(V,k) = T(V)/I(V,k).$$

For $k=2$ we get the usual exterior algebra. For $k=3$, the ideal $I(V,3)$ is generated by tensors of type

$$\begin{cases} v_1\otimes v_2\otimes v_3+v_2\otimes v_1\otimes v_3+v_3\otimes v_2\otimes v_1+v_1\otimes v_3\otimes v_2+v_2\otimes v_3\otimes v_1+v_3\otimes v_1\otimes v_2, \\ v_1^{\otimes^2}\otimes v_2+v_2\otimes v_1^{\otimes^2}, \\ v_1\otimes v_2^{\otimes^2}+v_2^{\otimes^2}\otimes v_1, \end{cases}$$

with distinct vectors v_1,v_2,v_3. If μ is the multiplication in $\Lambda(V,3)$, it satisfies

$$\begin{cases} \mu(v_1,v_2,v_3)+\mu(v_2,v_1,v_3)+\mu(v_3,v_2,v_1)+\mu(v_1,v_3,v_2)+\mu(v_2,v_3,v_1)+\mu(v_3,v_1,v_2)=0 \\ \mu(v_1,v_1,v_2)+\mu(v_2,v_1,v_1)=0. \end{cases}$$

with distinct vectors v_1,v_2,v_3. We deduce $\mu(v_1,v_1,v_1)=0$. If we claim now that μ is a multiplication of type $\bullet_{3,3}$, such an algebra is its exterior version.

3. A Poisson algebra of type $\bullet_{3,3}$ can be defined as a commutative algebra (V,μ) of type $\bullet_{3,3}$ with a Lie bracket satisfying

$$[\mu(X,Y,Z),T]=\mu([X,T],Y,Z)+\mu(X,[Y,T],Z)+\mu(X,Y,[Z,T]),$$

for any $X,Y,Z,T\in V$. If V is a $\mathbb{Z}_2$-graded vector space, we consider on $V=V_0\oplus V_1$ a graded Lie bracket which provides V with a super Lie algebra structure. Thus this bracket satisfies

$$\begin{cases} [X_1,X_2] &= -[X_2,X_1], \\ [X_1,Y_2] &= -[Y_2,X_1], \\ [Y_1,Y_2] &= [Y_2,Y_1], \end{cases}$$

for any $X_1,X_2\in V_0$ and $Y_1,Y_2\in V_1$. It also satisfies the graded Jacobi identity. A super-algebra Poisson structure of type $\bullet_{3,3}$ on $V=V_0\oplus V_1$ is given by a multiplication μ of type $\bullet_{3,3}$ and a graded Lie bracket satisfying

$$[\mu(X,Y,Z),T]=\mu([X,T],Y,Z)+\mu(X,[Y,T],Z)+\mu(X,Y,[Z,T])$$

An example is given by the F-algebras defined in [105] which are some generalization of the super-algebra associated to the super-symmetry. In fact such algebra (for $F=3$) is defined on a graded Lie algebra $(V=V_0\oplus V_1,[,])$ provided with a commutative multiplication of type $\bullet_{3,3}$, denoted $\{\ ,\ ,\}$ in this case, and satisfying

$$\{V_i,V_j,V_k\}=0$$

as soon as $(i,j,k)\neq(1,1,1)$,

$$\{V_1,V_1,V_1\}\subseteq V_0,$$

and the graded Leibniz relations

$$[X,\{Y_1,Y_2,Y_3\}]=\{[X,Y_1],Y_2,Y_3\}+\{Y_1,[X,Y_2],Y_3\}+\{Y_1,Y_2,[X,Y_3]\},$$

for any $X\in V_0$ et $Y_1,Y_2,Y_3\in V_1$,

$$[Y,\{Y_1,Y_2,Y_3\}]+[Y_1,\{Y_2,Y_3,Y\}]+[Y_2,\{Y_3,Y,Y_2\}]+[Y_3,\{Y,Y_1,Y_2\}]=0,$$

for any Y, $Y_1,Y_2,Y_3\in V_1$.

Now we will define a ternary and more generally a $(2k+1)$-ary product on the vector space $T_q^p(E)$ of tensors of type (p,q) which are contravariant of order p, covariant of order q and have total order $(p+q)$. This product is totally associative up to a permutation s_k of order k (we call this property a s_k-totally associativity). When $p=2$ and $q=1$, we obtain a $(2k+1)$-ary product on the space of bilinear maps on E with values on E, which is identified with the cubic matrices. If we call a l-matrix a square tableau with $l\times\cdots\times l$ entrances (if $l=3$ we have the cubic matrices and we speak about hypercubic matrices when $l>3$), then the $(2k+1)$-ary product on $T_q^p(E)$ gives a $(2k+1)$-product on the space of $(p+q)$-matrices. We describe also all these products which are s_k-totally associative. We compute the corresponding quadratic operads and their dual in Chapter 8.

7.7 Definition of n-ary σ-partially and σ-totally associative algebras

Definition 58 *For a permutation σ in Σ_n define a linear map*

$$\Phi_\sigma^V : V^{\otimes^n} \to V^{\otimes^n}$$

by

$$\Phi_\sigma^V(e_{i_1} \otimes \cdots \otimes e_{i_n}) = e_{i_{\sigma^{-1}(1)}} \otimes \cdots \otimes e_{i_{\sigma^{-1}(n)}}.$$

An n-ary algebra (V, μ) is **σ-partially associative** *if*

$$\sum_{p=0}^{n-1} (-1)^{p(n-1)} (-1)^{\varepsilon(\sigma^p)} \mu \circ (I_p \otimes (\mu \circ \Phi_{\sigma^p}^V) \otimes I_{n-p-1}) = 0, \tag{7.5}$$

and **σ-totally associative** *if*

$$\mu \circ (\mu \otimes I_{n-1}) = \mu \circ (I_p \otimes (\mu \circ \Phi_{\sigma^p}^V) \otimes I_{n-p-1}), \tag{7.6}$$

for all $p = 0, \cdots, n-1$.

Example. If $n = 3$ and $\sigma = \tau_{12}$ is the transposition exchanging 1 and 2 then a τ_{12}-totally associative algebra satisfies

$$\mu(\mu(e_1, e_2, e_3), e_4, e_5) = \mu(e_1, \mu(e_3, e_2, e_4), e_5) = \mu(e_1, e_2(\mu(e_3, e_4, e_5)),$$

and a τ_{12}-partially associative algebra satisfies

$$\mu(\mu(e_1, e_2, e_3), e_4, e_5) - \mu(e_1, \mu(e_3, e_2, e_4), e_5) + \mu(e_1, e_2(\mu(e_3, e_4, e_5)) = 0.$$

7.8 A $(2k+1)$-ary product on the vector space of tensors $T_1^2(E)$

7.8.1 The tensor space $T_1^2(E)$

Let E be a finite dimensional vector space over a field $\mathbb{K}$ of characteristic 0. We denote by $T_1^2(E) = E \otimes E \otimes E^*$ the space of tensors covariant of order 1 and contravariant of order 2. The space $T_1^2(E)$ is identified with the space of linear maps

$$\mathcal{L}(E \otimes E, E) = \{\varphi : E \otimes E \to E \text{ linear}\}.$$

Let $\{e_1, \cdots, e_n\}$ be a fixed basis of E. The structure constants $\left\{C_{ij}^k\right\}_{i,j,k \in \{1,\cdots,n\}}$ of $\varphi \in T_1^2(E)$ are defined by

$$\varphi(e_i \otimes e_j) = \sum_{k=1}^{n} C_{ij}^k e_k.$$

Definition 59 *The dual map of $\varphi \in T_1^2(E)$ is the tensor $\widetilde{\varphi} \in T_2^1(E)$ defined by*

$$\begin{array}{rccc} \widetilde{\varphi}: & E & \to & E \otimes E \\ & e_k & \mapsto & \displaystyle\sum_{1 \le i,j \le n} C_{ij}^k e_i \otimes e_j. \end{array}$$

In this definition, we identify $T_2^1(E)$ with $\mathcal{L}(E, E \otimes E)$. If φ is considered as a multiplication on E, then $\widetilde{\varphi}$ is a coproduct. For example, if φ is an associative product then $\widetilde{\varphi}$ is the corresponding coassociative coproduct (often denoted by Δ).

7.8.2 Definition of a 3-ary product on $T_1^2(E)$

Let $\varphi_1, \varphi_2, \varphi_3$ be in $T_1^2(E)$. We define a 3-ary product μ by

$$\mu(\varphi_1, \varphi_2, \varphi_3) = \varphi_1 \circ \widetilde{\varphi_2} \circ \varphi_3. \tag{7.7}$$

Since $\widetilde{\varphi_2}$ is a linear map from E to $E \otimes E$, then $\varphi_1 \circ \widetilde{\varphi_2} \circ \varphi_3 \in T_1^2(E)$ and μ is well defined. Let us compute its structure constants. We denote by $\{C_{ij}^k(l)\}_{i,j,k\in\{1,\cdots,n\}}$ the structure constants of φ_l $(l = 1, 2, 3)$.

$$\begin{aligned}\mu(\varphi_1, \varphi_2, \varphi_3)(e_i \otimes e_j) = \ & \varphi_1 \circ \widetilde{\varphi_2} \circ \varphi_3(e_i \otimes e_j)\\ &= \sum_{k=1}^{n} C_{ij}^k(3)\varphi_1 \circ \widetilde{\varphi_2}(e_k)\\ &= \sum_{k=1}^{n} \sum_{1\le l,m\le n} C_{ij}^k(3)C_{lm}^k(2)\varphi_1(e_l \otimes e_m)\\ &= \sum_{t=1}^{n}\sum_{k=1}^{n} \sum_{1\le l,m\le n} C_{ij}^k(3)C_{lm}^k(2)C_{lm}^t(1)e_t.\end{aligned}$$

Thus if $\mu(\varphi_1, \varphi_2, \varphi_3)(e_i \otimes e_j) = \sum_{t=1}^{n} A_{ij}^t(1,2,3)e_t$ we get

$$A_{ij}^t(1,2,3) = \sum_{1\le k,l,m\le n} C_{ij}^k(3)C_{lm}^k(2)C_{lm}^t(1).$$

Proposition 60 *The 3-ary product in $T_1^2(E)$ given by*

$$\mu(\varphi_1, \varphi_2, \varphi_3) = \varphi_1 \circ \widetilde{\varphi_2} \circ \varphi_3$$

satisfies

$$\begin{aligned}\mu(\mu(\varphi_1, \varphi_2, \varphi_3), \varphi_4, \varphi_5) &= \mu(\varphi_1, \mu(\varphi_4, \varphi_3, \varphi_2), \varphi_5)\\ &= \mu(\varphi_1, \varphi_2, \mu(\varphi_3, \varphi_4, \varphi_5)),\end{aligned}$$

that is, this product is τ_{13}-totally associative.

Proof. We have

$$\begin{aligned}\mu(\mu(\varphi_1, \varphi_2, \varphi_3), \varphi_4, \varphi_5)(e_i \otimes e_j) &= (\varphi_1 \circ \widetilde{\varphi_2} \circ \varphi_3) \circ \widetilde{\varphi_4} \circ \varphi_5(e_i \otimes e_j)\\ &= \sum_t \left[\sum_{k,l,m} C_{ij}^k(5)C_{lm}^k(4)A_{lm}^t(1,2,3)\right] e_t\\ &= \sum_t \left[\sum_{k,l,m}\sum_{u,r,s} \left(C_{ij}^k(5)C_{lm}^k(4)C_{lm}^u(3)C_{rs}^u(2)C_{rs}^t(1)\right)\right] e_t.\end{aligned}$$

Thus, for any $i, j, t \in \{1, \cdots, n\}$, the structure constant $A_{ij}^t((1,2,3),4,5)$ of this tensor is

$$A_{ij}^t((1,2,3),4,5) = \sum_{1\le \substack{k,l,m\\ u,r,s} \le n} C_{ij}^k(5)C_{lm}^k(4)C_{lm}^u(3)C_{rs}^u(2)C_{rs}^t(1).$$

Similarly

$$\begin{aligned}\mu(\varphi_1, \varphi_2, \mu(\varphi_3, \varphi_4, \varphi_5))(e_i \otimes e_j) &= \varphi_1 \circ \widetilde{\varphi_2} \circ (\varphi_3 \circ \widetilde{\varphi_4} \circ \varphi_5)(e_i \otimes e_j)\\ &= \sum_t \left[\sum_{u,r,s} A_{ij}^u(3,4,5)C_{rs}^u(2)C_{rs}^t(1)\right] e_t\\ &= \sum_t \left[\sum_{u,r,s}\left(\sum_{k,l,m} C_{ij}^k(5)C_{lm}^k(4)C_{lm}^u(3)\right) C_{rs}^u(2)C_{rs}^t(1)\right] e_t.\end{aligned}$$

Thus

$$A_{ij}^t(1,2,(3,4,5)) = \sum_{\substack{k,l,m\\u,r,s}} C_{ij}^k(5)C_{lm}^k(4)C_{lm}^u(3)C_{rs}^u(2)C_{rs}^t(1),$$

and

$$A_{ij}^t(1,2,(3,4,5)) = A_{ij}^t((1,2,3),4,5).$$

We also have

$$\begin{aligned}\mu(\varphi_1,\mu(\varphi_2,\varphi_3,\varphi_4),\varphi_5)(e_i\otimes e_j) &= \varphi_1\circ\widetilde{(\varphi_2\circ\widetilde{\varphi_3}\circ\varphi_4)}\circ\varphi_5(e_i\otimes e_j)\\ &= \sum_t\left[\sum_{k,l,m} C_{ij}^k(5)A_{lm}^k(2,3,4)C_{lm}^t(1)\right]e_t\\ &= \sum_t\left[\sum_{k,l,m}\sum_{u,r,s} C_{ij}^k(5)C_{lm}^u(4)C_{rs}^u(3)C_{rs}^k(2)C_{lm}^t(1)\right]e_t,\end{aligned}$$

and

$$A_{ij}^t(1,(2,3,4),5) = \sum_{\substack{k,l,m\\u,r,s}} C_{ij}^k(5)C_{lm}^u(4)C_{rs}^u(3)C_{rs}^k(2)C_{lm}^t(1).$$

This shows that

$$A_{ij}^t((1,2,3),4,5) = A_{ij}^t(1,(4,3,2),5).$$

Remarks.

1. We can define in this way other non equivalent products by:

$$\begin{cases}\mu_2(\varphi_1,\varphi_2,\varphi_3) = \varphi_3\circ\widetilde{\varphi_2}\circ\varphi_1,\\ \mu_3(\varphi_1,\varphi_2,\varphi_3) = \varphi_1\circ\widetilde{\varphi_2}\circ{}^t\varphi_3,\\ \mu_4(\varphi_1,\varphi_2,\varphi_3) = \varphi_3\circ\widetilde{\varphi_2}\circ{}^t\varphi_1,\end{cases}$$

where ${}^t\varphi(e_i\otimes e_j) = \varphi(e_j\otimes e_i)$.

2. If we identify a tensor φ with its structure constants $\{C_{ij}^k\}_{i,j,k\in\{1,\cdots,n\}}$ and if we consider the family $\{C_{ij}^k\}_{i,j,k\in\{1,\cdots,n\}}$ as a cubic matrix $\{C_{ijk}\}_{i,j,k\in\{1,\cdots,n\}}$ with 3-indexed entries, the product μ on $T_1^2(E)$ gives a 3-ary product on the cubic matrices. This last product has been studied in [1].

7.8.3 A $(2k+1)$-ary product on $T_1^2(E)$

Let $\varphi_1,\cdots,\varphi_{2k+1}$ be in $T_1^2(E)$. We define a $(2k+1)$-ary product μ_{2k+1} on $T_1^2(E)$ putting

$$\mu_{2k+1}(\varphi_1,\cdots,\varphi_{2k+1}) = \varphi_1\circ\widetilde{\varphi_2}\circ\cdots\circ\varphi_{2k-1}\circ\widetilde{\varphi_{2k}}\circ\varphi_{2k+1}.$$

Let s_k be the permutation of Σ_{2k+1} defined by

$$s_k(1,2,\cdots,2k+1) = (2k+1,2k,\cdots,2,1),$$

that is, $s_k = \tau_{1\,2k+1}\circ\tau_{2\,2k}\circ\cdots\circ\tau_{k-1\,k+3}\circ\tau_{k\,k+2} = \Pi_{i=1}^k\tau_{i\,2k+2-i}$. It satisfies $(s_k)^{2p} = Id$ and $(s_k)^{2p+1} = s_k$ for any p (it is a symmetry).

Recall that the $(2k+1)$-ary product μ_{2k+1} is a s_k-totally associative product if

$$\mu_{2k+1}\circ(\mu_{2k+1}\otimes I_{2k}) = \mu_{2k+1}\circ(I_p\otimes(\mu_{2k+1}\circ\Phi_{s_k^p})\otimes I_{2k-p}),$$

for $p=1,\cdots,2k$. In particular, we have

$$\mu_{2k+1}\circ(\mu_{2k+1}\otimes I_{2k}) = \mu_{2k+1}\circ(I_{2q}\otimes\mu_{2k+1}\otimes I_{2k-2q}),$$

for any $q=1,\cdots,k$.

Proposition 61 *The product μ_{2k+1} is s_k-totally associative.*

Proof. In fact if we put

$$\mu_{2k+1}(\varphi_1,\cdots,\varphi_{2k+1})(e_i\otimes e_j)=\sum_t A^t_{ij}(1,2,\cdots,2k+1)e_t,$$

then $A^t_{ij}(1,2,\cdots,2k+1)=$

$$\sum_{\substack{a_1,\cdots,a_{2k}\\ k_1,\cdots,k_k}} C^{k_1}_{ij}(2k+1)C^{k_1}_{a_1a_2}(2k)C^{k_2}_{a_1a_2}(2k-1)\cdots C^{k_k}_{a_{2k-1}a_{2k}}(2)C^t_{a_{2k-1}a_{2k}}(1).$$

More precisely the line of superscripts is

$$(k_1,k_1,k_2,k_2,\cdots,k_k,k_k,t),$$

and the line of subscripts is

$$((i,j),(a_1,a_2),(a_1,a_2),(a_3,a_4),(a_3,a_4),\cdots,(a_{2k-1},a_{2k}),(a_{2k-1},a_{2k})).$$

Let us consider

$$\mu_{2k+1}\circ(I_l\otimes(\mu_{2k+1}\circ\Phi_{s^l_k})\otimes I_{2k-l})(\varphi_1,\cdots,\varphi_{4k+1})(e_i\otimes e_j)=\sum B^t_{ij}e_t.$$

Thus for $l=2r$, we get

$$\begin{aligned} B^t_{ij}= \ & \sum C^{k_1}_{ij}(4k+1)C^{k_1}_{a_1a_2}(4k)C^{k_2}_{a_1a_2}(4k-1)\cdots C^{k_{k-r}}_{a_{2k-2r-1}a_{2k-2r}}(2k+l+2)\\ & A^{k_{k-r+1}}_{a_{2k-2r-1}a_{2k-2r}}(l+1,\cdots,2k+l+1)C^{k_{k-r+1}}_{a_{2k-2r+1}a_{2k-2r+2}}(l)\cdots C^t_{a_{4k-1}a_{4k}}(1),\end{aligned}$$

such that the line of superscripts is

$$(k_1,k_1,k_2,k_2,\cdots,k_{k-r},h_1,h_1,\cdots,h_k,h_k,k_{k-r+1},k_{k-r+1},\cdots,k_k,k_k,t),$$

where the terms $h_1,\cdots,h_k,k_{k-r+1}$ correspond to the factor

$$A^{k_{k-r+1}}_{a_{2k-2r-1}a_{2k-2r}}(l+1,\cdots,2k+l+1).$$

Such a line is the same as the line of superscripts of

$$\mu_{2k+1}\circ(\mu_{2k+1}\otimes I_{2k})(\varphi_1,\cdots,\varphi_{4k+1})(e_i\otimes e_j).$$

The line of subscripts is

$$\begin{aligned}&((i,j),(a_1,a_2),(a_1,a_2),\cdots,(a_{2k-2r-1},a_{2k-2r}),(a_{2k-2r-1},a_{2k-2r}),(\beta_1\beta_2),\cdots,\\&(\beta_{2k-1},\beta_{2k}),(a_{2k-2r-1},a_{2k-2r}),\cdots,(a_{4k-1},a_{4k})).\end{aligned}$$

So

$$\mu_{2k+1}\circ(\mu_{2k+1}\otimes I_{2k})=\mu_{2k+1}\circ(I_l\otimes(\mu_{2k+1}\circ\Phi_{s^l_k})\otimes I_{2k-l}),$$

for $l=2r$. Assume now that $l=2r+1$. In this case B^t_{ij} is of the form

$$\cdots C^{k_{k-r+1}}_{a_{2k-2r-1}a_{2k-2r}}(2k+l+2)A^{k_{k-r+1}}_{a_{2k-2r+1}a_{2k-2r+2}}(2k+l+1,\cdots,l+1)C^{k_{k-r+1}}_{a_{2k-2r+1}a_{2k-2r+2}}(l)\cdots$$

We find the same list of exponents and of indices that for $\mu_{2k+1}\circ(\mu_{2k+1}\otimes I_{2k})$. This finishes the proof.

Consequence 1. The product μ_{2k+1} on $T^2_1(E)$ induces directly a $(2k+1)$-ary products on cubic matrices.

Consequence 2. All the other products which are s_k-totally associative correspond to

$$\left\{\begin{array}{lll}\mu^2_{2k+1}(\varphi_1,\cdots,\varphi_{2k+1}) & = & \varphi_{2k+1}\circ\widetilde{\varphi_{2k}}\circ\cdots\widetilde{\varphi_2}\circ\varphi_1,\\ \mu^3_{2k+1}(\varphi_1,\cdots,\varphi_{2k+1}) & = & \mu_{2k+1}({}^t\varphi_1,\varphi_2,\cdots,\varphi_{2k+1}),\\ \mu^4_{2k+1}(\varphi_1,\cdots,\varphi_{2k+1}) & = & \mu^2_{2k+1}(\varphi_1,\cdots,\varphi_{2k},{}^t\varphi_{2k+1}).\end{array}\right.$$

and more generally

$$\mu_{2k+1}({}^t\varphi_1,\varphi_2,{}^t\varphi_3,\cdots,\varphi_{2k+1})$$

or

$$\mu_{2k+1}({}^t\varphi_1,\varphi_2,{}^t\varphi_3,\cdots,\varphi_{2k+1}).$$

7.9 Generalization: a $(2k+1)$-ary product on $T_q^p(E)$

7.9.1 The vector space $T_q^p(E)$

Let E be an m-dimensional $\mathbb{K}$-vector space. The vector space $T_q^p(E)$ is the space of tensors which are contravariant of order p and covariant of order q. If $\{e_1, \cdots, e_m\}$ is a fixed basis of E, a tensor t of $T_q^p(E)$ is written

$$t = \sum_{\substack{1 \leq i_k, j_l \leq n \\ 1 \leq k \leq p \\ 1 \leq l \leq q}} t_{i_1, \cdots, i_p}^{j_1, \cdots, j_q} e_{i_1} \otimes \cdots \otimes e_{i_p} \otimes e^{j_1} \otimes \cdots \otimes e^{j_q}$$

where $(e^1, \cdots, e^m)$ is the dual basis of $(e_1, \cdots, e_m)$. Since

$$T_q^p(E) = T_0^p(E) \otimes T_q^0(E),$$

the tensor space

$$T(E) = \sum_{p,q=0}^{\infty} T_q^p(E)$$

is an associative algebra with product

$$\begin{array}{ccc} T_q^p(E) \times T_m^l(E) & \rightarrow & T_{q+m}^{p+l}(E) \\ (K, L) & \mapsto & K \otimes L \end{array}.$$

But this product is not internal on each component $T_q^p(E)$. In this section we will define internal $(2p-1)$-ary-product on the components.

The vector space $T_q^p(E)$ is isomorphic to the space $\mathcal{L}(E^{\otimes p}, E^{\otimes q})$ of linear maps

$$t : E^{\otimes^p} \rightarrow E^{\otimes^q}.$$

We define the structure constants by

$$t(e_{i_1} \otimes \cdots \otimes e_{i_p}) = \sum C_{i_1 \cdots i_p}^{j_1 \cdots j_q} e_{j_1} \otimes \cdots \otimes e_{j_q}.$$

For such a map we define $\widetilde{t}$ by

$$\begin{array}{cccc} \widetilde{t} : & E^{\otimes^q} & \rightarrow & E^{\otimes^p} \\ & (e_{j_1} \otimes \cdots \otimes e_{j_q}) & \mapsto & \sum C_{i_1 \cdots i_p}^{j_1 \cdots j_q} e_{i_1} \otimes \cdots \otimes e_{i_p}. \end{array}$$

7.9.2 A $(2k+1)$-ary product on $T_q^p(E)$

Definition 60 *The map μ defined by:*

$$\mu(\varphi_1, \cdots, \varphi_{2k+1}) = \varphi_{2k+1} \circ \widetilde{\varphi_{2k}} \circ \varphi_{2k-1} \circ \cdots \circ \widetilde{\varphi_2} \circ \varphi_1, \tag{7.8}$$

for any $\varphi_1, \cdots, \varphi_{2k+1} \in T_r^s(E)$ is a $(2k+1)$-ary product on $T_r^s(E)$.

We take an odd number of maps φ_i so we get compositions of

$$\widetilde{\varphi_{j+1}} \circ \varphi_j : E^{\otimes^p} \rightarrow E^{\otimes^p}$$

for $j = 1, \cdots, 2k-1$ and finally compose it with $\varphi_{2k-1} : E^{\otimes^p} \rightarrow E^{\otimes^q}$ so μ is well defined.

Proposition 62 *The $(2k+1)$-ary product μ on $T_q^p(E)$ defined by (7.8) is s_k-totally associative.*

Proof. The proof is similar to the proof of Proposition 76 concerning an $(2k+1)$-ary product on $T_1^2(E)$. In fact we have

$$\mu(\varphi_1, \cdots, \varphi_{2p+1})(e_{i_1} \otimes \cdots \otimes e_{i_p}) = \sum A_{i_1 \cdots i_p}^{r_1 \cdots r_q} e_{r_1} \otimes \cdots \otimes e_{r_q},$$

and

$$A_{i_1 \cdots i_p}^{r_1 \cdots r_q} = C_{i_1 \cdots i_p}^{j_1 \cdots j_q}(2k+1) C_{l_1 \cdots l_p}^{j_1 \cdots j_q}(2k) C_{l_1 \cdots l_p}^{m_1 \cdots m_q}(2k-1) \cdots C_{s_1 \cdots s_p}^{r_1 \cdots r_q}(1),$$

that is, the line of superscripts is

$$(j_1 \cdots j_q)(j_1 \cdots j_q)(m_1 \cdots m_q)(m_1 \cdots m_q) \cdots (n_1 \cdots n_q)(n_1 \cdots n_q)(r_1 \cdots r_q),$$

and the line of subscripts is

$$(i_1 \cdots i_p)(l_1 \cdots l_p)(l_1 \cdots l_p) \cdots (s_1 \cdots s_p)(s_1 \cdots s_p).$$

Using the same arguments as in Proposition 76, replacing pairs by p-uples and q-uples, we obtain the announced result.

Remark. We can use the same trick as in Consequence 2 to find other s_k-totally associative products on $T_q^p(E)$.

Applications. This product can be translated as a product of "hypercubic matrices", that is, square tableaux of length $p+q$. This generalizes in a natural way the classical associative product of matrices.

7.10 Current $(2k+1)$-ary s_k-totally associative algebras

The problem is to find a category of $(2k+1)$-ary algebras such that its tensor product with a $(2k+1)$-ary s_k-totally associative algebra gives a $(2k+1)$-ary s_k-totally associative algebra with an obvious operation on the tensor product. Such a tensor product will be called *current* $(2k+1)$*-ary* s_k*-totally associative algebra*. We first focus on the ternary case and $s_1 = \tau_{13}$.

Let (V, μ) be a 3-ary algebra where μ is a τ_{13}-totally associative product on V (for example $V = T_1^2(E)$ and μ is defined by (7.7)) so μ satisfies Equation (7.6) for $\sigma = \tau_{13}$, that is,

$$\mu(\mu(e_1, e_2, e_3), e_4, e_5) = \mu(e_1, \mu(e_4, e_3, e_2), e_5) = \mu(e_1, e_2(\mu(e_3, e_4, e_5)),$$

for any e_1, e_2, e_3 in V. Let $(W, \tilde{\mu})$ be a 3-ary algebra. Then the tensor algebra $(V \otimes W, \mu \otimes \tilde{\mu})$ is a 3-ary τ_{13}-totally associative algebra if and only if the map $\mu \otimes \tilde{\mu}$ define by

$$(\mu \otimes \tilde{\mu})(v_1 \otimes w_1 \otimes v_2 \otimes w_2 \otimes v_3 \otimes w_3) = \mu(v_1, v_2, v_3) \otimes \tilde{\mu}(w_1, w_2, w_3)$$

satisfies the τ_{13}-totally associativity relation. But

$$\begin{cases} (\mu \otimes \tilde{\mu}) \circ (\mu \otimes \tilde{\mu} \otimes I_4) = \mu \circ (\mu \otimes I_2) \otimes \tilde{\mu} \circ (\tilde{\mu} \otimes I_2), \\ (\mu \otimes \tilde{\mu}) \circ (I_2 \otimes (\mu \otimes \tilde{\mu}) \circ \Phi_{\tau_{13}}^{V \otimes W} \otimes I_2) = \mu \circ (I \otimes \mu \circ \Phi_{\tau_{13}}^{V} \otimes I) \otimes \tilde{\mu} \circ (I \otimes \tilde{\mu} \circ \Phi_{\tau_{13}}^{W} \otimes I), \\ (\mu \otimes \tilde{\mu}) \circ (I_4 \otimes \mu \otimes \tilde{\mu}) = \mu \circ (I_2 \otimes \mu) \otimes \tilde{\mu} \circ (I_2 \otimes \tilde{\mu}), \end{cases}$$

so $(\mu \otimes \tilde{\mu}) \circ (\mu \otimes \tilde{\mu} \otimes I_4) - (\mu \otimes \tilde{\mu}) \circ (I_4 \otimes \mu \otimes \tilde{\mu}) = 0$ is equivalent to

$$\mu \circ (\mu \otimes I_2) \otimes \tilde{\mu} \circ (\tilde{\mu} \otimes I_2) - \mu \circ (I_2 \otimes \mu) \otimes \tilde{\mu} \circ (I_2 \otimes \tilde{\mu}) = 0. \tag{7.9}$$

But $\mu \circ (\mu \otimes I_2) = \mu \circ (I_2 \otimes \mu)$. Thus Equation (7.9) is equivalent to

$$\mu \circ (\mu \otimes I_2) \otimes [\tilde{\mu} \circ (\tilde{\mu} \otimes I_2) - \tilde{\mu} \circ (I_2 \otimes \tilde{\mu})] = 0,$$

and

$$\tilde{\mu} \circ (\tilde{\mu} \otimes I_2) = \tilde{\mu} \circ (I_2 \otimes \tilde{\mu}).$$

Similarly

$$\begin{aligned}&(\mu\otimes\tilde{\mu})\circ(\mu\otimes\tilde{\mu}\otimes I_4)-(\mu\otimes\tilde{\mu})\circ(I_2\otimes(\mu\otimes\tilde{\mu})\circ\Phi^{V\otimes W}_{\tau_{13}}\otimes I_2)\\&\quad=\mu\circ(\mu\otimes I_2)\otimes\left[\tilde{\mu}\circ(\tilde{\mu}\otimes I_2)-\tilde{\mu}\circ(I\otimes\tilde{\mu}\circ\Phi^{W}_{\tau_{13}}\otimes I)\right]=0,\end{aligned}$$

which leads to

$$\tilde{\mu}\circ(\tilde{\mu}\otimes I_2)=\tilde{\mu}\circ(I\otimes\tilde{\mu}\circ\Phi^{W}_{\tau_{13}}\otimes I).$$

So $\mu\otimes\tilde{\mu}$ is τ_{13}-totally associative if and only if $\tilde{\mu}$ is τ_{13}-totally associative.

Proposition 63 *Let (V,μ) be a 3-ary τ_{13}-totally associative algebra and $(W,\tilde{\mu})$ be a 3-ary algebra. Then $(V\otimes W,\mu\otimes\tilde{\mu})$ is a 3-ary τ_{13}-totally associative algebra if and only if $(W,\tilde{\mu})$ is also of this type.*

This result can be extended for $(2k+1)$-ary s_k-totally associative algebras.

Proposition 64 *Let (V,μ) be a $(2k+1)$-ary s_k-totally associative algebra and $(W,\widetilde{\mu})$ be a $(2k+1)$-ary algebra. Then $(V\otimes W,\mu\otimes\tilde{\mu})$ is a $(2k+1)$-ary s_k-totally associative algebra if and only if $(W,\tilde{\mu})$ is also of this type.*

Proof. The product μ is a $(2k+1)$-ary s_k-totally associative product, so it satisfies

$$\begin{aligned}\mu\circ(\mu\otimes I_{2k})&=\mu\circ(I_{2q}\otimes\mu\otimes I_{2k-2q})\\&=\mu\circ(I_{2q+1}\otimes\mu\circ\Phi^{V}_{s_k^q}\otimes I_{2k-2q-1}),\end{aligned}$$

for any $q=0,\cdots,k$. The system

$$\begin{aligned}&(\mu\otimes\widetilde{\mu})\circ((\mu\otimes\widetilde{\mu})\otimes I_{4k})-(\mu\otimes\widetilde{\mu})\circ(I_{4q}\otimes(\mu\otimes\widetilde{\mu})\circ\Phi^{V\otimes W}_{s_k^q}\otimes I_{4k-2q})\\&\quad=\mu\circ(\mu\otimes I_{2k})\otimes\widetilde{\mu}\circ(\widetilde{\mu}\otimes I_{2k})\\&\qquad-\mu\circ(I_q\otimes\mu\circ\Phi^{V}_{s_k^q}\otimes I_{2k-q})\otimes\tilde{\mu}\circ(I_q\otimes\widetilde{\mu}\circ\Phi^{W}_{s_k^q}\otimes I_{2k-q})\\&\quad=0,\end{aligned}$$

for any $q=0,\cdots,k$ is equivalent to

$$\mu\circ(\mu\otimes I_{2k})\otimes\left[\widetilde{\mu}\circ(\widetilde{\mu}\otimes I_{2k})-\tilde{\mu}\circ(I_q\otimes\widetilde{\mu}\circ\Phi^{W}_{s_k^q}\otimes I_{2k-q})\right]=0,$$

for any $q=0,\cdots,k$. Then $\mu\otimes\tilde{\mu}$ is $(2k+1)$-ary s_k-totally associative if and only if

$$\widetilde{\mu}\circ(\widetilde{\mu}\otimes I_{2k})-\tilde{\mu}\circ(I_q\otimes\widetilde{\mu}\circ\Phi^{W}_{s_k^q}\otimes I_{2k-q})=0$$

for any $q=0,\cdots,k$, that is, $\tilde{\mu}$ is a $(2k+1)$-ary s_k-totally associative product.

Chapter 8

(Non-)Koszulity of operads for n-ary algebras

The aim of this chapter is to write correctly and explicitly the definition of the quadratic dual of a quadratic operad and to apply this definition to the case of the corresponding multiplication algebra being an n-ary product. In fact, in many works concerning such a product, the definition of the duality is a direct extension of the definition of the duality for binary products. But in the case of binary product, the definition of duality can be simplified, forgetting for example all the degree of the binary maps because these degree are even and they have no influence in the computations. Unfortunately, the definition of the duality for n-ary multiplication has been done by a simple extension of the simplified definition. This implies some mistakes in the computation of the dual operads and, in particular, on homological properties of the associated free algebras. Then, at first, we replace the definition of the duality in the graded complex framework. We define four families of n-ary algebras namely $t\mathcal{A}ss^n_d$ which are totally associative n-ary algebras with an operation of degree d, $p\mathcal{A}ss^n_d$, partially associative n-ary algebras with operation in degree d and $\widetilde{t\mathcal{A}ss^n_d}, \widetilde{p\mathcal{A}ss^n_d}$ which are respectively the suspension of $t\mathcal{A}ss^n_{-d+n-2}$ and the desuspension of $p\mathcal{A}ss^n_{-d+n-2}$. We give the link between these four families and focus on $\widetilde{\mathcal{A}ss} = \widetilde{t\mathcal{A}ss^2_0} = \widetilde{p\mathcal{A}ss^2_0}$ which can be understood as an associative algebra with a wrong sign, that is, a binary algebra, where the identity $x(yz) = -(xy)z$ replaces the associativity. We call such an algebra anti-associative. We determine explicitly the dual operads for these operads so for the particular cases of n-ary partially and totally associative algebras when the product is of degree odd or even. The main result is the proof of the koszulity and non-Koszulity of these operads which is summarized in Figure 8.1. We prove the non-koszulity only for $n \leq 7$ although we expect it to be true for arbitrary n. In particular, the dual operad for 3-ary partially associative algebras of degree 0 is the operad of 3-ary totally associative of degree 1 and we conclude that these operads are non-Koszul. To obtain this result we compute the generating functions of these operads and use the functional equation which should be satisfied by the generating functions of an operad and its dual if they are Koszul. We then focus on algebras with one anti-associative operation. Since the corresponding operad is non-Koszul, the deformation cohomology differs from the standard one. We describe the relevant part of the deformation cohomology for this type of algebras using the minimal model for the anti-associative operad and compare it to the standard cohomology. In the remaining sections we discuss free partially associative algebras and formulate open problems.

8.1 Duality for quadratic operads revisited

All ideas recalled in this section are already present in [46], but we want to emphasize some specific features of the non-binary case related to a degree shift in the Koszul dual which seem to be overlooked, see Remark 62 below.

Fix a natural $n \geq 2$ and assume $E = \{E(a)\}_{a \geq 2}$ is a Σ-module such that $E(a) = 0$ if $a \neq n$. We will study operads $\mathcal{P}$ of the form $\mathcal{P} = \Gamma(E)/(R)$, where $\Gamma(E)$ is the free operad generated by E and (R) the operadic ideal generated by a subspace $R \subset \Gamma(E)(2n-1)$. Operads of this type are called *quadratic*, or *binary* quadratic if $n = 2$.

Let $E^\vee = \{E^\vee(a)\}_{a\geq 2}$ be a Σ-module with

$$E^\vee(a) := \begin{cases} \mathrm{sgn}_a \otimes \uparrow^{a-2} E(a)^\#, & \text{if } a = n \text{ and} \\ 0, & \text{otherwise} \end{cases}$$

where $\uparrow^{a-2}$ denotes the suspension iterated $a-2$ times, sgn_a the signum representation of the symmetric group Σ_a, and $\#$ the linear dual of a graded vector space with the induced representation. Recall that $V^\# := \mathit{Hom}(V, \mathbb{K})$, so $(V^\#)_d = (V_{-d})^\#$. There is a non-degenerate, Σ_{2n-1}-equivariant pairing

$$\langle - | - \rangle : \Gamma(E^\vee)(2n-1) \otimes \Gamma(E)(2n-1) \to \mathbb{K} \tag{8.1}$$

determined by requiring that

$$\langle \uparrow^{n-2} e' \circ_i \uparrow^{n-2} f' \mid e'' \circ_j f'' \rangle := \delta_{ij} (-1)^{(i+1)(n+1)} e'(e'') f'(f'') \in \mathbb{K},$$

for arbitrary $e', f' \in E(n)^\#$, $e'', f'' \in E(n)$.

Definition 61 *The* Koszul *or* quadratic *dual of the quadratic operad $\mathcal{P} = \Gamma(E)/(R)$ as above is the quotient*

$$\mathcal{P}^! := \Gamma(E^\vee)/(R^\perp),$$

where $R^\perp \subset \Gamma(E^\vee)(2n-1)$ is the annihilator of $R \subset \Gamma(E)(2n-1)$ in the pairing (8.1), and $(R^\perp)$ the operadic ideal generated by $R^\perp$.

Remark 62 If $\mathcal{P}$ is a quadratic operad generated by an operation of arity n and degree d, then the generating operation of $\mathcal{P}^!$ has the same arity but degree $-d+n-2$, i.e. for $n \neq 2$ (the non-binary case) the Koszul duality *may not* preserve the degree of the generating operation! There seemed to be a common misunderstanding that the Koszul duality preserves the degree of generating operations even in the non-binary case. This resulted in several wrong statements about the Koszulity of n-ary algebras.

Recall [91, Definition II.3.15] that the *operadic suspension* $\mathbf{s}\mathcal{P}$ of an operad $\mathcal{P} = \{\mathcal{P}(a)\}_{a\geq 1}$ is the operad $\mathbf{s}\mathcal{P} = \{\mathbf{s}\mathcal{P}(a)\}_{a\geq 1}$, where $\mathbf{s}\mathcal{P}(a) := \mathrm{sgn}_a \otimes \uparrow^{a-1} \mathcal{P}(a)$, with the structure operations induced by those of $\mathcal{P}$ in the obvious way taking into account, as always, the Koszul sign rule requiring that whenever we interchange two "things" of odd degrees, we multiply the sign by -1. The *cooperadic suspension* $\mathbf{s}\mathcal{C}$ of a cooperad $\mathcal{C}$ is defined analogously.

The *cobar construction* [91, Definition II.3.9] of a cooperad $\mathcal{C}$ is a dg-operad $\Omega(\mathcal{C})$ of the form $\Omega(\mathcal{C}) = (\Gamma(\downarrow \mathbf{s}\mathcal{C}), \partial_\Omega)$. Here $\mathbf{s}$ denotes the cooperadic suspension recalled above and $\downarrow$ the component-wise desuspension. The differential ∂_Ω is induced by the structure operations of the cooperad $\mathcal{C}$. If $\mathcal{P} = \{\mathcal{P}(a)\}_{a\geq 1}$ is an operad with finite-dimensional components, the component-wise linear dual $\mathcal{P}^\# = \{\mathcal{P}(a)^\#\}_{a\geq 1}$ is a cooperad. The composition $\mathsf{D}(\mathcal{P}) := \Omega(\mathcal{P}^\#)$ of the linear dual with the cobar construction is the *dual bar construction* of $\mathcal{P}$. For $\mathcal{P}$ quadratic, there exist a natural map $\mathsf{D}(\mathcal{P}^!) \to \mathcal{P}$ of dg-operads. Let us recall the central notion of [46].

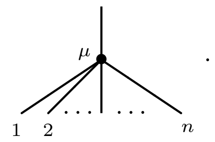
Definition 63 *A quadratic operad $\mathcal{P}$ is* Koszul *if the natural map $\mathsf{D}(\mathcal{P}^!) \to \mathcal{P}$ is a homology equivalence.*

The notation and terminology concerning the (co)bar construction has not been standardized. For instance, the use of the (co)operadic suspension $\mathbf{s}$ in our definition of $\Omega(\mathcal{C})$ is purely conventional and guarantees that $\Omega(-)$ does not change the degree of elements of arity 2. In recent literature, the Koszul dual is sometimes defined as a functor $\mathcal{P} \mapsto \mathcal{P}^{¡}$ from the category of quadratic operads to the category of quadratic *co*operads. Our $\mathcal{P}^!$ recalled above is then the composition of this co-operadic Koszul dual with the component-wise linear dual, $\mathcal{P}^! = (\mathcal{P}^{¡})^\#$. In our choice of conventions we, however, tried to stay as close to [46] as possible.

8.2 Four families of n-ary algebras

We introduce four families of quadratic operads and describe their Koszul duals. These families cover all examples of 'n-ary algebras' with one operation without symmetry which we were able to find in the literature.

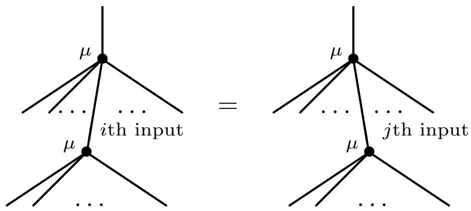
Let V be a graded vector space, $n \geq 2$, and $\mu : V^{\otimes n} \to V$ a degree d multilinear operation symbolized by



We say that $A = (V, \mu)$ is a *degree d totally associative n-ary algebra* if, for each $1 \leq i, j \leq n$,

$$\mu\left(\mathbb{1}^{\otimes i-1} \otimes \mu \otimes \mathbb{1}^{\otimes n-i}\right) = \mu\left(\mathbb{1}^{\otimes j-1} \otimes \mu \otimes \mathbb{1}^{\otimes n-j}\right),$$

where $\mathbb{1} : V \to V$ is the identity map. Graphically, we demand that



for each i, j for which the above compositions make sense. Observe that degree 0 totally associative 2-algebras are ordinary associative algebras.

In the following definitions, $\Gamma(\mu)$ will denote the free operad on the Σ-module E_μ with

$$E_\mu(a) = \begin{cases} \text{the regular representation } \mathbb{K}[\Sigma_n] \text{ generated by } \mu, & \text{if } a = n \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

Definition 64 *We denote $t\mathcal{A}ss^n_d$ the operad for totally associative n-ary algebras with operation in degree d, that is,*

$$t\mathcal{A}ss^n_d := \Gamma(\mu)/(R_{t\mathcal{A}ss^n_d})$$

with μ an arity n generator of degree d and

$$R_{t\mathcal{A}ss^n_d} := \mathit{Span}\left\{\mu \circ_i \mu - \mu \circ_j \mu, \text{ for } i, j = 1, \ldots, n\right\}.$$

We call $A = (V, \mu)$ a *degree d partially associative n-ary algebra* if the following single axiom is satisfied:

$$\sum_{i=1}^{n} (-1)^{(i+1)(n-1)} \mu\left(\mathbb{1}^{\otimes i-1} \otimes \mu \otimes \mathbb{1}^{\otimes n-i}\right) = 0. \tag{8.2}$$

Degree 0 partially associative 2-ary algebras are classical associative algebras. More interesting observation is that degree $(n-2)$ partially associative n-ary algebras are the same as A_∞-algebras $A = (V, \mu_1, \mu_2, \ldots)$ [83, §1.4] which are "meager" in that they satisfy $\mu_k = 0$ for $k \neq n$. Symmetrizations of these *meager A_∞-algebras* are *n-Lie algebras* in the sense of [63].

Definition 65 *We denote $p\mathcal{A}ss^n_d$ the operad for partially associative n-ary algebras with operation in degree d. Explicitly,*

$$p\mathcal{A}ss^n_d := \Gamma(\mu)/\Big(\sum_{i=1}^{n} (-1)^{(i+1)(n-1)} \mu \circ_i \mu\Big)$$

with μ a generator of degree d and arity n.

It follows from the above remarks that $t\mathcal{A}ss_0^2 = p\mathcal{A}ss_0^2 = \mathcal{A}ss$, where $\mathcal{A}ss$ is the operad for associative algebras. We are going to introduce the remaining two families of operads. Recall that $\mathbf{s}$ denotes the operadic suspension and $\mathbf{s}^{-1}$ the obvious inverse operation.

Definition 66 *We define* $t\widetilde{\mathcal{A}ss}_d^n := \mathbf{s}\, t\mathcal{A}ss_{d-n+1}^n$ *and* $p\widetilde{\mathcal{A}ss}_d^n := \mathbf{s}^{-1} p\mathcal{A}ss_{d+n-1}^n$.

We leave as an exercise to verify that $t\widetilde{\mathcal{A}ss}_d^n$-algebras are structures $A = (V, \mu)$, where $\mu : V^{\otimes n} \to V$ is a degree d linear map satisfying, for each $1 \leq i, j \leq n$,

$$(-1)^{i(n+1)}\mu\left(\mathbb{1}^{\otimes i-1} \otimes \mu \otimes \mathbb{1}^{\otimes n-i}\right) = (-1)^{j(n+1)}\mu\left(\mathbb{1}^{\otimes j-1} \otimes \mu \otimes \mathbb{1}^{\otimes n-j}\right).$$

Likewise, $p\widetilde{\mathcal{A}ss}_d^n$-algebras are similar structures, but this time satisfying

$$\sum_{i=1}^{n} \mu\left(\mathbb{1}^{\otimes i-1} \otimes \mu \otimes \mathbb{1}^{\otimes n-i}\right) = 0.$$

Definition 67 *Let* $\widetilde{\mathcal{A}ss} := t\widetilde{\mathcal{A}ss}_0^2 = p\widetilde{\mathcal{A}ss}_0^2$. *Explicitly,* $\widetilde{\mathcal{A}ss}$*-algebras are structures* $A = (V, \mu)$ *with a degree* 0 *bilinear operation* $\mu : V \otimes V \to V$ *satisfying*

$$\mu(\mu \otimes \mathbb{1}) + \mu(\mathbb{1} \otimes \mu) = 0$$

or, in elements

$$a(bc) + (ab)c = 0, \tag{8.3}$$

for $a, b, c \in V$. *We call these objects* anti-associative *algebras.*

Anti-associative algebras can be viewed as associative algebras with the associativity taken with the opposite sign which explains their name. Similarly, $t\mathcal{A}ss_1^2 = p\mathcal{A}ss_1^2$-algebras are associative algebras with operation of degree 1. The corresponding, essentially equivalent, operads are the simplest examples of non-Koszul operads, as we will see in Section 8.3. The proof of the following proposition is an exercise.

Proposition 65 *For each* $n \geq 2$ *and* d, $(t\mathcal{A}ss_d^n)^! = p\mathcal{A}ss_{-d+n-2}^n$, $(p\mathcal{A}ss_d^n)^! = t\mathcal{A}ss_{-d+n-2}^n$, $(t\widetilde{\mathcal{A}ss}_d^n)^! = p\widetilde{\mathcal{A}ss}_{-d+n-2}^n$ *and* $(p\widetilde{\mathcal{A}ss}_d^n)^! = t\widetilde{\mathcal{A}ss}_{-d+n-2}^n$,

8.3 Koszulity - the case study

This section is devoted to the following statement organized in the tables of Figure 8.1. We, however, know to prove the non-Koszulity direction ("no" in Figure 8.1) only for $n \leq 7$ although we expect it to be true for arbitrary n, see also the notes following Theorem 68, Remark 73 and the first problem of Section 8.6.

Theorem 68 *The operad* $t\mathcal{A}ss_d^n$ *is Koszul if and only if* d *is even. The operad* $p\mathcal{A}ss_d^n$ *is Koszul if and only if* n *and* d *have the same parity. The operad* $t\widetilde{\mathcal{A}ss}_d^n$ *is Koszul if and only if* n *and* d *have different parities. The operad* $p\widetilde{\mathcal{A}ss}_d^n$ *is Koszul if and only if* d *is odd.*

The Koszulity part ("yes" in the tables of Figure 8.1) will follow from [67] and relations between the operads $t\mathcal{A}ss_d^n$, $p\mathcal{A}ss_d^n$, $t\widetilde{\mathcal{A}ss}_d^n$ and $p\widetilde{\mathcal{A}ss}_d^n$, see Proposition 66. The non-Koszulness part ("no" in Figure 8.1) will, for $n \leq 7$, follow in a similar fashion from Proposition 67. Since we do not know how to extend the proof of Proposition 67 for arbitrary n, the non-Koszulity part of Theorem 68 is, for $n \geq 8$, only conjectural.

In particular, the operads $\widetilde{\mathcal{A}ss}$ and $t\mathcal{A}ss_1^2 = p\mathcal{A}ss_1^2$ are not Koszul. Let us formulate useful

Lemma 11 *Let* $\mathcal{P}_d^n$ *be one of the operads above. Then* $\mathcal{P}_d^n$ *is Koszul if and only if* $\mathcal{P}_{d+2}^n$ *is Koszul, that is, only the parity of* d *matters.*

$t\mathcal{A}ss^n_d$	n even	n odd
d even	yes	yes
d odd	no	no

$p\mathcal{A}ss^n_d$	n even	n odd
d even	yes	no
d odd	no	yes

$\widetilde{t\mathcal{A}ss}^n_d$	n even	n odd
d even	no	yes
d odd	yes	no

$\widetilde{p\mathcal{A}ss}^n_d$	n even	n odd
d even	no	no
d odd	yes	yes

Figure 8.1: Koszulness of the operads $t\mathcal{A}ss^n_d$, $p\mathcal{A}ss^n_d$, $\widetilde{t\mathcal{A}ss}^n_d$ and $\widetilde{p\mathcal{A}ss}^n_d$. "Yes" means that the corresponding operad is Koszul, "no" that it is not Koszul.

Proof. There is a 'twisted' isomorphism

$$\varphi : \mathcal{P}^n_d \xrightarrow{\cong} \mathcal{P}^n_{d+2}, \tag{8.4}$$

i.e. a sequence of equivariant isomorphisms $\varphi(a) : \mathcal{P}^n_d(a) \to \mathcal{P}^n_{d+2}(a)$, $a \geq 1$, that commute with the $\circ_i$-operations such that the component $\varphi(k(n-1)+1)$ is of degree $2k$, for $k \geq 0$.

To construct such an isomorphism, consider an operation μ' of arity n and degree d, and another operation μ'' of the same arity but of degree $d+i$. We leave as an exercise to verify that the assignment $\mu' \mapsto \mu''$ extends to a twisted isomorphism $\omega : \Gamma(\mu') \to \Gamma(\mu'')$ if and only if i is even.

Let $\mathcal{P}^n_d = \Gamma(\mu')/(R')$ and $\mathcal{P}^n_{d+2} = \Gamma(\mu'')/(R'')$. It is clear that the twisted isomorphism $\omega : \Gamma(\mu') \to \Gamma(\mu'')$ preserves the ideals of relations, so it induces a twisted isomorphism (8.4). A moment's reflection convinces one that φ induces similar twisted isomorphisms of the Koszul duals and the bar constructions. This, by Definition 63, gives the lemma.

Proposition 66 *The operads marked "yes" in the tables of Figure 8.1 are Koszul.*

Proof. The operads $t\mathcal{A}ss^n_0$ are Koszul for all $n \geq 2$ by [67, § 7.2] (see also [47] for the case n even and $d = 0$). So, by Lemma 11, the operads $t\mathcal{A}ss^n_d$ are Koszul for all even d and $n \geq 2$, which gives the two "yes" in the upper left table of Figure 8.1.

The "yes" in the upper right table follow from the "yes" in the upper left table, the fact that an operad is Koszul if and only if its dual operad is Koszul, and the isomorphism $(p\mathcal{A}ss^n_d)^! = t\mathcal{A}ss^n_{-d+n-2}$ established in Proposition 65. The "yes" in the lower two tables in Figure 8.1 follow from the upper two tables and the fact that the suspension preserves Koszulity.

Before we attend to the "no" part of Theorem 68, we recall an important criterion for Koszulity due to [46] which easily generalizes to the non-binary case. The *Poincaré* or *generating series* of a graded operad $\mathcal{P}_* = \{\mathcal{P}_*(a)\}_{a\geq 1}$ is defined by

$$g_{\mathcal{P}}(t) := \sum_{a\geq 1} \frac{1}{a!}\chi(\mathcal{P}(a))t^a,$$

where $\chi(\mathcal{P}(a))$ denotes the Euler characteristic of the graded vector space $\mathcal{P}_*(n)$,

$$\chi(\mathcal{P}(a)) := \sum_i (-1)^i \dim(\mathcal{P}_i(a)).$$

Observe that if $\mathcal{P}$ is not concentrated in degree zero, the Euler characteristic and thus also the Poincaré series may involve negative signs.

Theorem 69 ([46]) *If a quadratic operad $\mathcal{P}$ is Koszul, then its Poincaré series and the Poincaré series of its dual $\mathcal{P}^!$ are tied by the functional equation*

$$g_{\mathcal{P}}(-g_{\mathcal{P}^!}(-t)) = t. \tag{8.5}$$

Our first task will therefore be to describe the Poincaré series of the family $t\mathcal{A}ss^n_d$ which generates, via the duality and suspension, all the remaining operads.

Lemma 12 *The generating function for the operad $t\mathcal{A}ss^n_d$ is*

$$g_{t\mathcal{A}ss^n_d}(t) := \begin{cases} \dfrac{t}{1-t^{n-1}}, & \text{if } d \text{ is even, and} \\ t - t^n + t^{2n-1}, & \text{if } d \text{ is odd.} \end{cases} \tag{8.6}$$

Proof. The components of the operad $t\mathcal{A}ss^n_d$ are trivial in arities different from $k(n-1)+1$, $k \geq 0$. The piece $t\mathcal{A}ss^n_d(k(n-1)+1)$ is generated by all possible $\circ_i$-compositions involving k instances of the generating operation μ, modulo the relations

$$\mu \circ_i \mu - \mu \circ_j \mu, \text{ for } i,j = 1,\ldots,n \tag{8.7}$$

which enable one to replace each $\mu \circ_i \mu$, $2 \leq i \leq n$, by $\mu \circ_1 \mu$.

If the degree d is *even*, the operad $t\mathcal{A}ss^n_d$ is evenly graded, so the associativity [85, p. 1473, Eqn. (1)] of the $\circ_i$-operations does not involve signs. Therefore an arbitrary $\circ_i$-composition of k instances of μ can be brought to the form

$$\eta_k := (\cdots((\mu \circ_1 \mu) \circ_1 \mu) \circ_1 \cdots) \circ_1 \mu.$$

We see that $t\mathcal{A}ss^n_d(k(n-1)+1)$ is spanned by the set $\{\eta_k \circ \sigma;\ \sigma \in \Sigma_{k(n-1)+1}\}$, so

$$\dim(t\mathcal{A}ss^n_d(k(n-1)+1)) = (k(n-1)+1)!$$

and, by definition,

$$g_{t\mathcal{A}ss^n_d}(t) = \sum_{k\geq 0} t^{k(n-1)+1} = \frac{t}{1-t^{n-1}},$$

which verifies the even case of (8.6).

The *odd* case is subtler since the associativity [85, p. 1473, Eqn. (1)] may involve nontrivial signs. As in the even case we calculate that

$$\dim(t\mathcal{A}ss^n_d(k(n-1)+1)) = (k(n-1)+1)! \text{ for } k = 0,1,2, \tag{8.8}$$

because these small arities do not require the associativity.

If $k \geq 3$, we can still to bring each $\circ_i$-composition of k instances of μ to the form of the 'canonical' generator η_k, but we may get a nontrivial sign which may moreover depend on the way we applied the associativity. Relation (8.7) implies that

$$(\mu \circ_1 \mu) \circ_1 \mu = (\mu \circ_n \mu) \circ_1 \mu \tag{8.9}$$

in $t\mathcal{A}ss^n_d(3n-2)$. Applying (8.7) and the associativity [85, p. 1473, Eqn. (1)] several times, we get that

$$\begin{aligned}(\mu \circ_1 \mu) \circ_1 \mu &= \mu \circ_1 (\mu \circ_1 \mu) = \mu \circ_1 (\mu \circ_n \mu) = (\mu \circ_1 \mu) \circ_n \mu = (\mu \circ_n \mu) \circ_n \mu \\ &= \mu \circ_n (\mu \circ_1 \mu) = \mu \circ_n (\mu \circ_n \mu) = (\mu \circ_n \mu) \circ_{2n-1} \mu \\ &= (\mu \circ_1 \mu) \circ_{2n-1} \mu.\end{aligned} \tag{8.10}$$

Since the degree of μ is odd, the first line of the associativity [85, p. 1473, Eqn. (1)] implies

$$(\mu \circ_1 \mu) \circ_{2n-1} \mu = -(\mu \circ_n \mu) \circ_1 \mu$$

therefore (8.9) and (8.10) combine into

$$(\mu \circ_1 \mu) \circ_1 \mu = -(\mu \circ_1 \mu) \circ_1 \mu.$$

This means that $(\mu \circ_1 \mu) \circ_1 \mu = 0$ so $t\mathcal{A}ss^n_d(3n-2) = 0$. Since $t\mathcal{A}ss^n_d(k(n-1)+1)$ is, for $k \geq 3$, generated by $t\mathcal{A}ss^n_d(3n-2)$, we conclude that $t\mathcal{A}ss^n_d(k(n-1)+1) = 0$ for $k \geq 3$ which, along with (8.8), verifies the odd case of (8.7).

Remark 70 The Poincaré series of an operad $\mathcal{P}$ and its suspension $\mathbf{s}\mathcal{P}$ are related by $g_{\mathbf{s}\mathcal{P}}(t) = -g_{\mathcal{P}}(-t)$. Lemma 12 thus implies that the generating series of the operad $t\widetilde{\mathcal{A}ss}^n_d = \mathbf{s}t\mathcal{A}ss^n_{d-n+1}$ equals

$$g_{t\widetilde{\mathcal{A}ss}^n_d}(t) := \begin{cases} t + (-1)^d t^n + t^{2n-1}, & \text{if } n \text{ and } d \text{ have the same parity, and} \\ \dfrac{t}{1-(-1)^d t^{n-1}}, & \text{if } n \text{ and } d \text{ have different parities.} \end{cases}$$

We do not know explicit formulas for the Poincaré series of $p\mathcal{A}ss^n_d$ and $p\widetilde{\mathcal{A}ss}^n_d$ except in the case $n = 2$ when these operads coincide with the corresponding (anti)-associative operads.

Example 71 It easily follows from the above calculations that, for the anti-associative operad $\widetilde{\mathcal{A}ss}$, one has

$$\widetilde{\mathcal{A}ss}(1) \cong \mathbb{K},\ \widetilde{\mathcal{A}ss}(2) \cong \mathbb{K}[\Sigma_2] \text{ and } \widetilde{\mathcal{A}ss}(3) \cong \mathbb{K}[\Sigma_3],$$

while $\widetilde{\mathcal{A}ss}(a) = 0$ for $a \geq 4$.

Let us return to our task of proving the non-Koszulity of the "no" cases in the tables of Figure 8.1. Our strategy will be to interpret (8.5) as saying that $-g_{\mathcal{P}^!}(-t)$ is a formal inverse of $g_{\mathcal{P}}(t)$ at 0. Since $g'_{\mathcal{P}}(0) = 1$, this unique formal inverse exists. In the particular case of $\mathcal{P} = t\mathcal{A}ss^n_d$, with d odd, this means that $-g_{p\mathcal{A}ss^n_{-d+n-2}}(-t)$ should be compared to a formal inverse of $g_{t\mathcal{A}ss^n_d}(t) = t - t^n + t^{2n-1}$. A simple degree count shows that $g_{p\mathcal{A}ss^n_{-d+n-2}}(t)$ is of the form

$$\begin{cases} t - A_1 t^n + A_2 t^{2n-1} - A_3 t^{3n-2} + \cdots, & \text{for } n \text{ even and} \\ t + A_1 t^n + A_2 t^{2n-1} + A_3 t^{3n-2} + \cdots, & \text{for } n \text{ odd,} \end{cases}$$

for some *non-negative* integers $A_1, A_2, A_3, \ldots$, therefore $-g_{p\mathcal{A}ss^n_{-d+n-2}}(-t)$ is in both cases the formal power series

$$t + A_1 t^n + A_2 t^{2n-1} + A_3 t^{3n-2} + \cdots \tag{8.11}$$

with non-negative coefficients. If we show that the formal inverse of $t - t^n + t^{2n-1}$ is not of this form, by Theorem 69 the corresponding operad $t\mathcal{A}ss^n_d$ is not Koszul.

Example 72 The Poincaré series of the operad $t\mathcal{A}ss^2_1$ is, by Lemma 12,

$$g_{t\mathcal{A}ss^2_1}(t) = t - t^2 + t^3.$$

One can compute the formal inverse of this function as

$$t + t^2 + t^3 - 4t^5 - 14t^6 - 30t^7 - 33t^8 + 55t^9 + \cdots.$$

The presence of negative coefficients implies that the operad $t\mathcal{A}ss^2_1$ is not Koszul, neither is the anti-associative operad $\widetilde{\mathcal{A}ss} = t\widetilde{\mathcal{A}ss}^2_0 = \mathbf{s}^{-1}t\mathcal{A}ss^2_1$.

Likewise, the Poincaré series of the operad $t\mathcal{A}ss^3_1$ equals

$$g_{t\mathcal{A}ss^3_1}(t) = t - t^3 + t^5$$

and we computed, using `Matematica`, the initial part of the formal inverse as

$$t + t^3 + 2t^5 + 4t^7 + 5t^9 - 13t^{11} - 147t^{13} + \cdots$$

The existence of negative coefficients again implies that the operad $t\mathcal{A}ss^3_1$ is not Koszul. The formal inverse of

$$g_{t\mathcal{A}ss^4_1}(t) = t - t^4 + t^7$$

up to the first negative term is

$$t + t^4 + 3t^7 + 11t^{10} + 42t^{13} + 153t^{16} + 469t^{19} + 690t^{22} - 5967t^{25} + \cdots$$

so $t\mathcal{A}ss^4_1$ is not Koszul.

The complexity of the calculation of the relevant initial part of the inverse of $g_{t\mathcal{A}ss_1^n}(t) = t - t^n + t^{2n-1}$ grows rapidly with n. We have, however, the following:

Proposition 67 *For $n \leq 7$, the formal inverse of $t - t^n + t^{2n-1}$ has at least one negative coefficient. Therefore the operads $t\mathcal{A}ss_d^n$ for d odd and $n \leq 7$ are not Koszul.*

Proof. The function $g(z) := z - z^n + z^{2n-1}$ is analytic in the complex plane $\mathbb{C}$. Its analytic inverse $g^{-1}(z)$ is a not-necessarily single-valued analytic function defined outside the points in which the derivative $g'(z)$ vanishes. Let us denote by $\mathfrak{Z}$ the set of these points, i.e.

$$\mathfrak{Z} := \{z \in \mathbb{C};\ g'(z) = 0\}.$$

The key observation is that, for $n \leq 7$, the equation $g'(z) = 0$ has no real solutions, $\mathfrak{Z} \cap \mathbb{R} = \emptyset$. Indeed, one has to solve the equation

$$g'(z) = 1 - nz^{n-1} + (2n-1)z^{2n-2} = 0 \tag{8.12}$$

which, after the substitution $w := y^{n-1}$ leads to the quadratic equation

$$1 - nw + (2n-1)w^2 = 0$$

whose discriminant $n^2 - 8n + 4$ is, for $n \leq 7$, negative.

Let $f(z)$ be the power series representing the branch at 0 of $g^{-1}(z)$ such that $f(0) = 0$. It is clear that $f(t)$ is precisely the formal inverse of $g(t)$ at 0. Suppose that

$$f(z) = z + a_2z^2 + a_3z^3 + a_4z^4 + \cdots,$$

with all coefficients $a_2, a_3, a_4, \dots$ non-negative real numbers. Since $\mathfrak{Z} \neq \emptyset$ and obviously $0 \notin \mathfrak{Z}$, the radius of convergence of $f(z)$ at 0, which equals the radius of the maximal circle centered at 0 whose interior does not contain points in $\mathfrak{Z}$, is some number r with $0 < r < \infty$. Let $\mathfrak{z} \in \mathfrak{Z}$ be such that $|\mathfrak{z}| = r$. Since all coefficients of the power series f are positive, we have

$$|f(\mathfrak{z})| \leq f(|\mathfrak{z}|) = f(r),$$

so the function $f(r)$ must have singularity at the *real* point $r \in \mathbb{R}$, i.e. $g'(z)$ must vanish at r. This contradicts the fact that $g'(z) = 0$ has no real solutions.

Remark 73 Equation (8.12) has, for $n = 8$, two real solutions, $\mathfrak{z}_1 = \sqrt[7]{1/3}$ and $\mathfrak{z}_2 = \sqrt[7]{1/5}$. This means that the inverse function of $z - z^n + z^{2n-1}$ has two positive real poles and the arguments used in our proof of Proposition 67 do not apply.

We verified Proposition 67 using `Matematica`. The first negative coefficient in the inverse of $t - t^n + t^{2n-1}$ was at the power t^{57} for $n = 5$, at t^{161} for $n = 6$, and at t^{1171} for $n = 7$. For $n = 8$ we did not find any negative term of degree less than 10 000. It is indeed possible that all coefficients of the inverse of $t - t^8 + t^{15}$ are positive.

Proposition 67 together with the fact that the suspension and the !-dual preserves Koszulity imply the “no” entries of the tables in Figure 8.1 for $n \leq 7$.

8.4 Cohomology of algebras over non-Koszul operads – an example

In this section we study anti-associative algebras introduced in Definition 67, i.e. structures $A = (V, \mu)$ with a degree-0 bilinear anti-associative multiplication $\mu : V^{\otimes 2} \to V$. We describe the ‘standard’ cohomology $H^*_{\widetilde{\mathcal{A}ss}}(A; A)_{\mathrm{st}}$ of an anti-associative algebra A with coefficients in itself and compare it to the relevant part of the deformation cohomology $H^*_{\widetilde{\mathcal{A}ss}}(A; A)$ based on the minimal model of the anti-associative operad $\widetilde{\mathcal{A}ss}$.

Since $\widetilde{\mathcal{A}ss}$ is, by Theorem 68, not Koszul, these two cohomologies differ. While the standard cohomology has no sensible meaning, the deformation cohomology coincides with the triple cohomology [32, 33] and governs deformations of anti-associative algebras.

It was explained at several places [84, 85, 90, 87] how a, not-necessarily acyclic, quasi-free resolution $(\mathcal{P}, \partial = 0) \xleftarrow{\rho} (\mathscr{R}, \partial)$ of an operad $\mathcal{P}$, which we assume for simplicity non-dg and concentrated in degree 0, determines a cohomology theory for $\mathcal{P}$-algebras with coefficients in itself. If $\mathcal{P}$ is quadratic and if we take as $(\mathscr{R}, \partial)$ the dual bar construction (recalled in Section 8.1) of the quadratic dual $\mathcal{P}^!$, we get the 'standard' cohomology $H^*_{\mathcal{P}}(A; A)_{\rm st}$ as the cohomology of the 'standard' cochain complex

$$C^1_{\mathcal{P}}(A;A)_{\rm st} \xrightarrow{\delta^1_{\rm st}} C^2_{\mathcal{P}}(A;A)_{\rm st} \xrightarrow{\delta^2_{\rm st}} C^3_{\mathcal{P}}(A;A)_{\rm st} \xrightarrow{\delta^3_{\rm st}} C^4_{\mathcal{P}}(A;A)_{\rm st} \xrightarrow{\delta^4_{\rm st}} \cdots$$

in which $C^p_{\mathcal{P}}(A;A)_{\rm st} := \mathit{Hom}(\mathcal{P}^!(p) \otimes_{\Sigma_p} V^{\otimes p}, V)$, $p \geq 1$, and the differential $\delta^*_{\rm st}$ is induced from the structure of $\mathcal{P}^!$ and A, see [33, Section 8] or [91, Definition II.3.99]. This type of (co)homology was considered in the seminal paper [46].

The deformation (also called, in [84], the *cotangent*) cohomology uses the minimal model of $\mathcal{P}$ in place of $(\mathscr{R}, \partial)$. Recall [85, p. 1479] that the *minimal model* of an operad $\mathcal{P}$ is a homology isomorphism

$$(\mathcal{P}, 0) \xleftarrow{\rho} (\Gamma(E), \partial)$$

of dg-operads such that the image of ∂ consists of decomposable elements of the free operad $\Gamma(E)$ (the minimality). It is known [91, Section II.3.10] that each operad with $\mathcal{P}(1) \cong \mathbb{K}$ admits a minimal model unique up to isomorphism. The *deformation cohomology* $H^*_{\mathcal{P}}(A; A)$ is the cohomology of the complex

$$C^1_{\mathcal{P}}(A;A) \xrightarrow{\delta^1} C^2_{\mathcal{P}}(A;A) \xrightarrow{\delta^2} C^3_{\mathcal{P}}(A;A) \xrightarrow{\delta^3} C^4_{\mathcal{P}}(A;A) \xrightarrow{\delta^4} \cdots$$

in which $C^1_{\mathcal{P}}(A;A) := \mathit{Hom}(V, V)$ and

$$C^p_{\mathcal{P}}(A;A) := \mathit{Hom}\big(\textstyle\bigoplus_{q \geq 2} E_{p-2}(q) \otimes_{\Sigma_q} V^{\otimes q}, V\big), \quad \text{for } p \geq 2.$$

The differential δ^* is defined by the formula which can be found in [90, Section 2] or in the introduction to [87]. If $\mathcal{P}$ is quadratic Koszul, the dual bar construction of $\mathcal{P}^!$ is, by [85, Proposition 2.6], isomorphic to the minimal model of $\mathcal{P}$, thus the standard and deformation cohomologies coincide, giving rise to the 'standard' constructions such as the Hochschild, Harrison or Chevalley-Eilenberg cohomology.

Neither $H^*_{\mathcal{P}}(A; A)_{\rm st}$ nor $H^*_{\mathcal{P}}(A; A)$ have the 0th term. A natural H^0 exists only for algebras for which the concept of unitality makes sense. This is not always the case. Assume, for example, that an anti-associative algebra $A = (V, \mu)$ has a unit, i.e. and element $1 \in V$ such that $1a = a1 = a$, for all $a \in V$. Then the anti-associativity (8.3) with $c = 1$ gives $ab + ab = 0$, so $ab = 0$ for each $a, b \in V$.

Let us describe the standard cohomology $H^*_{\widetilde{\mathcal{A}ss}}(A; A)_{\rm st}$ of an anti-associative algebra $A = (V, \mu)$. The operad $\widetilde{\mathcal{A}ss}$ is, by Proposition 65, self-dual and it follows from the description of $\widetilde{\mathcal{A}ss} = \widetilde{\mathcal{A}ss}^!$ given in Example 71 that $H^*_{\widetilde{\mathcal{A}ss}}(A; A)_{\rm st}$ is the cohomology of

$$C^1_{\widetilde{\mathcal{A}ss}}(A;A)_{\rm st} \xrightarrow{\delta^1_{\rm st}} C^2_{\widetilde{\mathcal{A}ss}}(A;A)_{\rm st} \xrightarrow{\delta^2_{\rm st}} C^3_{\widetilde{\mathcal{A}ss}}(A;A)_{\rm st} \xrightarrow{\delta^3_{\rm st}} 0 \xrightarrow{0} 0 \xrightarrow{0} \cdots$$

in which $\mathbb{C}p := \mathit{Hom}(V^{\otimes p}, V)$ for $p = 1, 2, 3$, and all higher $\mathbb{C}p$'s are trivial. The two nontrivial pieces of the differential are basically the Hochschild differentials with "wrong" signs of some terms:

$$\delta^1(\varphi)(a,b) := a\varphi(b) - \varphi(ab) + \varphi(a)b, \text{ and}$$
$$\delta^2(f)(a,b,c) := af(b,c) + f(ab,c) + f(a,bc) + f(a,b)c,$$

for $\varphi \in \mathit{Hom}(V, V)$, $f \in \mathit{Hom}(V^{\otimes 2}, V)$ and $a, b, c \in V$. We abbreviated $\mu(a, b) = ab$, $\mu(a, \varphi(b)) = a\varphi(b)$, &c. One sees, in particular, that $H^p_{\widetilde{\mathcal{A}ss}}(A; A)_{\rm st} = 0$ for $p \geq 4$.

Let us describe the relevant part of the deformation cohomology of A. It can be shown that $\widetilde{\mathcal{A}ss}$ has the minimal model

$$(\widetilde{\mathcal{A}ss}, 0) \xleftarrow{\rho} (\Gamma(E), \partial)$$

with the generating Σ-module $E = \{E(a)\}_{a \geq 2}$ such that

– $E(2)$ is generated by a degree 0 bilinear operation $\mu_2 : V \otimes V \to V$,

– $E(3)$ is generated by a degree 1 trilinear operation $\mu_3 : V^{\otimes 3} \to V$,

– $E(4) = 0$, and

– $E(5)$ is generated by four 5-linear degree 2 operations $\mu_5^1, \mu_5^2, \mu_5^3, \mu_5^4 : V^{\otimes 5} \to V$,

so the minimal model of $\widetilde{\mathcal{A}ss}$ is of the form

$$(\widetilde{\mathcal{A}ss}, 0) \xleftarrow{\alpha} (\Gamma(\mu_2, \mu_3, \mu_5^1, \mu_5^2, \mu_5^3, \mu_5^4, \ldots), \partial).$$

Notice the gap in the arity 4 generators! We do not know the exact form of the pieces $E(a)$, $a \geq 6$, of the generating Σ-module E, but we know that they do not contain elements of degrees ≤ 2. Their Euler characteristics can be read off from the inverse of the Poincaré series $g_{\widetilde{\mathcal{A}ss}}(t) = t + t^2 + t^3$, and one gets

$$\begin{aligned} &\chi(E(2)) = 1,\ \chi(E(3)) = -1,\ \chi(E(4)) = 0,\ \chi(E(5)) = 4, \\ &\chi(E(6)) = -14,\ \chi(E(7)) = 30,\ \chi(E(8)) = -33,\ \chi(E(9)) = -55, \ldots \end{aligned}$$

The differential ∂ of the relevant generators is given by:

$$\begin{aligned}
\partial(\mu_2) &:= 0, \\
\partial(\mu_3) &:= \mu_2 \circ_1 \mu_2 + \mu_2 \circ_2 \mu_2, \\
\partial(\mu_5^1) &:= (\mu_2 \circ_2 \mu_3) \circ_4 \mu_2 - (\mu_3 \circ_3 \mu_2) \circ_4 \mu_2 + (\mu_2 \circ_1 \mu_2) \circ_3 \mu_3 - (\mu_3 \circ_1 \mu_2) \circ_3 \mu_2 \\
&\quad + (\mu_2 \circ_1 \mu_3) \circ_1 \mu_2 - (\mu_3 \circ_1 \mu_2) \circ_1 \mu_2 + (\mu_2 \circ_1 \mu_3) \circ_4 \mu_2 - (\mu_3 \circ_2 \mu_2) \circ_4 \mu_2, \\
\partial(\mu_5^2) &:= (\mu_3 \circ_1 \mu_2) \circ_1 \mu_2 - (\mu_2 \circ_1 \mu_3) \circ_1 \mu_2 + (\mu_2 \circ_1 \mu_3) \circ_3 \mu_2 - (\mu_3 \circ_2 \mu_2) \circ_3 \mu_2 \\
&\quad + (\mu_2 \circ_2 \mu_3) \circ_3 \mu_2 - (\mu_3 \circ_3 \mu_2) \circ_3 \mu_2 + (\mu_2 \circ_1 \mu_2) \circ_3 \mu_3 - (\mu_3 \circ_1 \mu_2) \circ_4 \mu_2, \\
\partial(\mu_5^3) &:= (\mu_3 \circ_2 \mu_2) \circ_4 \mu_2 - (\mu_2 \circ_2 \mu_3) \circ_2 \mu_2 + (\mu_3 \circ_2 \mu_2) \circ_2 \mu_2 - (\mu_2 \circ_1 \mu_2) \circ_2 \mu_3 \\
&\quad + (\mu_2 \circ_1 \mu_3) \circ_3 \mu_2 - (\mu_2 \circ_1 \mu_3) \circ_1 \mu_2 + (\mu_2 \circ_1 \mu_2) \circ_1 \mu_3 - (\mu_3 \circ_1 \mu_2) \circ_2 \mu_2, \text{ and} \\
\partial(\mu_5^4) &:= (\mu_3 \circ_1 \mu_2) \circ_3 \mu_2 - (\mu_3 \circ_3 \mu_2) \circ_3 \mu_2 + (\mu_2 \circ_2 \mu_3) \circ_3 \mu_2 - (\mu_3 \circ_2 \mu_2) \circ_3 \mu_2 \\
&\quad + (\mu_2 \circ_1 \mu_2) \circ_2 \mu_3 - (\mu_2 \circ_1 \mu_3) \circ_2 \mu_2 + (\mu_2 \circ_1 \mu_2) \circ_1 \mu_3 - (\mu_2 \circ_1 \mu_3) \circ_1 \mu_2.
\end{aligned}$$

One can make the formulas clearer by using the nested bracket notation. For instance, μ_2 will be represented by $(\bullet\bullet)$, μ_3 by $(\bullet\bullet\bullet)$, μ_5^2 by $(\bullet\bullet\bullet\bullet\bullet)^2$, $\mu_3 \circ_2 \mu_2$ by $(\bullet(\bullet\bullet)\bullet)$, &c. With this shorthand, the formulas for the differential read

$$\begin{aligned}
\partial(\bullet\bullet) &:= 0, \\
\partial(\bullet\bullet\bullet) &:= ((\bullet\bullet)\bullet) + (\bullet(\bullet\bullet)), \\
\partial(\bullet\bullet\bullet\bullet)^1 &:= (\bullet(\bullet\bullet(\bullet\bullet))) - (\bullet\bullet(\bullet(\bullet\bullet))) + ((\bullet\bullet)(\bullet\bullet\bullet)) - ((\bullet\bullet)(\bullet\bullet)\bullet) \\
&\quad + (((\bullet\bullet)\bullet\bullet)\bullet) - (((\bullet\bullet)\bullet)\bullet\bullet) + ((\bullet\bullet\bullet)(\bullet\bullet)) - (\bullet(\bullet\bullet)(\bullet\bullet)), \\
\partial(\bullet\bullet\bullet\bullet)^2 &:= (((\bullet\bullet)\bullet)\bullet\bullet) - (((\bullet\bullet)\bullet\bullet)\bullet) + ((\bullet\bullet(\bullet\bullet))\bullet) - (\bullet(\bullet(\bullet\bullet))\bullet) \\
&\quad + (\bullet(\bullet(\bullet\bullet)\bullet)) - (\bullet\bullet((\bullet\bullet)\bullet)) + ((\bullet\bullet)(\bullet\bullet\bullet)) - ((\bullet\bullet)\bullet(\bullet\bullet)), \\
\partial(\bullet\bullet\bullet\bullet)^3 &:= (\bullet(\bullet\bullet)(\bullet\bullet)) - (\bullet((\bullet\bullet)\bullet\bullet)) + (\bullet((\bullet\bullet)\bullet)\bullet) - ((\bullet(\bullet\bullet\bullet))\bullet) \\
&\quad + ((\bullet\bullet(\bullet\bullet))\bullet) - (((\bullet\bullet)\bullet\bullet)\bullet) + (((\bullet\bullet\bullet)\bullet)\bullet) - ((\bullet(\bullet\bullet))\bullet\bullet), \text{ and} \\
\partial(\bullet\bullet\bullet\bullet)^4 &:= ((\bullet\bullet)(\bullet\bullet)\bullet) - (\bullet\bullet((\bullet\bullet)\bullet)) + (\bullet(\bullet(\bullet\bullet)\bullet)) - (\bullet(\bullet(\bullet\bullet))\bullet) \\
&\quad + ((\bullet(\bullet\bullet\bullet))\bullet) - ((\bullet(\bullet\bullet)\bullet)\bullet) + (((\bullet\bullet\bullet)\bullet)\bullet) - (((\bullet\bullet)\bullet\bullet)\bullet).
\end{aligned}$$

Let us indicate how we obtained the above formulas. We observed first that the degree-one subspace $\Gamma(\mu_2, \mu_3)(5)_1 \subset \Gamma(\mu_2, \mu_3)(5)$ is spanned by $\circ_i$-compositions of two μ_2's and one μ_3, i.e., in the bracket language, by nested bracketings of five $\bullet$'s with two binary and one ternary bracket. These elements are in one-to-one correspondence with the edges of the 5th Stasheff associahedron K_5 shown in Figure 8.2, see [91, Section II.1.6].

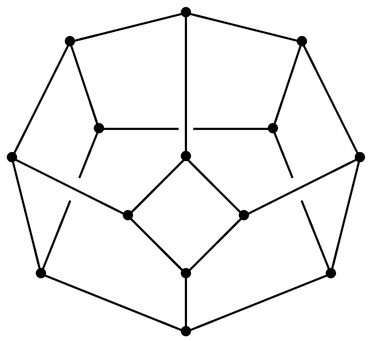


Figure 8.2: Stasheff's associahedron K_5.

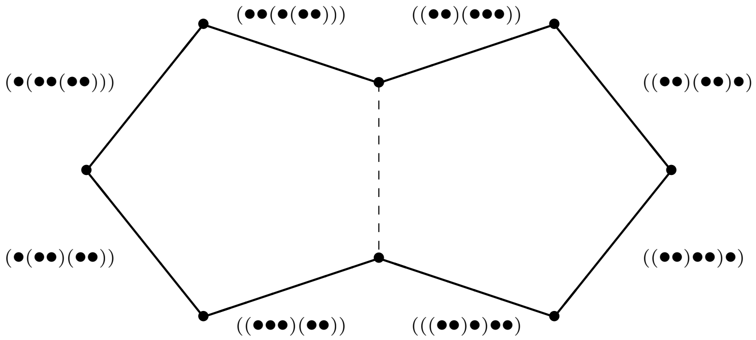


Figure 8.3: An closed edge path of length 8 in K_5 defining $\partial(\mu_5^1)$.

Let $x_e \in \Gamma(\mu_2, \mu_3)(5)_1$ be the element indexed by an edge e of K_5. Clearly $\partial(x_e) = x_a + x_b$, where a, b are the endpoints of e and $x_a, x_b \in \Gamma(\mu_2)(5)_0$ the elements given by the nested bracketings of five $\bullet$'s with three binary brackets corresponding to these endpoints. We concluded that the ∂-cycles in $\Gamma(\mu_2, \mu_3)(5)_1$ are generated by closed edge-paths of even length in K_5; the cycle corresponding to such a path $P = (e_1, e_2, \ldots, e_{2r})$ being

$$\sum_{1 \leq i \leq 2r} (-1)^{i+1} x_{e_i}.$$

Examples of these paths are provided by two adjacent pentagons in K_5 such as the ones shown in Figure 8.3. There are also three edge paths of length 4 given by the three square faces of K_5, but the corresponding cohomology classes have already been killed by the ∂-images of the compositions $\mu_3 \circ_i \mu_3$, $i = 1, 2, 3$. We showed that there are four linearly independent edge paths of length 8 that, together with the three squares, generate all edge paths of even length in K_5. The generators $\mu_5^1, \mu_5^2, \mu_5^3, \mu_5^4$ correspond to these paths.

Also for $a \geq 6$ the 1-dimensional ∂-cycles in $\Gamma(\mu_2, \mu_3, \mu_5^1, \mu_5^2, \mu_5^3, \mu_5^4)(a)_1$ are given by closed edge paths of even length in the associahedron K_a but one can show that they are all generated by the squares and the images of the paths as in Figure 8.3 under the face inclusions $K_5 \hookrightarrow K_a$. Therefore $(\Gamma(\mu_2, \mu_3, \mu_5^1, \mu_5^2, \mu_5^3, \mu_5^4), \partial)$ is acyclic in degree 1, so $\mu_5^1, \mu_5^2, \mu_5^3, \mu_5^4$ are the only degree two generators of the minimal model of $\widetilde{\mathcal{A}ss}$.

From the above description of the minimal model of $\widetilde{\mathcal{A}ss}$ one easily gets the relevant part

$$\mathbb{C}1 \xrightarrow{\delta^1} \mathbb{C}2 \xrightarrow{\delta^2} \mathbb{C}3 \xrightarrow{\delta^3} \mathbb{C}4 \xrightarrow{\delta^4} \cdots$$

of the complex defining the deformation cohomology of an anti-associative algebra $A = (V, \mu)$. One has

– $\mathbb{C}1 = \mathit{Hom}(V, V)$

– $\mathbb{C}2 = \mathit{Hom}(V^{\otimes 2}, V)$

– $\mathbb{C}3 = \mathit{Hom}(V^{\otimes 3}, V)$, and

– $\mathbb{C}4 = Hom(V^{\otimes 5}, V) \oplus Hom(V^{\otimes 5}, V) \oplus Hom(V^{\otimes 5}, V) \oplus Hom(V^{\otimes 5}, V)$.

Observe that $C^p_{\widetilde{\mathcal{A}ss}}(A; A)_{\rm st} = \mathbb{C}p$ for $p = 1, 2, 3$, while $\mathbb{C}4$ consists of 5-linear maps. The differential δ^p agrees with $\delta^p_{\rm st}$ for $p = 1, 2$ while, for $g \in \mathbb{C}3$, one has

$$\delta^3(g) = (\delta^3_1(g), \delta^3_2(g), \delta^3_3(g), \delta^3_4(g)),$$

where

$$\begin{aligned}
\delta^3_1(g)(a, b, c, d, e) &:= af(b, c, de) - f(a, b, c(de)) + (ab)f(c, d, e) - f(ab, cd, e)\\
&\quad + f(ab, c, d)e - f((ab)c, d, e) + f(a, b, c)(de) - f(a, bc, de),\\
\delta^3_2(g)(a, b, c, d, e) &:= f((ab)c, d, e) - f(ab, c, d)e + f(a, b, cd)e - f(a, b(cd), e)\\
&\quad + af(b, cd, e) - f(a, b, (cd)e) + (ab)f(c, d, e) - f(ab, c, de),\\
\delta^3_3(g)(a, b, c, d, e) &:= f(a, bc, de) - af(bc, d, e) + f(a, (bc)d, e) - a(f(b, c, d)e)\\
&\quad + f(a, b, cd)e - f(ab, c, d)e + (f(a, b, c)d)e - f(a(bc), d, e), \text{ and}\\
\delta^3_4(g)(a, b, c, d, e) &:= f(ab, cd, e) - f(a, b, (cd)e) + af(b, cd, e) - f(a, b(cd), e)\\
&\quad + (af(b, c, d))e - f(a, bc, d)e + (f(a, b, c)d)e - f(ab, c, d)e,
\end{aligned}$$

for $a, b, c, d, e \in V$. The following proposition follows from [84, Section 4].

Proposition 68 *The cohomology $H^*_{\widetilde{\mathcal{A}ss}}(A; A)$ governs deformations of anti-associative algebras. This means that $H^2_{\widetilde{\mathcal{A}ss}}(A; A)$ parametrizes isomorphism classes of infinitesimal deformations and $H^3_{\widetilde{\mathcal{A}ss}}(A; A)$ contains obstructions to extensions of partial deformations.*

8.5 Free partially associative n-algebras

In this section we describe free partially associative algebras in the situation when the corresponding operad is Koszul. We thus fill in the gap in the proof of [47, Theorem 13] and extend Gnedbaye's description of free $p\mathcal{A}ss^n_d$-algebras to all cases when d and n have the same parity.

Let $p\mathcal{A}ss^n_d(V)$ be the free $p\mathcal{A}ss^n_d$-algebra generated by a graded vector space V. It obviously decomposes as

$$p\mathcal{A}ss^n_d(V) = \bigoplus_{l \geq 0} p\mathcal{A}ss^n_d(V)_l,$$

where $p\mathcal{A}ss^n_d(V)_l \subset p\mathcal{A}ss^n_d(V)$ is the subspace generated by elements obtained by applying the structure n-ary multiplication μ to elements of V l-times. For instance, $p\mathcal{A}ss^n_d(V)_0 \cong V$ and $p\mathcal{A}ss^n_d(V)_1 \cong V^{\otimes n}$.

Denote by $\mathcal{T}^n_l$, $l \geq 1$, the set of planar directed (= rooted) trees with $l(n-1)+1$ leaves whose vertices have precisely n incoming edges (see [88, Section 4] or [91, II.1.5] for terminology). We extend the definition to $l = 0$ by putting $\mathcal{T}^n_0 := \{ \mid \}$, the one-point set consisting of the exceptional tree with one leg and no internal vertex. Clearly, each tree in $\mathcal{T}^n_l$ has exactly l vertices. For each l there is a natural epimorphism

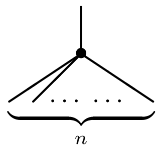
$$\omega : \mathcal{T}^n_l \times V^{\otimes l(n-1)+1} \twoheadrightarrow p\mathcal{A}ss^n_d(V)_l \tag{8.13}$$

given by interpreting the trees in $\mathcal{T}^n_l$ as the 'pasting schemes' for the iterated multiplication μ. More precisely, if $T \in \mathcal{T}^n_l$ and $v_1, \ldots, v_{l(n-1)+1} \in V$, then

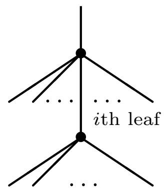
$$\omega(T \times (v_1, \ldots, v_{l(n-1)+1})) \in p\mathcal{A}ss^n_d(V)_l$$

is obtained by decorating the vertices of T by μ, the leaves of T by elements $v_1, \ldots, v_{l(n-1)+1}$, and performing the indicated composition, observing the Koszul sign rule in the nontrivially graded cases.

Let $\mathcal{S}^n_l$ be the subset of $\mathcal{T}^n_l$ of trees having the property that the first incoming edge of each vertex is a leaf. Therefore $\mathcal{S}^n_0 = \mathcal{T}^n_0 = \{ \mid \}$, $\mathcal{S}^n_1$ is the one-point set consisting of the n-corolla

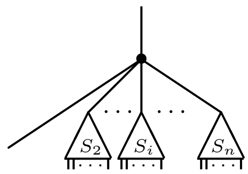


and $\mathcal{S}_2^n$ has $n-1$ elements



$, 2 \leq i \leq n.$

It is clear that, for $l \geq 3$, $\mathcal{S}_l^n$ consists of trees of the form



(8.14)

where $S_i \in \mathcal{S}_{l_i}^n$ for $2 \leq i \leq n$ and $l_2 + \cdots + l_n = l - 1$. The main result of this section is

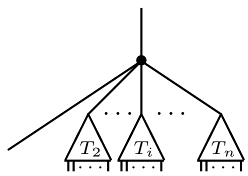
Theorem 74 *Assume that n and d are of the same parity, so the operad $p\mathcal{A}ss_d^n$ is Koszul by Theorem 68. Then the restriction (denoted by the same symbol)*

$$\omega : \mathcal{S}_l^n \times V^{\otimes l(n-1)+1} \longrightarrow p\mathcal{A}ss_d^n(V)_l \tag{8.15}$$

of the epimorphism (8.13) is an isomorphism, for each $l \geq 0$.

Proof. Axiom (8.2) for partially associative algebras implies that each iterated multiplication in $p\mathcal{A}ss_d^n(V)_l$ can be brought into a linear combination of multiplications with the pasting schemes in $\mathcal{S}_l^n$, i.e. that the map (8.15) is an epimorphism. Let us prove this statement by induction.

There is nothing to prove for $l = 0, 1$. Assume that we have established the claim for all $0 \leq l \leq k$, $k \geq 1$, and prove it for $l = k$. Let μ_T be an iterated multiplication with the pasting scheme $T \in \mathcal{T}_k^n$. There are two possibilities. The *first case:* the tree T is of the form



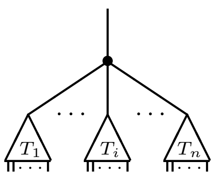
(8.16)

for some $T_i \in \mathcal{T}_{l_i}^n$, $2 \leq i \leq n$, with $l_2 + \cdots + l_n = k - 1$. Then

$$\mu_T = \mu(\mathbb{1} \otimes \mu_{T_2} \otimes \cdots \otimes \mu_{T_n}),$$

where μ_{T_i} denotes the iterated multiplication with the pasting scheme T_i. By induction, each $\mu_{T_2}, \ldots, \mu_{T_n}$ is a linear combination of iterated multiplications whose pasting schemes belong to the subsets $\mathcal{S}_{l_2}^n, \ldots, \mathcal{S}_{l_n}^n$, respectively. The observation that the tree (8.16) belongs to $\mathcal{S}_k^n$ if $T_i \in \mathcal{S}_{l_i}^n$ for each $2 \leq i \leq n$ completes the induction step for this case.

In the *second case,* the tree T has the form



where $T_i \in \mathcal{T}^n_{l_i}$ for $1 \leq i \leq n$, $l_1 + \cdots + l_n = k-1$ and $l_1 \geq 1$. Now

$$\mu_T = \mu(\mu_{T_1} \otimes \cdots \otimes \mu_{T_n})$$

and we may assume, by induction, that $T_i \in \mathcal{S}^n_{l_i}$ for each $1 \leq i \leq n$. In particular, T_1 is as in (8.14) with $S_j \in \mathcal{S}^n_{l'_j}$, $2 \leq j \leq n$ such that $l'_2 + \cdots + l'_n = l_1 - 1$, thus

$$\mu_T = \mu(\mu \otimes 1\!\!1^{\otimes n-1})(1\!\!1 \otimes \mu_{S_2} \otimes \cdots \otimes \mu_{S_n} \otimes \mu_{T_2} \otimes \cdots \otimes \mu_{T_n}).$$

By (8.2), one may replace the factor $\mu(\mu \otimes 1\!\!1^{\otimes n-1})$ by the linear combination

$$-\sum_{i=2}^{n} (-1)^{(i+1)(n-1)} \mu \left(1\!\!1^{\otimes i-1} \otimes \mu \otimes 1\!\!1^{\otimes n-i}\right)$$

which brings μ_T also in the second case to the desired form and finishes the induction step.

To prove that the map (8.15) it is an isomorphism, it suffices now to compare the dimensions of $\mathcal{S}^n_l \times V^{\otimes l(n-1)+1}$ and $p\mathcal{A}ss^n_d(V)_l$. It follows from the description [91, Proposition II.1.25] of the free operad algebra that, for each $l \geq 0$,

$$p\mathcal{A}ss^n_d(V)_l \cong p\mathcal{A}ss^n_d(l(n-1)+1) \otimes \Sigma_{l(n-1)+1} V^{\otimes l(n-1)+1}.$$

Theorem 74 will thus be established if we prove that

$$S^n_l := \mathrm{card}(\mathcal{S}^n_l) \text{ equals } A^n_l := \frac{1}{(l(n-1)+1)!} \dim(p\mathcal{A}ss^n_d(l(n-1)+1)),$$

for each $l \geq 0$. It easily follows from (8.14) that the sequence $\{S^n_l\}_{l\geq 0}$ is determined by the recursion $S^n_0 = 1$ and

$$S^n_l := \sum_{\substack{0 \leq l_2, \cdots, l_n \leq l-1 \\ l_2 + \cdots + l_{n-1} = l-1}} S^n_{l_2} \cdots S^n_{l_n} \text{ for } l \geq 1. \tag{8.17}$$

In Proposition 69 below, which is of independent interest, we prove that the sequence $\{A^n_l\}_{l\geq 0}$ satisfies the same recursion. This finishes the proof.

Recursion (8.17) appeared, with $p_l^{<n-1>}$ in place of S^n_l, in [47, Section 3.4]. Theorem 74 gives a realization of free $p\mathcal{A}ss^n_d$-algebras in the Koszul case ($n \equiv d \bmod 2$) by putting

$$p\mathcal{A}ss^n_d(V) := \bigoplus_{l \geq 0} \mathcal{S}^n_l \times V^{\otimes l(n-1)+1}.$$

We leave as an exercise to describe the structure n-ary multiplication of $p\mathcal{A}ss^n_d(V)$ in this language, see [47].

Proposition 69 *The Poincaré series of the operad $p\mathcal{A}ss^n_d$ is, in the Koszul case (with n and d of the same parity), given by*

$$g_{p\mathcal{A}ss^n_d}(t) = \sum_{l \geq 0} (-1)^{ln} A^n_l t^{l(n-1)+1} \tag{8.18}$$

where the coefficients $\{A^n_l\}_{l\geq 0}$ are defined recursively by $A^n_0 := 1$ and

$$A^n_l := \sum_{\substack{0 \leq l_2, \cdots, l_n \leq l-1 \\ l_2 + \cdots + l_{n-1} = l-1}} A^n_{l_2} \cdots A^n_{l_n} \textit{ for } l \geq 1. \tag{8.19}$$

Proof. One can easily check that the recursive definition (8.19) of the coefficients of $f(t) := g_{p\mathcal{A}ss^n_d}(t)$ is equivalent to the functional equation

$$f(t) = t\left(1 + (-1)^n f(t)^{n-1}\right)$$

which in turn immediately implies that $f(t)$ is the unique formal solution of

$$g_{t\mathcal{A}ss^n_{-d+n-2}}(-f(-t)) = t,$$

where the Poincaré series $g_{t\mathcal{A}ss^n_{-d+n-2}}(t)$ is as in the first line of (8.6) because $-d+n-2$ is even. Since we are in the Koszul case, the above display means, by Theorem 69, that $f(t)$ is the Poincaré series of $(t\mathcal{A}ss^n_{-d+n-2})^! = p\mathcal{A}ss^n_d$. This proves the proposition.

The description of the Poincaré series of $p\mathcal{A}ss^n_d$ for n and d of the same arity given in Proposition 69 implies that the Poincaré series of $p\widetilde{\mathcal{A}ss}^n_d$ for d odd equals

$$g_{p\widetilde{\mathcal{A}ss}^n_d}(t) = \sum_{l\geq 0}(-1)^l A^n_l t^{l(n-1)+1},$$

with $\{A^n_l\}_{l\geq 0}$ having the meaning as in (8.18).

Example 75 Using `Matematica`, we calculated initial values of the series $\{A^3_l\}_{l\geq 0}$ as $A^3_0 = 1$, $A^3_1 = 1$, $A^3_2 = 2$, $A^3_3 = 5$, $A^3_4 = 14$, $A^3_5 = 42$, $A^3_6 = 132$, $A^3_7 = 429$, $A^3_8 = 1\ 430$, $A^3_9 = 4\ 862$, $A^3_{10} = 16\ 796$, &c.

Remark 76 In the non-Koszul case (n and d of different parity), the map (8.15) of Theorem 74, while always being an epimorphism, need not be a monomorphism. This means that there may be "unexpected relations" in the free algebra $p\mathcal{A}ss^n_d(V)$ and Gnedbaye's description of free partially associative n-algebras is no longer true.

For instance, while S^3_3 (the dimension of $\mathbb{S}^n_3$ for $n=3$) equals 5, the dimension of $p\mathcal{A}ss^3_0(7)$ equals $7!\cdot 4$, so $p\mathcal{A}ss^3_0(V)_3$ has one copy $V^{\otimes 7}$ less than $\mathbb{S}^3_3 \times V^{\otimes 7}$. Therefore the map (8.15) has, for $n=3$, $d=0$ and $l=3$, a nontrivial kernel. The Poincaré series of $p\mathcal{A}ss^3_0$ is was calculated in [58] as

$$g_{p\mathcal{A}ss^3_0}(t) = t + t^3 + 2t^5 + 4t^7 + 5t^9 + 6t^{11} + 7t^{13} + \cdots .$$

8.6 Open problems

The first question which our paper leaves open is the Koszulity of the operads $t\mathcal{A}ss^n_d$ with d odd and $n \geq 8$. The method used in the proof of Proposition 67 does not apply to these cases and indeed, our numerical tests mentioned in Remark 73 suggest that it may happen that all coefficients in the formal inverse of $t - t^n + t^{2n-1}$ are non-negative.

Even if this happens, it would not necessarily mean that the operad $t\mathcal{A}ss^n_d$ is Koszul, only that subtler methods must be applied to that case. For instance, one may try to compare the coefficients of this formal inverse to the dimensions of the components of the dual operad $(t\mathcal{A}ss^n_d)^!$.

Understanding these components is, of course, equivalent to finding a basis for the free partially associative algebras in the non-Koszul cases. This problem was solved, in [58], for free $p\mathcal{A}ss^3_0$-algebras, for $n \geq 4$ it remains open.

The last problem we want to formulate here is to find more about the minimal model of the anti-associative operads $\widetilde{\mathcal{A}ss}$, or even to describe it completely. As far as we know, no complete description of the minimal model of a non-Koszul operad is known. Since $\widetilde{\mathcal{A}ss}$ is the simplest non-Koszul operad, it is the first obvious candidate to attack. A related task is to find as much as information about minimal models of the remaining non-Koszul n-ary operads as possible.

Part II

Affine and complex structures on Lie algebras, Γ-symmetric spaces

This part is devoted to the study of some geometrical structures on Lie algebras. More particulary, we are interested in

- **The problem of existence of complex structures on nilpotent Lie algebras.** The classification of 6-dimensional nilpotent real Lie algebras provided with a complex structure is known. But this classification is based on the general classification of real and complex Lie algebras. Such a classification does not exist for dimensions greater than or equal to 8. Thus our approach to this problem must be different. First of all we prove directly that filiform algebras (that is, nilpotent Lie algebras with a maximal nilindex) have no complex structures. This family corresponds to nilpotent Lie algebras with Goze's invariant equal to $(n-1,1)$, where n is the dimension of the Lie algebra. Then we are interested in quasi filiform algebras, that is, with a Goze's invariant equal to $(n-2,1,1)$. In this case, the approach is based on the notion of generalized complex structures of Gualtieri and Cavalcanti (note that in their works, these authors reproduce the main result for filiform algebras). In this case we prove that any quasi filiform algebra provided with a complex structure is 6-dimensional and we describe all these algebras (in fact, we obtain only one class).

- **The problem of existence of affine structure on nilpotent Lie algebras.** We approached this problem in the first part considering the pre-Lie algebras. In fact, a pre-Lie algebra is a Lie-admissible algebra whose corresponding Lie algebra admits an affine structure. The problem of existence of affine structures on nilpotent Lie algebras is always open. Contrary to what people expected, the counter-example of Benoist shows that there is a nilpotent Lie algebras with no affine structure. In Chapter 10 we begin to determinate all the affine structures (complete or not) on the 3-dimensional abelian Lie algebra and we classify all the affine structure on the 3-torus. This completes the classical work of Goldmann. After that we study general results on existence problems. We begin by proving that any non-characteristically (that is, with a non nilpotent derivation) filiform Lie algebra admits an affine structure. This permits to describe a non complete affine representation on the Heisenberg group. We construct also current affine Lie algebras, that is, Lie algebras with an affine structure obtained by a tensor product of a pre-Lie algebra with an algebra associated with the current operad. At the end of this chapter we are interested in contact Lie algebras with affine structures. This is motivated by the classical result that states that any symplectic Lie algebra admits a natural affine structure. This is not the case in the contact situation (cf the counter-example of Benoist). We describe an obstruction to extend a symplectic affine structure to a contact Lie algebra obtained by a central extension.

- **The geometry of symmetric spaces.** In a recent paper, Bathurin and Goze generalize the notion of symmetric spaces and propose a new class of reductive space called Γ-symmetric spaces. The most interesting examples are the flag manifolds. It is not a symmetric space but we can find in this space a lot of commuting symmetries which are in correspondance with an abelian group (Γ). Here we are interested in the riemannian case and the main result is the classification of compact $\mathbb{Z}_2^2$-symmetric spaces.

Chapter 9

Complex structures on nilpotent Lie algebras

The study of invariant complex structures on real connected Lie Groups is reduced to the study of linear operators J on the corresponding Lie algebra which satisfies the Nijenhuis condition and $J^2 = -Id$. When the Lie algebra is even dimensional, real and reductive, the existence of such structures follows from the work of Morimoto [1]. On the other hand, there exist solvable and nilpotent Lie algebras which can not be provided with a complex structure. Some recent works present classifications of nilpotent or solvable Lie algebras with complex structures in small dimensions (dimension four for the solvable case [2] and six for the nilpotent case [3]). In this chapter, we are interested, at first, by biinvariant complex structures, that is, complex structures which commute with the adjoint operators. These structures induce holomorphic structures on the corresponding connected Lie groups. We establish this classification up to dimension 8. Next, we focus our attention on the problem of existence of complex structures on real nilpotent Lie algebras. As every six dimensional nilpotent Lie algebra does not admit complex structure (following the work of Salomon), we are conducted to determine the classes of nilpotent Lie algebra that are not provided with such a structure. We prove two results :

- any filiform $2p$-dimensional Lie algebras (that is, nilpotent Lie algebras with a maximal nilindex equal to $2p-1$) does not admit complex structures. This result is proved by considering a decomposition of $2p$-dimensional filiform algebras into a sum of two p-dimensional subalgebras.

- a $2p$-dimensional quasi-filiform Lie algebra (that is, nilpotent Lie algebra with nilindex equal to $2p-2$) admits a complex structure if and only if $p = 2$ or 3 and, if $p = 3$, this Lie algebra is isomorphic to the quasi-filiform filiform algebra given by

$$\begin{cases} [X_0, X_i] = X_{i+1}, \quad i = 1, 2, 3, \\ [X_1, X_2] = X_5, \\ [X_1, X_5] = X_4. \end{cases}$$

This result is proved using the theory of Generalized complex structures (see the work of Calvancanti and Gualtieri [20]).

9.1 Complex structures on real Lie algebras

9.1.1 Definition

Let $\mathfrak{g}$ be a real Lie algebra.

Definition 77 *A complex structure on $\mathfrak{g}$ is an endomorphism J of $\mathfrak{g}$ such that*

(1) $J^2 = -Id$,

(2) $[JX, JY] = [X, Y] + J[JX, Y] + J[X, J(Y)], \forall X, Y \in \mathfrak{g}$.

(The second condition is called the Nijenhuis condition of integrability).

The underlying real vector space V of the Lie algebra $\mathfrak{g}$ can be provided with a complex vector space structure by putting

$$(a+ib)\cdot v = a\cdot v + b\cdot J(v),$$

for any $v \in V$ and $a, b \in \mathbb{R}$. We denote by V_J this complex vector space. We have

$$\dim_{\mathbb{C}} V_J = \frac{1}{2}\dim_{\mathbb{R}} V = \frac{1}{2}\dim_{\mathbb{R}} \mathfrak{g}.$$

Definition 78 *The complex structure J is* **bi-invariant** *if it satisfies*

(3) $[J, adX] = 0$, $\forall X \in \mathfrak{g}$.

Remark that (3) implies (2).

If J is bi-invariant, then V_J is a complex Lie algebra, denoted $\mathfrak{g}_J$, because in this case we have

$$[(a+ib)\,X, (c+id)\,Y] = (a+ib)\,(c+id)\,[X, Y]\,.$$

9.1.2 Decomposition associated to a complex structure

Let J be a complex structure on the real Lie algebra $\mathfrak{g}$. We can extend the endomorphism J in a natural way on the complexification $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}$ of $\mathfrak{g}$. It induces on $\mathfrak{g}_{\mathbb{C}}$ a direct vectorial sum

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}^{i} \oplus \mathfrak{g}_{\mathbb{C}}^{-i},$$

where

$$\mathfrak{g}_{\mathbb{C}}^{\varepsilon i} = \{X \in \mathfrak{g}_{\mathbb{C}} \ / \ J(X) = \varepsilon i X\}\,, \text{ with } \varepsilon = \pm 1.$$

The Nijenhuis condition (2) implies that $\mathfrak{g}_{\mathbb{C}}^{\varepsilon i}$ is a complex subalgebra of $\mathfrak{g}_{\mathbb{C}}$. If σ denotes the conjugation on $\mathfrak{g}_{\mathbb{C}}$ defined by $\sigma\,(X+iY) = X - iY$, then

$$\mathfrak{g}_{\mathbb{C}}^{-i} = \sigma(\mathfrak{g}_{\mathbb{C}}^{i}).$$

Proposition 70 *A $2n$-dimensional real Lie algebra $\mathfrak{g}$ is provided with a complex structure if and only if the complexification $\mathfrak{g}_{\mathbb{C}}$ admits the decomposition*

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{h} \oplus \sigma(\mathfrak{h}),$$

where $\mathfrak{h}$ is a n-dimensional complex subalgebra of $\mathfrak{g}_{\mathbb{C}}$.

If J is bi-invariant, Condition (3) implies that $\mathfrak{g}_{\mathbb{C}}^{\varepsilon i}$ is an ideal of $\mathfrak{g}_{\mathbb{C}}$. In fact, if $X \in \mathfrak{g}_{\mathbb{C}}^{\varepsilon i}$ and $Y \in \mathfrak{g}_{\mathbb{C}}^{-\varepsilon i}$ we have

$$J\,[X, Y] = [JX, Y] = \varepsilon i\,[X, Y] = [X, JY] = -\varepsilon i\,[X, Y]\,.$$

Then

$$[X, Y] = 0.$$

Proposition 71 *A $2n$-dimensional real Lie algebra $\mathfrak{g}$ has a bi-invariant complex structure if and only if the complexification $\mathfrak{g}_{\mathbb{C}}$ is a direct sum of ideals I and $\sigma\,(I)$, that is,*

$$\mathfrak{g}_{\mathbb{C}} = I \oplus \sigma(I).$$

Of course every n-dimensional complex Lie algebra $\mathfrak{h}$ comes from a $2n$-dimensional real Lie algebra endowed with a bi-invariant complex structure.

9.2 Bi-invariant complex structures

9.2.1 Nilpotent case

Let $\mathfrak{g}$ be a $2n$-dimensional real nilpotent Lie algebra with a bi-invariant complex structure. Then $\mathfrak{g}_{\mathbb{C}} = I \oplus \sigma(I)$ where I is a n-dimensional complex ideal of $\mathfrak{g}_{\mathbb{C}}$. This describes entirely the structure of the complexifications $\mathfrak{g}_{\mathbb{C}}$ of the real Lie algebras $\mathfrak{g}$ provided with bi-invariant complex structure.

Let $c(\mathfrak{n})$ be the characteristic sequence of the complex nilpotent algebra $\mathfrak{n}$ ([4]). It is defined by

$$c(\mathfrak{n}) = \quad Max\{c(X) \ \mid \ X \in \mathfrak{n} - [\mathfrak{n}, \mathfrak{n}]\},$$

where $c(X) = (c_1(X), \cdots, c_k(X), 1)$ is the sequence of similitude invariants of the nilpotent operator adX. We deduce that

$$c\left(\mathfrak{g}_{\mathbb{C}}\right) = \left(c_1, c_1, c_2, c_2, \cdots, 1, 1\right).$$

Since the Jordan normal form of the nilpotent operators adX does not depend of the field of scalars, we have

Proposition 72 *If $\mathfrak{g}$ is a $2n$-dimensional real nilpotent Lie algebra which admits a bi-invariant complex structure, then its characteristic sequence is of type*

$$\left(c_1, c_1, c_2, c_2, \cdots, c_j, c_j, \cdots, 1, 1\right).$$

A nilpotent Lie algebra is of maximal class (or filiform in Vergne's terminology [5],[6]) if the descending sequence of derived ideals $\mathcal{C}^i\mathfrak{g}$ satisfies:

$$\left\{ \begin{array}{l} \dim \mathcal{C}^1\mathfrak{g} = \dim \mathfrak{g} - 2, \\ \dim \mathcal{C}^i\mathfrak{g} = \dim \mathfrak{g} - i - 1, \text{ for any } i \geq 2. \end{array} \right.$$

Corollary 79 *A filiform Lie algebra has no bi-invariant complex structure.*

In fact, the characteristic sequence of a filiform Lie algebra is $(2n-1, 1)$. Thus, by Proposition 72, a filiform Lie algebra can not admit a bi-invariant complex structure.

Let $\{U_j\}$ and $\{V_j\}$ be the basis of the complex ideals I and $\sigma(I)$. With respect to $\mathfrak{g}$, we can write $U_l = X_l - iY_l$, $V_l = X_l + iY_l$, where $\{X_1, \cdots, X_n, Y_1, \cdots, Y_n\}$ is a real basis of $\mathfrak{g}$. As $[U_l, V_l] = 0$, we obtain

$$[X_l - iY_l, X_k + iY_k] = [X_l, X_k] + [Y_l, Y_k] + i\left([X_l, Y_k] - [Y_l, X_k]\right) = 0$$

Thus

$$\left\{ \begin{array}{l} [X_l, X_k] = -\left[Y_l, Y_k\right], \\ [X_l, Y_k] = [Y_l, X_k], \qquad k = 1, \cdots, n \quad ; \quad l = 1, \cdots, n. \end{array} \right.$$

Likewise $[U_l, U_k] \in I$ and $[V_l, V_k] \in \sigma\left(I\right)$ imply

$$\left\{ \begin{array}{l} [X_l, X_k] - [Y_l, Y_k] \in \mathbb{C}\left\{X_1, \cdots, X_k\right\}, \\ [X_l, Y_k] + [Y_l, Y_k] \in \mathbb{C}\left\{Y_1, \cdots, Y_k\right\}. \end{array} \right.$$

If we suppose that I admits a real basis we would get

$$\left\{ \begin{array}{l} [X_l, X_k] = \sum_{j=1}^n C_{lk}^j X_j \quad , \quad [Y_l, Y_k] = -\sum_{j=1}^n C_{lk}^j Y_j, \\ [X_l, Y_k] = \sum_{j=1}^n D_{lk}^j Y_j \quad , \quad [Y_l, X_k] = \sum_{j=1}^n D_{lk}^j Y_j, \end{array} \right.$$

with C_{lk}^j and D_{lk}^j in $\mathbb{R}$. Considering $\mathfrak{g}_0 = \mathbb{R}\left\{X_1, \cdots, X_n\right\}$ and $\mathfrak{g}_1 = \mathbb{R}\left\{Y_1, \cdots, Y_n\right\}$, the previous relations show that $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ and

$$\left\{ \begin{array}{l} [\mathfrak{g}_0, \mathfrak{g}_0] \subset \mathfrak{g}_0, \\ [\mathfrak{g}_1, \mathfrak{g}_1] \subset \mathfrak{g}_0, \\ [\mathfrak{g}_0, \mathfrak{g}_1] \subset \mathfrak{g}_1, \end{array} \right.$$

which gives a structure of $\mathbb{Z}_2$-graded Lie algebra on $\mathfrak{g}$.

9.2.2 On the classification of nilpotent Lie algebras with bi-invariant complex structures

If $\mathfrak{g}$ is a $2n$-dimensional real nilpotent Lie algebra with a bi-invariant complex structure, $\mathfrak{g}_{\mathbb{C}} = I \oplus \sigma(I)$. Then the classification of associated complexification $\mathfrak{g}_{\mathbb{C}}$ corresponds to the classification of complex nilpotent Lie algebra I. By this time, this classification is known only up to dimension 7, and for special classes up to dimension 8. To obtain a general list of these algebras is therefore a hopeless pretention. From [3] we can extract the list for the 6-dimensional case. From [4] we can present the classification of complexifications $\mathfrak{g}_{\mathbb{C}}$ of real nilpotent Lie algebras with bi-invariant complex structure up the dimension 14. Here we present briefly the classification in the real case up to dimension 8.

Dimension 2 : $\mathfrak{g}_2^1$ is abelian.

Dimension 4 : $\mathfrak{g}_4^1$ is abelian.

In fact, $\mathfrak{g}_{\mathbb{C}} = I \oplus \sigma(I)$ and I is an abelian ideal.

Dimension 6 : - $\mathfrak{g}_6^1$ is abelian.

- $\mathfrak{g}_6^2 : \begin{cases} [X_2, X_3] = -[Y_2, Y_3] = X_1, \\ [X_2, Y_3] = [Y_2, X_3] = Y_1. \end{cases}$

Remark.

1. The complexification of $\mathfrak{g}_6^2$ is isomorphic to $\mathfrak{h}_3 \oplus \mathfrak{h}_3$ where $\mathfrak{h}_3$ is the 3-dimensional Heisenberg Lie algebra.
2. The real nilpotent Lie algebra $\mathfrak{h}_3 \oplus \mathfrak{h}_3$ has no bi-invariant complex structure.

   In fact, if $\{X_1, X_2, X_3, X_4, X_5, X_6\}$ is a basis of $\mathfrak{h}_3 \oplus \mathfrak{h}_3$ with brackets
   $$[X_1, X_2] = X_3, \quad [X_4, X_5] = X_6,$$
   then every isomorphism J which commutes with adX for every X satisfies $J(X_3) = \alpha X_3$ and $J(X_6) = \beta X_6$. Then $J^2 = -Id$ implies $\alpha^2 = \beta^2 = -1$ and $\alpha, \beta \in \mathbb{R}$ which is impossible.
3. The Lie algebra $\mathfrak{g}_6^2$ is 2-step nilpotent and it is a Lie algebra of type H (see [7]).

Dimension 8 : - $\mathfrak{g}_8^1$ is abelian.

- $\mathfrak{g}_8^2 = \mathfrak{g}_6^2 \oplus \mathfrak{g}_2^1$,

- $\mathfrak{g}_8^3 : \begin{cases} [X_2, X_4] = X_1, & [Y_2, Y_4] = -X_1, \\ [X_2, Y_4] = +Y_1, & [Y_2, X_4] = +Y_1, \\ [X_3, X_4] = X_2, & [Y_3, Y_4] = -X_2, \\ [X_3, Y_4] = Y_2, & [Y_3, X_4] = Y_2, \end{cases}$

- $\mathfrak{g}_8^4 : \begin{cases} [X_2, X_3] = X_1, & [Y_2, Y_3] = -X_1, \\ [X_2, Y_3] = Y_1, & [Y_2, X_3] = Y_1, \\ [X_2, X_4] = Y_1, & [Y_2, Y_4] = -Y_1, \\ [X_2, Y_4] = -X_1, & [Y_2, X_4] = -X_1. \end{cases}$

9.2.3 On the classification of solvable Lie algebras with bi-invariant complex structures

As known, there is, up to an isomorphism, only one 2-dimensional non nilpotent solvable Lie algebra, denoted by $\mathfrak{r}_2^2$ and defined by $[X_1, X_2] = X_1$. This Lie algebra admit no bi-invariant complex structure. Thus, a non abelian solvable Lie algebra admitting such a structure is at least four dimensional.

Theorem 80 *Any 4-dimensional real solvable Lie algebras which admits a bi-invariant complex structure is isomorphic to one of the following algebras:*

- $\mathfrak{r}_4^1 = \mathfrak{g}_4^1$ *the abelian Lie algebra,*

- $\mathfrak{r}_4^2 = \begin{cases} [X_1, X_3] = X_3, \\ [X_1, X_4] = X_4, \\ [X_2, X_3] = -X_4, \\ [X_2, X_4] = X_3. \end{cases}$

In the last case, any bi-invariant complex structure satisfy

$$\begin{cases} J(X_1) = aX_1 + bX_2, \\ J(X_2) = -bX_1 + aX_2, \\ J(X_3) = aX_3 + bX_4, \\ J(X_4) = -bX_3 + aX_4. \end{cases}$$

Its complexification $\mathfrak{r}_{4\mathbb{C}}^2$ is isomorphic to $\mathfrak{r}_2^2 \times \mathfrak{r}_2^2$. Remark that the real Lie algebra $\mathfrak{r}_2^2 \times \mathfrak{r}_2^2$ which has the same complexification as $\mathfrak{r}_4^2$ is not provided with a bi-invariant complex structure.

9.3 Non Existence of complex structures over filiform algebras

Let $\mathfrak{n}$ be a n-dimensional filiform Lie algebra. Then there exists a basis $\{X_1, X_2, \cdots, X_n\}$ which is adapted to the flag

$$\mathfrak{g} \supset \mathcal{C}^1\mathfrak{g} \supset \mathcal{C}^2\mathfrak{g} \supset \cdots \supset \mathcal{C}^{n-1}\mathfrak{g} = \{0\}\,;$$

with $\dim \mathcal{C}^1\mathfrak{g} = n-2$, $\dim \frac{\mathcal{C}^i\mathfrak{g}}{\mathcal{C}^{i-1}\mathfrak{g}}$ =1, $i \geq 1$, and satisfies:

$$\begin{cases} [X_1, X_i] = X_{i+1}, \quad i = 2, \cdots, n-1, \\ [X_i, X_j] = \sum_{k \geq i+j} C_{ij}^k X_k. \end{cases}$$

The change of basis $Y_1 = X_1$, $Y_i = tX_i$ for $i \geq 2$ and $t \neq 0$ shows that this Lie algebra is isomorphic to the following:

$$\begin{cases} [X_1, X_i] = X_{i+1}, \quad i = 2, \cdots, n-1 \\ [X_i, X_j] = t\sum C_{ij}^k X_k \end{cases}$$

The simplest example of filiform algebra is given by the n-dimensional Lie algebra, denoted L_n and defined by the brackets:

$$[X_1, X_i] = X_{i+1}, \quad i = 2, \cdots, n-1,$$

where the non-defined brackets are zero or obtained by antisymmetry.

9.3.1 Complex structures and filiform Lie algebras

Proposition 73 *The real nilpotent filiform Lie algebra L_{2n} admits no complex structure.*

Proof . Let T be a linear isomorphism of the real vector space L_{2n} satisfying the Nijenhuis condition:

$$[T(X), T(Y)] = [X, Y] + T\,[X, T(Y)] + T\,[T(X), Y]\,,$$

where [,] is the bracket of L_{2n}. Consider the basis $\{X_1, \cdots, X_{2n}\}$ of L_{2n} satisfying

$$\begin{cases} [X_1, X_i] = X_{i+1}, & i = 2, \cdots, 2n-1, \\ [X_i, X_j] = 0, & i, j \neq 1. \end{cases}$$

We have

$$\begin{aligned} [T(X_{2n-1}), T(X_{2n})] &= [X_{2n-1}, X_{2n}] + T\,[X_{2n-1}, T(X_{2n})] + T\,[T(X_{2n-1}), X_{2n}] \\ &= T\,[X_{2n-1}, T(X_{2n})]\,. \end{aligned}$$

Since

$$[X_{2n-1}, X_1] = -X_{2n},$$

we obtain

$$[T(X_{2n-1}), T(X_{2n})] = T\,[X_{2n-1}, T(X_{2n})] = -aT(X_{2n}),$$

where

$$T(X_{2n}) = aX_1 + \sum_{i\geq 2} a_i X_i.$$

The nilpotency of L_{2n} implies that the constant a which appears as an eigenvalue of $ad(T(X_{2n-1}))$ is zero. Then

$$[T(X_{2n-1}), T(X_{2n})] = 0.$$

This implies that

$$[T(X_i), T(X_{2n})] = T\,[X_i, T(X_{2n})] = 0,$$

for $i = 2, \cdots, 2n$. If $T(X_{2n}) \notin Z(L_{2n}) = \mathbb{R}\{X_{2n}\}$, then $T(X_i) = \sum_{j\geq 2} a_{ij} X_j$ for $j = 2, \cdots, 2n$. Since T is non singular we have

$$T(X_1) = a_{11} X_1 + \sum_{i\geq 2} a_{i1} X_i,$$

with $a_{11} \neq 0$. The condition $T^2 = -Id$ implies $a_{11}^2 = -1$. Since $a_{11} \in \mathbb{R}$, this gives a contradiction. Thus $T(X_{2n}) \in Z(L_{2n}) = \mathbb{R}\{X_{2n}\}$ and it follows that $T(X_{2n}) = \alpha X_{2n}$. As above, we obtain $\alpha^2 = -1$ which is impossible. So the proposition is proved.

Now we prove that any deformation of the model filiform L_{2n} can not be provided with an invariant complexe structure too.

Theorem 81 *There is no complex structure over a real filiform Lie algebra.*

Proof. Let $\mathfrak{g}$ be a $2n$-dimensional real filiform Lie algebra and let $\mathfrak{g}_{\mathbb{C}}$ be its complexification. If there a complex structure J on $\mathfrak{g}$, then $\mathfrak{g}_{\mathbb{C}}$ admits the following decomposition

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_1 \oplus \sigma(\mathfrak{g}_1),$$

where $\mathfrak{g}_1$ is a n-dimensional subalgebra of $\mathfrak{g}_{\mathbb{C}}$ and σ the conjugated automorphism. Since the Lie algebra $\mathfrak{g}_{\mathbb{C}}$ is filiform, there is an adapted basis $\{X_1, .., X_{2n}\}$ satisfying

$$\begin{cases} [X_1, X_i] = X_{i+1}, \quad 2 \leq i \leq 2n-1, \\ [X_2, X_3] = \sum_{k\geq 5} C_{23}^k X_k, \\ \mathcal{C}^i(\mathfrak{g}) = \mathbb{R}\{X_{i+2}, \cdots, X_n\}. \end{cases}$$

In particular we have

$$\dim \frac{\mathfrak{g}_{\mathbb{C}}}{\mathcal{C}^1(\mathfrak{g}_{\mathbb{C}})} = 2, \quad \dim \frac{\mathcal{C}^i(\mathfrak{g}_{\mathbb{C}})}{\mathcal{C}^{i+1}(\mathfrak{g}_{\mathbb{C}})} = 1 \quad \text{for} \quad i \geq 1$$

The ordered sequence of the dimension of Jordan blocks of the nilpotent operator adX_1 is $(n-1, 1)$. Such a vector X_1 is called characteristic vector.

Lemma 13 *Every characteristic vector can be written $Y = \alpha X_1 + U$ where U is in the complex vector space generated by $\{X_2, \cdots, X_{2n}\}$ and $\alpha \neq 0$.*

It follows that the set of characteristic vectors of $\mathfrak{g}_{\mathbb{C}}$ is the open set $\mathfrak{g}_{\mathbb{C}} - \mathbb{C}\{X_2, .., X_{2n}\}$.

Lemma 14 *Either $\mathfrak{g}_1$ or $\sigma(\mathfrak{g}_1)$ contains a characteristic vector of $\mathfrak{g}_{\mathbb{C}}$.*

Observe that otherwise we would have $\mathfrak{g}_1 \subset \mathbb{C}\{X_2,..,X_{2n}\}$ and $\sigma(\mathfrak{g}_1) \subset \mathbb{C}\{X_2,..,X_{2n}\}$, which contradicts the previous decomposition. Thus $\mathfrak{g}_1$ or $\sigma(\mathfrak{g}_1)$ is a $n-$dimensional complex filiform Lie algebra. But if $Y \in \mathfrak{g}_1$ is a characteristic vector of $\mathfrak{g}_1$, then $\sigma(Y)$ is a characteristic vector of $\sigma(\mathfrak{g}_1)$ with the same characteristic sequence. Then every $2n$-dimensional filiform Lie algebra appears as a direct vectorial sum of two n-dimensional filiform Lie algebras. We shall prove that it is impossible. More precisely we have

Proposition 74 *For $n \geq 3$, a $2n-$dimensional complex filiform Lie algebra is never a vectorial direct sum of two $n-$dimensional filiform subalgebras.*

Proof. Let $\mathfrak{g}_{\mathbb{C}}$ be a filiform Lie algebra of dimension $2n$ such that $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$. From the previous lemma, one of them for example $\mathfrak{g}_1$, contains a characteristic vector. Let X_1 be this vector an $\{X_1, X_2, .., X_{2n}\}$ the corresponding basis. This implies that $\mathfrak{g}_1 \cap \mathbb{C}\{X_2,..,X_{2n}\} = \mathfrak{g}_1 \cap \mathbb{C}\{X_{n+1},..,X_{2n}\}$. But $\mathfrak{g}_2$ cannot contain characteristic vector of $\mathfrak{g}$, if not $\mathfrak{g}_2 \cap \mathbb{C}\{X_2,..,X_{2n}\} = \mathfrak{g}_2 \cap \mathbb{C}\{X_{n+1},..,X_{2n}\}$ and this is a contradiction with $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$. Then $\mathfrak{g}_2 \subset \mathbb{C}\{X_2,..,X_{2n}\}$. But using the expression of the brackets in the adapted basis, this is impossible as soon as $n > 2$.

The four dimensional case can be treated directly. Up to isomorphism, there is only one 4 dimensional filiform Lie algebra, L_4 which has no complex structures. Note that, with respect to the adapted basis, this algebra admits the decomposition $L_4 = \mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$,where $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are the abelian subalgebras generated respectively by $\{X_1, X_4\}$ and $\{X_2, X_3\}$.

9.3.2 Consequence.

Let J be a complex structure on a Lie algebra $\mathfrak{g}$ whose multiplication is denoted by μ. We consider the Chevalley cohomology of $\mathfrak{g}$. The coboundary operator is denoted by δ_μ.

Proposition 75 *We have*

$$\delta_\mu J = \mu_J$$

where μ_J is the Lie algebra multiplication, isomorphic to μ, defined by

$$\mu_J(X,Y) = J^{-1}(\mu(J(X), J(Y)).$$

In fact, the Nijenhuis condition is written:

$$\mu(JX, JY) = \mu(X,Y) + J\mu(JX,Y) + J\mu(X, J(Y)).$$

Then

$$J^{-1}\mu(JX, JY) = J^{-1}\mu(X,Y) + \mu(JX,Y) + \mu(X, J(Y)),$$

that is, since $J^2 = -Id$

$$\begin{aligned} J^{-1}\mu(JX, JY) &= -J\mu(X,Y) + \mu(JX,Y) + \mu(X, J(Y)) \\ &= \delta_\mu J(X,Y). \end{aligned}$$

Corollary 82 *If $\mathfrak{g}$ is a filiform Lie algebra, then there is no 2-coboundaries for the Chevalley cohomology such that*

$$\delta_\mu(J) = \mu_J$$

where $J^2 = -Id$.

9.4 Complex Structures on quasi-filiform Lie algebras

In the previous section we have proved that there is no filiform algebras provided with a complex structure. In this chapter we consider other class of nilpotent Lie algebras, called quasi filiform, characterized by the characteristic sequence (often called the Goze invariant), which is equal to $(n-2,1,1)$. Its nilindex is equal to $n-2$ (for filiform algebras it is equal to $n-1$). We show that quasi filiform with complex structure have dimension greater than or equal to 6.

9.4.1 Generalized complex structures on Lie algebras

Generalized complex structures can be defined in the general context of smooth manifolds, nevertheless, throughout this paper we study generalized complex structures on real Lie groups. We recall the definition in this context, which is purely algebraic.

Let $\mathfrak{g}$ be a real $2n$-dimensional Lie algebra and $\mathfrak{g}^*$ its dual space which can be identified with the space of left-invariant differential 1-forms on a connected Lie group with Lie algebra $\mathfrak{g}$. We consider the coboundary operator d map $\mathfrak{g}^*$ onto $\Lambda^2(\mathfrak{g}^*)$ and defined by $d\alpha(X,Y) = -\alpha[X,Y]$ where $\alpha \in \mathfrak{g}^*$ and $[,]$ is the Lie bracket of $\mathfrak{g}$. We define a bracket on $\mathfrak{g} \oplus \mathfrak{g}^*$ called the Courant bracket which is generally written:

$$[X+\xi, Y+\eta]_c = [X,Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}d(I_X\eta - I_Y\xi),$$

where $X, Y \in \mathfrak{g}$, $\xi, \eta \in \mathfrak{g}^*$ and $I_X\eta$ represents the inner product of X on η and $\mathcal{L}_X\eta$ denotes the Lie derivative of η by X. As $I_X\eta - I_Y\xi$ is a constant, then $d(I_X\eta - I_Y\xi) = 0$.

This operation is skew-symmetric and satisfies Jacobi identity (remark that, in the context of smooth manifolds, this bracket is defined in the sum of tangent and cotangent bundles and does not satisfy Jacobi identity). Thereby $\mathfrak{g} \oplus \mathfrak{g}^*$ endowed with the Courant bracket is a $4n$-dimensional real Lie algebra. It is moreover a quadratic Lie algebra. Indeed, the space $\mathfrak{g} \oplus \mathfrak{g}^*$ admits a natural non-degenerate inner product of signature $(2n, 2n)$ defined by:

$$\langle X+\xi, Y+\eta\rangle = \frac{1}{2}(\xi(Y) + \eta(X)).$$

Definition 83 *Let $\mathfrak{g}$ be a real Lie algebra of dimension $2n$. A generalized complex structure on $\mathfrak{g}$ is a linear endomorphism $\mathcal{J}$ of $\mathfrak{g} \oplus \mathfrak{g}^*$ such that:*

1. $\mathcal{J}^2 = -Id$,
2. *$\mathcal{J}$ is orthogonal with respect to the scalar product $\langle\ ,\ \rangle$, that is,*
$$\langle \mathcal{J}(X+\xi), \mathcal{J}(Y+\eta)\rangle = \langle X+\xi, Y+\eta\rangle \quad \forall X, Y \in \mathfrak{g}, \forall \xi, \eta \in \mathfrak{g}^*,$$
3. *The $+i$-eigenspace L of $\mathcal{J}$ is required to be involutive with respect to the Courant bracket, that is, $[L, L]_c \subset L$.*

The subalgebra L of $\mathfrak{g} \oplus \mathfrak{g}^*$ is an isotropic space, which means that

$$\langle X+\xi, Y+\eta\rangle = 0,$$

for all $X+\xi, Y+\eta \in L$. Since its dimension is $2n$, L is maximal isotropic. We consider the projection of L on $\mathfrak{g}$ and denote by k the codimension of this projection. It is clear that

$$0 \leq k \leq n.$$

Definition 84 *The type of a generalized complex structure is the codimension of the projection of L on $\mathfrak{g}$.*

Example. Let $\mathfrak{g}$ be a real Lie algebra of dimension $2n$ provided with a complex structure $J : \mathfrak{g} \to \mathfrak{g}$ satisfying $J^2 = -Id$ and the Nijenhuis condition $N(J)(X, Y) = 0$ for all $X, Y \in \mathfrak{g}$. We define a generalized complex structure associated to $\mathcal{J}$

$$\mathcal{J}_J : \mathfrak{g} \oplus \mathfrak{g}^* \to \mathfrak{g} \oplus \mathfrak{g}^*$$

by setting

$$\mathcal{J}_J(X + \xi) = -J(X) + J^*(\xi), \quad \forall X \in \mathfrak{g},\ \forall \xi \in \mathfrak{g}^*.$$

Conditions 1 and 2 of Definition 83 are easy to check. It remains to prove Condition 3. We denote by T_+ (resp. T_-) the eigenspace of J associated to the eigenvalue $+i$ (resp. $-i$) so the $+i$-eigenspace of $\mathcal{J}_J$ is given by:

$$L = T_- \oplus (T_+)^*.$$

Remark that the involutivity of L with respect to the Courant bracket means that T_- is a subalgebra of $\mathfrak{g}$. Therefore $\mathcal{J}_J$ is a generalized complex structure of type n.

Example. Let $\mathfrak{g}$ be a $2n$-dimensional real Lie algebra endowed with a symplectic structure ω, that is, ω is a skew-symmetric 2-form satisfying

$$\begin{cases} \omega^n = \omega \wedge \omega \cdots \wedge \omega \neq 0, \\ d\omega(X, Y, Z) = \omega([X, Y], Z) + \omega([Y, Z], X) + \omega([Z, X], Y) = 0. \end{cases} \tag{9.1}$$

This form can be identified to an isomorphism, also named ω:

$$\omega : \mathfrak{g} \longrightarrow \mathfrak{g}^*$$

and given by $\omega(X) = I_X\omega$. Thereby, we define the generalized complex structure

$$\mathcal{J}_\omega : \mathfrak{g} \oplus \mathfrak{g}^* \to \mathfrak{g} \oplus \mathfrak{g}^*$$

by putting

$$\mathcal{J}_\omega(X + \xi) = \omega(X) + \omega^{-1}(\xi), \quad \forall X \in \mathfrak{g},\ \forall \xi \in \mathfrak{g}^*.$$

This generalized complex structure is of type 0 since its $+i$-eigenspace is

$$L = \{X - i\, I_X\omega\,,\ X \in \mathfrak{g} \otimes \mathbb{C}\}.$$

In the two previous examples, complex and symplectic geometry appear as extremal cases of generalized complex structures. According to [20], any generalized complex structure of type k can be written as a direct sum of a complex structure of dimension k and a symplectic structure of dimension $2n - 2k$. We deduce that every generalized complex structure of the type 0 arises from a complex structure and every generalized complex structure of the type n is given by a symplectic form.

9.4.2 Spinorial approach

Let T be the tensor algebra of $\mathfrak{g} \oplus \mathfrak{g}^*$ and I the ideal generated by $\{X + \xi \otimes X + \xi - \langle X + \xi, X + \xi\rangle \cdot 1\ ,\ X + \xi \in \mathfrak{g} \oplus \mathfrak{g}^*\}$. The factor algebra $C = T/I$ is called the Clifford algebra of $\mathfrak{g} \oplus \mathfrak{g}^*$ associated to the scalar product $\langle\,,\,\rangle$. As C is a simple associative algebra, all its simple representations are equivalent ([22]). A representation $\phi : C \to End_{\mathbb{R}}(S)$ of the Clifford algebra on the vector space S is called a spin representation if it is simple. In this case, S is the space of spinors.

Henceforth, we will consider $S = \wedge\mathfrak{g}^*$ with the spin representation given by the Clifford action:

$$\begin{array}{rccl} \circ : & \mathfrak{g} \oplus \mathfrak{g}^* \times \wedge\mathfrak{g}^* & \to & \wedge\mathfrak{g}^* \\ & (X + \xi, \rho) & \mapsto & (X + \xi) \circ \rho = i_X\rho + \xi \wedge \rho. \end{array}$$

If $\rho \in \wedge\mathfrak{g}^*$ a nonzero spinor, we define its null space $L_\rho \subset \mathfrak{g} \oplus \mathfrak{g}^*$ as follows:

$$L_\rho = \{X + \xi \in \mathfrak{g} \oplus \mathfrak{g}^* :\ (X + \xi) \circ \rho = 0\}.$$

It is clear that L_ρ is an isotropic space. We say that ρ is a pure spinor when L_ρ is maximal isotropic. Conversely, if L is a maximal isotropic space, we can consider the set U_L of pure spinors ρ such that $L = L_\rho$. If L is the $+i$-eigenspace of a generalized complex structure, it is proved that the set of pure spinors U_L is a line generated by:

$$\rho = \Omega\ e^{B+i\omega}$$

with B, ω real 2-forms and $\Omega = \theta_1 \wedge \cdots \wedge \theta_k$, where $\theta_1, \ldots, \theta_k$ are complex forms. Moreover, we deduce from (Proposition III.2.3) that $L \cap \overline{L} = \{0\}$ if and only if:

$$\omega^{2n-2k} \wedge \Omega \wedge \overline{\Omega} \neq 0, \tag{9.2}$$

where L is the $+i$-eigenspace of the generalized complex structure. It is also proved that the involutivity condition among L is equivalent to the following integrability condition:

$$\exists X + \xi \in \mathfrak{g} \oplus \mathfrak{g}^* \ / \ d\rho = (X + \xi) \circ \rho. \tag{9.3}$$

9.4.3 Application to nilpotent Lie algebras

Let us consider a real nilpotent Lie algebra $\mathfrak{g}$ of even dimension. The central descending serie is defined by:

$$\begin{cases} \mathfrak{g}^0 & = \mathfrak{g}, \\ \mathfrak{g}^i & = \left[\mathfrak{g}^{i-1}, \mathfrak{g}\right]. \end{cases}$$

We denote by m the nilpotency index of $\mathfrak{g}$. In $\mathfrak{g}^*$, we consider now the increasing series of subspaces V_i where V_i is the annihilator of $\mathfrak{g}^i$, that is to say:

$$\begin{cases} V_0 = \{0\}, \\ V_i = \left\{\varphi \in \mathfrak{g}^* \text{ such that } \forall \mathrm{X} \in \mathfrak{g}^{\mathrm{i}}, \varphi(\mathrm{X}) = 0\right\}. \end{cases}$$

It is clear that $V_m = \mathfrak{g}^*$. Those subspaces can also be expressed as

$$V_i = \{\varphi \in \mathfrak{g}^* \text{ such that } \forall \mathrm{X} \in \mathfrak{g}, \mathrm{I_X d}\varphi \in \mathrm{V_{i-1}}\}.$$

Definition 85 *Let α be p-form of $\mathfrak{g}$. The nilpotent degree of α, denoted by nil(α), is the smallest integer i such that $\alpha \in \wedge^p V_i$.*

Suppose that $\mathfrak{g}$ admits a generalized complex structure of type k. We shall do a special choice of the forms $\{\theta_1, \ldots, \theta_k\}$, at first we order these forms according to their nilpotent degree and we choose them in such a way that $\{\theta_j : \mathrm{nil}(\theta_j) > i\}$ are linearly independent modulo V_i, then the decomposition $\Omega = \theta_1 \wedge \cdots \wedge \theta_k$ satisfies

a) $\mathrm{nil}(\theta_i) \leq \mathrm{nil}(\theta_j)$ for $i < j$,

b) for each i, the forms $\{\theta_j : \mathrm{nil}(\theta_j) > i\}$ are linearly independent modulo V_i.

Such a decomposition will be called appropriate.

Theorem 86 *If $\mathfrak{g}$ is a nilpotent Lie algebra provided with a generalized complex structure, the corresponding pure spinor must be a closed differential form.*

Corollary 87 *If we choose an appropriate decomposition of Ω, then*

a) $d\theta_i \in \mathcal{I}(\{\theta_j : \mathit{nil}(\theta_j) < \mathit{nil}(\theta_i)\})$. *In particular*

$$d\theta_i \in \mathcal{I}(\theta_1 \ldots \theta_{i-1}).$$

b) If dim $\left(\frac{V_{j+1}}{V_j}\right) = 1$ *then, either there exists* θ_i *with nilpotent degree* j*, or no* θ_i *has nilpotent degree* $j+1$.

Remark. Suppose there exists a $j > 0$ such that:

$$\dim\left(\frac{V_{i+1}}{V_i}\right) = 1, \quad \forall i \geq j;$$

It is a consequence of the previous corollary that if none θ_i has nilpotent degree $s \geq j$ then there can be none with nilpotent degree upper than s. Using this fact, we can find an upper bound for all the nilpotent degrees. From Corollary 87, we deduce that $\mathrm{nil}(\theta_1) = 1$. If $j > 1$, we can see that $\mathrm{nil}(\theta_2) \leq j$. Otherwise it would not exist any θ_i of nilpotent degree j and neither of upper degree. This leads to a contradiction with $\mathrm{nil}(\theta_2) > j$. Likewise, we prove by induction that $\mathrm{nil}(\theta_i) \leq j + i - 2$. For $j = 1$, we see in the same way that $\mathrm{nil}(\theta_i) \leq i$.

Theorem 88 *Let* $\mathfrak{g}$ *be a real nilpotent Lie algebra of dimension* $2n$ *endowed with a generalized complex structure of type* $k > 1$*. If there exists* $j > 0$ *such that:*

$$dim\left(\frac{V_{i+1}}{V_i}\right) = 1, \quad \forall i \geq j,$$

then k *is bounded above by:*

$$k \leq \begin{cases} 2n - nil(\mathfrak{g}) + j - 2 & if \quad j > 1, \\ 2n - nil(\mathfrak{g}) & if \quad j = 1. \end{cases}$$

Proof. Suppose $j > 1$. According to the above remark, $\mathrm{nil}(\theta_k) \leq j + k - 2$ and thus all the $\theta_1, \ldots, \theta_k$ belong to V_{j+k-2}. Since $\Omega \wedge \overline{\Omega} \neq 0$, we have

$$\dim V_{j+k-2} \geq 2k.$$

On the other hand, $\dim V_{j+k-2} = 2n - \dim\left(\dfrac{\mathfrak{g}^*}{V_{j+k-2}}\right)$ and

$$\frac{\mathfrak{g}^*}{V_{j+k-2}} \simeq \frac{V_{\mathrm{nil}(\mathfrak{g})}}{V_{\mathrm{nil}(\mathfrak{g})-1}} \oplus \cdots \oplus \frac{V_{j+k-1}}{V_{j+k-2}},$$

so the dimension of V_{j+k-2} is equal to $2n - \mathrm{nil}(\mathfrak{g}) + j + k - 2$. By replacing in the above inequality we finally obtain:

$$k \leq 2n - \mathrm{nil}(\mathfrak{g}) + j - 2.$$

For $j = 1$, we deduce the required result by using similar arguments and considering that $\mathrm{nil}(\theta_k) \leq k$.

Application to filiform Lie algebras. The main result stated in [61] for filiform Lie algebras is a consequence of the previous theorem. In fact, by taking $m = 2n - 1$ and $j = 1$ we obtain that $k < 2$. Thereby, there is no generalized complex structure of type n excepted for $n = 1$ but in this case the algebra is abelian. In the next section, we are going to study the quasi-filiform case for which $m = 2n - 2$.

9.4.4 Classification of naturally graded quasi-filiform Lie algebras

Let $\mathfrak{g}$ be nilpotent Lie algebra with nilindex m. This algebra is naturally filtered by the descending sequence of derived ideals:

$$\mathfrak{g}^0 = \mathfrak{g} \supset \mathfrak{g}^1 \supset \mathfrak{g}^2 \supset \cdots \supset \mathfrak{g}^k \supset \cdots \supset \mathfrak{g}^m = \{0\}.$$

We can associate a graded Lie algebra to $\mathfrak{g}$ which is usually denoted by $\mathrm{gr}(\mathfrak{g})$, and defined by:

$$\mathrm{gr}(\mathfrak{g}) = \sum_{i=1}^{m} \frac{\mathfrak{g}^{i-1}}{\mathfrak{g}^i} = \sum_{i=1}^{m} W_i,$$

with the brackets:

$$[X+\mathfrak{g}^i, Y+\mathfrak{g}^j] = [X,Y]+\mathfrak{g}^{i+j}, \quad \forall X \in \mathfrak{g}^{i-1},\ \forall Y \in \mathfrak{g}^{j-1}.$$

By definition, a Lie algebra is naturally graded if $\mathfrak{g}$ and $\mathrm{gr}(\mathfrak{g})$ are isomorphic Lie algebras. We say that the algebra $\mathfrak{g}$ has the form $\{p_1,\ldots,p_m\}$ when $\dim W_i = p_i$. Clearly, the graded Lie algebra $\mathrm{gr}(\mathfrak{g})$ has the same form as $\mathfrak{g}$.
Note that a nilpotent Lie algebra is filiform if and only if it has the form $\{2,1,1,\ldots,1\}$. Therefore, the graded algebra of a filiform Lie algebra is also filiform.

Definition 89 *Let $\mathfrak{g}$ be a nilpotent Lie algebra , $\mathfrak{g}$ is said quasi-filiform if its nilindex m is equal to dim $\mathfrak{g}-2$.*

If $\mathfrak{g}$ is quasi-filiform, there are two possibilities:

1. either $\mathfrak{g}$ has the form $t_1 = \{p_1 = 3,\, p_2 = 1,\, p_3 = 1,\, \ldots, p_m = 1\}$.
2. or $\mathfrak{g}$ has the form $t_r = \{p_1 = 2,\, p_2 = 1, \ldots, p_{r-1} = 1,\, p_r = 2,\, p_{r+1} = 1, \ldots, p_m = 1\}$ where $r \in \{2,\ldots,m\}$.

Proposition 76 *Let $\mathfrak{g}$ be a quasi-filiform naturally graded Lie algebra of dimension $2n$. If $\mathfrak{g}$ has the form t_r with $r \in \{1,\ldots,2n-2\}$ then there exists a homogeneous basis $\{X_0, X_1, X_2, \ldots, X_{2n-1}\}$ of $\mathfrak{g}$ in which X_0 and X_1 belong to W_1, $X_i \in W_i$ for $i \in \{2,\ldots,2n-2\}$ and $X_{2n-1} \in W_r$. Furthermore, $\mathfrak{g}$ is given in this basis by one of the algebras below.*

1. *If $\mathfrak{g}$ has the form t_1*
   $\mathfrak{L}_{2n-1}\oplus\mathbb{R} \quad (n \geq 2)$
   $$\{[X_0, X_i] = X_{i+1}, \quad 1 \leq i \leq 2n-3.$$
2. *If $\mathfrak{g}$ has the form t_r with $r \in \{2,\ldots,2n-2\}$*
   (a) $\mathfrak{L}_{2n,r}; \quad n \geq 3,\ r \text{ odd},\ 3 \leq r \leq 2n-3$
   $$\begin{cases} [X_0, X_i] = X_{i+1}, & i = 1,\ldots,2n-3,\\ [X_i, X_{r-i}] = (-1)^{i-1}X_{2n-1}, & i = 1,\ldots,\frac{r-1}{2}. \end{cases}$$
   (b) $\mathfrak{T}_{2n,2n-3}; \quad n \geq 3$
   $$\begin{cases} [X_0, X_i] = X_{i+1}, & i = 1,\ldots,2n-4,\\ [X_0, X_{2n-1}] = X_{2n-2}, & \\ [X_i, X_{2n-3-i}] = (-1)^{i-1}X_{2n-1}, & i = 1,\ldots,n-2,\\ [X_i, X_{2n-2-i}] = (-1)^{i-1}(n-1-i)X_{2n-2}, & i = 1,\ldots,n-2. \end{cases}$$
   (c) $\mathfrak{n}_6^{10}$
   $$\begin{cases} [X_0, X_i] = X_{i+1}, \quad i = 1,2,3,\\ [X_1, X_2] = X_5,\\ [X_1, X_5] = X_4, \end{cases}$$

The non written brackets are zero, excepted those that follow from antisymmetry.

In order to obtain this classification, we have to revised the complex one ([37]). For example, if $\mathfrak{g}$ is a quasi-filiform Lie algebra of dimension 6 with the form t_3, then there exists a basis $\{X_0, X_1, \ldots, X_5\}$ satisfying

$$\begin{cases} [X_0, X_i] = X_{i+1},\ i = 1,2,3,\\ [X_1, X_3] = bX_4,\\ [X_1, X_2] = bX_3 - X_5,\\ [X_5, X_1] = aX_4. \end{cases}$$

When $a = b = 0$, $\mathfrak{g}$ and $\mathfrak{L}_{6,3}$ are isomorphic algebras. In other way, by making the change of basis

$$Y_0 = \alpha X_0,\ Y_1 = \beta X_1 + X_0,\ Y_2 = \alpha\beta X_2,\ Y_3 = \alpha^2\beta X_3,\ Y_4 = \alpha^3\beta X_4,\ Y_5 = -\alpha\beta^2 X_5,$$

with $\beta = \begin{cases} -\frac{1}{b-\sqrt{|a|}} & \text{if} \quad b \neq \sqrt{|a|}, \\ -\frac{1}{2\sqrt{|a|}} & \text{if} \quad b = \sqrt{|a|}, \end{cases}$ and $\alpha = b\beta + 1$, we obtain the brackets

$$\begin{cases} [Y_0, Y_i] = Y_{i+1},\ i = 1, 2, 3, \\ [Y_1, Y_3] = Y_4, \\ [Y_1, Y_2] = Y_3 + Y_5, \\ [Y_5, Y_1] = \delta Y_4,\ \delta = \pm 1. \end{cases}$$

With another change of basis, we can see that $\mathfrak{g}$ is isomorphic to the algebras $\mathfrak{T}_{6,3}$ for $\delta = 1$ and $\mathfrak{n}_6^{10}$ for $\delta = -1$. We remark that, those algebras $\mathfrak{T}_{6,3}$ and $\mathfrak{n}_6^{10}$ are isomorphic as complex algebras. Beyond dimension 6, the way of construction in the complex case gives the real classification.

Corollary 90 *Let $\mathfrak{g}$ be a quasi-filiform Lie algebra of dimension $2n$. Then there exists a basis $\{X_0, X_1, X_2, \dots, X_{2n-1}\}$ of $\mathfrak{g}$ such that:*

1. *If* $\mathrm{gr}(\mathfrak{g}) \simeq \mathfrak{L}_{2n-1} \oplus \mathbb{R} \quad (n \geq 2)$,

$$\begin{cases} [X_0, X_i] = X_{i+1}, & 1 \leq i \leq 2n-3, \\ [X_i, X_j] = \sum_{k=i+j+1}^{2n-2} C_{i,j}^k X_k, & 1 \leq i < j \leq 2n-3-i, \\ [X_i, X_{2n-1}] = \sum_{k=i+2}^{2n-2} C_{i,2n-1}^k X_k, & 1 \leq i \leq 2n-4. \end{cases}$$

2. *If* $\mathrm{gr}(\mathfrak{g}) \simeq \mathfrak{L}_{2n,r} \quad n \geq 3,\ r : \text{odd},\ 3 \leq r \leq 2n-3$,

$$\begin{cases} [X_0, X_i] = X_{i+1}, & i = 1, \dots, 2n-3, \\ [X_0, X_{2n-1}] = \sum_{k=r+2}^{2n-2} C_{0,2n-1}^k X_k, & \\ [X_i, X_j] = \sum_{k=i+j+1}^{2n-1} C_{i,j}^k X_k, & 1 \leq i < j \leq r-i-1, \\ [X_i, X_j] = \sum_{k=i+j+1}^{2n-2} C_{i,j}^k X_k, & 1 \leq i < j \leq 2n-3-i,\ r < i+j, \\ [X_i, X_{2n-1}] = \sum_{k=r+i+1}^{2n-2} C_{i,2n-1}^k X_k, & 1 \leq i \leq 2n-3-r, \\ [X_1, X_{r-1}] = X_{2n-1}, & \\ [X_i, X_{r-i}] = (-1)^{(i-1)} X_{2n-1} + \sum_{k=r+1}^{2n-2} C_{i,r-i}^k X_k, & 2 \leq i \leq \frac{r-1}{2}. \end{cases}$$

3. *If* $\mathrm{gr}(\mathfrak{g}) \simeq \mathfrak{T}_{2n,2n-3} \quad n \geq 3$,

$$\begin{cases} [X_0, X_i] = X_{i+1}, & i = 1, \dots, 2n-4, \\ [X_0, X_{2n-1}] = X_{2n-2}, & \\ [X_i, X_j] = \sum_{k=i+j+1}^{2n-1} C_{i,j}^k X_k, & 1 \leq i < j \leq 2n-4-i, \\ [X_1, X_{2n-4}] = X_{2n-1}, & \\ [X_i, X_{2n-3-i}] = (-1)^{(i-1)} X_{2n-1} + C_{i,2n-3-i}^{2n-2} X_{2n-2}, & 2 \leq i \leq n-2. \end{cases}$$

4. *If* $\mathrm{gr}(\mathfrak{g}) \simeq \mathfrak{n}_6^{10}$ *then* $\mathfrak{g} \simeq \mathfrak{n}_6^{10}$,

$$\begin{cases} [X_0, X_i] = X_{i+1}, \quad i = 1, 2, 3, \\ [X_1, X_2] = X_5, \\ [X_1, X_5] = X_4. \end{cases}$$

Such a basis $\{X_0, X_1, X_2, \dots, X_{2n-1}\}$ is called an adapted basis of $\mathfrak{g}$.

9.4.5 Complex structures on quasi-filiform Lie algebras

The aim of this section is to find the quasi-filiform Lie algebras endowed with a complex structure or equivalently a generalized complex structure of type $k = n$. If $\mathfrak{g}$ has the form t_1, Theorem 88 says that $k = n = 2$ so the algebra $\mathfrak{g}$ is isomorphic to $\mathfrak{L}_3 \oplus \mathbb{R}$. We can check that this algebra admits a complex structure associated to the pure spinor

$$\Omega = (\omega_0 + i\omega_1) \wedge (\omega_2 + i\omega_3),$$

where $\{\omega_0, \omega_1, \omega_2, \omega_3\}$ denotes the dual basis corresponding to the homogeneous basis $\{X_0, X_1, X_2, X_3\}$ of Proposition 76. Assume that $\mathfrak{g}$ is a quasi-filiform Lie algebra the form t_r with $r \geq 3$. According to Theorem 88, $n = k \leq r$.

Lemma 15 *Let $\mathfrak{g}$ be a quasi-filiform Lie algebra of the form t_r with $r \geq 3$. If $\mathfrak{g}$ admits a generalized complex structure of the type k, then we can choose the complex forms $\theta_1, \dots, \theta_k$ corresponding to the generalized complex structure, satisfying either:*

$$\mathit{nil}(\theta_1) = 1,\ \mathit{nil}(\theta_2) = r,\ \mathit{nil}(\theta_3) = r + 1, \dots, \mathit{nil}(\theta_k) = r + k - 2$$

or

$$\mathit{nil}(\theta_1) = 1,\ \mathit{nil}(\theta_2) = r,\ \mathit{nil}(\theta_3) = r \dots, \mathit{nil}(\theta_k) = r + k - 3$$

and in this case, $k < r$.

Proof. Let us consider an appropriate decomposition $\{\theta_1 \dots \theta_k\}$. From Corollary 87, we deduce that $\mathrm{nil}(\theta_1) = 1$ and $\mathrm{nil}(\theta_2) \in \{1, 2, r\}$. As $\dim V_1 = 2$ and $\dim V_2 = 3$, Condition (9.2) implies $\mathrm{nil}(\theta_2) = r$. According to Corollary 87:

$$\mathrm{nil}(\theta_{i-1}) \leq \mathrm{nil}(\theta_i) \leq r + i - 2 \quad i = 3, \dots, k.$$

Indeed, there are two possible values for $\mathrm{nil}(\theta_3)$:

1. $\mathrm{nil}(\theta_3) = r + 1$
   If we suppose that $\mathrm{nil}(\theta_4) = \mathrm{nil}(\theta_3) = r + 1$, the forms θ_4 and θ_3 belong to V_{r+1} and as they are linearly independent modulo V_r, this leads to $\dim\left(\frac{V_{r+1}}{V_r}\right) \geq 2$ in contradiction with $\dim\left(\frac{V_{r+1}}{V_r}\right) = 1$. We deduce that $\mathrm{nil}(\theta_4) = r + 2$ and by the same way we obtain:

   $$\mathrm{nil}(\theta_i) = r + i - 2, \quad \text{for } i = 3, \dots, k.$$

2. $\mathrm{nil}(\theta_3) = r$
   By using similar arguments, we can prove that $\mathrm{nil}(\theta_i) = r + i - 3$ for $i = 3, \dots, k$. In this case, we remark that, when $k = r$, the nilindex of θ_r is equal to $2r - 3$ and then $\dim V_{2r-3} \geq 2r$. This is impossible because $\dim V_{2r-3} = 2r - 1$. Indeed $k < r$.

Example 91 *Let us consider a quasi-filiform Lie algebra $\mathfrak{g}$ of dimension 6 defined in the basis $\{X_0, X_1, \dots, X_5\}$ by:* ▌

$$\begin{cases} [X_0, X_i] = X_{i+1}, & i = 1, 2, 3, \\ [X_1, X_2] = X_5, & \\ [X_1, X_5] = \delta X_4, & \delta \in \{0, 1, -1\}. \end{cases}$$

Let us suppose that $\mathfrak{g}$ admits a complex structure which determines a generalized complex structure of the type $k = 3$ and with the spinor:

$$\Omega = \theta_1 \wedge \theta_2 \wedge \theta_3,$$

where θ_1, θ_2 and θ_3 are complex forms. Note that this algebra has the form t_3 and according to the previous lemma the corresponding nilindices are given by:

$$\mathit{nil}(\theta_1) = 1,\ \mathit{nil}(\theta_2) = 3,\ \mathit{nil}(\theta_3) = 4.$$

The complex forms θ_1, θ_2 and θ_3 can be written as:

$$\begin{aligned}
\theta_1 &= \lambda_0\omega_0 + \lambda_1\omega_1,\\
\theta_2 &= \beta_0\omega_0 + \beta_1\omega_1 + \beta_2\omega_2 + \beta_3\omega_3 + \beta_5\omega_5,\\
\theta_3 &= \gamma_0\omega_0 + \gamma_1\omega_1 + \gamma_2\omega_2 + \gamma_3\omega_3 + \gamma_4\omega_4 + \gamma_5\omega_5,
\end{aligned}$$

where $\omega_1, \cdots, \omega_5$ is the dual basis of $X_0, \cdots, X_6$, $\lambda_i, \beta_i, \gamma_i \in \mathbb{C}$, γ_4 is non-zero and β_3, β_5 can not be simultaneously zero. Moreover, the condition $\theta_1 \wedge \overline{\theta_1} \neq 0$ means that the imaginary part of $\lambda_0\overline{\lambda_1}$ is non-zero. Corollary 87 leads to:

$$\begin{cases} \theta_1 \wedge d\theta_2 = 0, \\ \theta_1 \wedge \theta_2 \wedge d\theta_3 = 0, \end{cases}$$

that is,

$$\begin{cases}
\beta_5\lambda_0 - \beta_3\lambda_1 & = 0,\\
-\gamma_3\beta_3\lambda_1 + \gamma_4\beta_2\lambda_1 + \gamma_5\beta_3\lambda_0 & = 0,\\
\gamma_4(\beta_5\lambda_1 + \delta\beta_3\lambda_0) & = 0,\\
-\gamma_3\beta_5\lambda_1 - \delta\gamma_4\beta_2\lambda_0 + \gamma_5\beta_5\lambda_0 & = 0.
\end{cases}$$

From the first and third equations, we deduce:

$$\lambda_1^2 + \delta\lambda_0^2 = 0.$$

For $\delta = 0$, this is in contradiction with $\theta_1 \wedge \overline{\theta_1} \neq 0$. If $\delta = -1$, we deduce that $\lambda_1 = \pm\lambda_0$ and since the spinor is uniquely defined up to a multiplicative constant, we can take $\theta_1 = \omega_0 \pm \omega_1$ in contradiction with $\theta_1 \wedge \overline{\theta_1} \neq 0$. Finally, when $\delta = 1$, the spinor $\Omega = (\omega_0 + i\omega_1) \wedge (\omega_3 + i\omega_5) \wedge (\omega_2 + i\omega_4)$ is associated to a complex structure of $\mathfrak{g}$. We conclude that the Lie algebra $\mathfrak{g}$ admits a complex structure if and only if $\delta = 1$.

Theorem 92 *Let $\mathfrak{g}$ be a real quasi-filiform Lie algebra endowed with a complex structure. Then $\mathfrak{g}$ is isomorphic either to the four-dimensional algebra $\mathfrak{L}_3 \oplus \mathbb{R}$ or to the algebra $\mathfrak{n}_6^{10}$ of dimension 6.*

Proof. Let $\mathfrak{g}$ be a $2n$-dimensional real quasi-filiform Lie algebra of the form t_r with $r \in \{1, 3, \ldots, 2n-3\}$. Let us suppose that $\mathfrak{g}$ admits a complex structure which determines a generalized complex structure of the type $k = n$.
For $r = 1$, $\mathfrak{g}$ is isomorphic to $\mathfrak{L}_3 \oplus \mathbb{R}$ and this algebra admits a complex structure. Henceforth, we assume $r \in \{3, \ldots, 2n-3\}$. Applying Theorem 88 and the inequality

$$\mathrm{nil}(\theta_k) = \mathrm{nil}(\theta_n) \leq \mathrm{nil}(\mathfrak{g})$$

in each of the possibilities of Lemma 15, we deduce:

1. If $\mathrm{nil}(\theta_3) = r + 1$ then $\mathrm{nil}(\theta_k) = r + k - 2$ and:

$$n = k \leq r \leq n \;\Rightarrow\; r = n.$$

2. If $\mathrm{nil}(\theta_3) = r$ then $\mathrm{nil}(\theta_k) = r + k - 3$ and in this case, $k < r$ so

$$n = k < r \leq n + 1 \;\Rightarrow\; r = n + 1.$$

Moreover, since the graduate algebra $gr(\mathfrak{g})$ must be isomorphic to one of the algebras of Proposition 76 we obtain the following possibilities:

1. $gr(\mathfrak{g}) \sim \mathfrak{L}_{2n,r}$; $\quad n \geq 3$, r odd, $3 \leq r \leq 2n-3$.
   (a) If $\mathrm{nil}(\theta_3) = r + 1$ then $gr(\mathfrak{g}) \sim \mathfrak{L}_{2n,n}$ with $n \geq 3$ odd.
   For $n = 3$, $\mathfrak{g}$ is the algebra of Example 91 with $\delta = 0$ and it does not admit a complex structure.

Suppose $n > 3$. If $\{X_0, X_1, \ldots, X_{2n-1}\}$ is an adapted basis of $\mathfrak{g}$ and $\{\omega_0, \omega_1, \ldots, \omega_{2n-1}\}$ is its dual basis, then

$$\left\{\begin{array}{l} \theta_1 = \lambda_1^0\omega_0 + \lambda_1^1\omega_1, \\ \theta_2 = \sum\limits_{k=0}^{n} \lambda_2^k\omega_k + \lambda_2^{2n-1}\omega_{2n-1}. \end{array}\right.$$

Since $\theta_1 \wedge d\theta_2 = 0$, grouping together $\omega_0 \wedge \omega_1 \wedge \omega_{n-1}$, $\omega_0 \wedge \omega_2 \wedge \omega_{n-2}$ and $\omega_0 \wedge \omega_3 \wedge \omega_{n-3}$ in $\theta_1 \wedge d\theta_2$, it results:

$$\lambda_2^n = \lambda_2^{2n-1} = 0.$$

This is impossible since $\mathrm{nil}(\theta_2) = n$. Hence, there are no complex structure, excepted for $n = 3$. Suppose $\mathrm{nil}(\theta_3) = r$ then $gr(\mathfrak{g}) \sim \mathfrak{L}_{2n,n+1}$ with $n \geq 4$ even. We can write θ_1 and θ_2 as:

$$\left\{\begin{array}{l} \theta_1 = \lambda_1^0\omega_0 + \lambda_1^1\omega_1, \\ \theta_2 = \sum\limits_{k=0}^{n+1} \lambda_2^k\omega_k + \lambda_2^{2n-1}\omega_{2n-1}. \end{array}\right.$$

where $\{\omega_0, \omega_1, \ldots, \omega_{2n-1}\}$ is the dual basis of an adapted basis of $\mathfrak{g}$. In the equation $\theta_1 \wedge d\theta_2 = 0$, the coefficients of $\omega_0 \wedge \omega_1 \wedge \omega_n$ and $\omega_0 \wedge \omega_2 \wedge \omega_{n-1}$ give:

$$\left\{\begin{array}{l} \lambda_1^0\lambda_2^{2n-1} - \lambda_1^1\lambda_2^{n+1} = 0, \\ \lambda_1^1\lambda_2^{2n-1} = 0. \end{array}\right.$$

Since $\theta_1 \wedge \overline{\theta_1} \neq 0$, we deduce that $\lambda_2^{n+1} = \lambda_2^{2n-1} = 0$ contradicting $\mathrm{nil}(\theta_2) = n + 1$.

2. $gr(\mathfrak{g}) \sim \mathfrak{T}_{2n,2n-3}$; $\quad n \geq 3$

   (a) If $\mathrm{nil}(\theta_3) = r + 1$ then $gr(\mathfrak{g}) \sim \mathfrak{T}_{6,3}$. In this case $\mathfrak{g}$ is isomorphic to the algebra of Example 91 with $\delta = -1$ which do not admit any complex structure.

   (b) When $\mathrm{nil}(\theta_3) = r$, $gr(\mathfrak{g}) \sim \mathfrak{T}_{8,5}$ and there is an adapted basis $\{X_0, X_1, \ldots, X_7\}$ of $\mathfrak{g}$ satisfying:

$$\left\{\begin{array}{ll} [X_0, X_i] = X_{i+1}, & i = 1, \ldots, 4, \\ [X_0, X_7] = X_6, & \\ [X_1, X_i] = \sum_{k=i+2}^{7} C_{1,i}^k X_k, & i = 2, 3, \\ [X_1, X_4] = X_7, & \\ [X_1, X_5] = 2X_6, & \\ [X_2, X_4] = -X_6, & \end{array}\right.$$

   In the dual basis $\{\omega_0, \omega_1, \ldots, \omega_7\}$, we can write θ_1, θ_2 and θ_3 as:

$$\left\{\begin{array}{l} \theta_1 = \lambda_1^0\omega_0 + \lambda_1^1\omega_1, \\ \theta_2 = \sum\limits_{k=0}^{5} \lambda_2^k\omega_k + \lambda_2^7\omega_7, \\ \theta_3 = \sum_{k=0}^{5} \lambda_3^k\omega_k + \lambda_3^7\omega_7. \end{array}\right.$$

   According to Corollary 87, $\theta_1 \wedge d\theta_2 = 0$ and $\theta_1 \wedge d\theta_3 = 0$. Thus:

$$\left\{\begin{array}{ll} \lambda_1^0\lambda_2^7 - \lambda_1^1\lambda_2^5 & = 0, \\ \lambda_1^0\lambda_3^7 - \lambda_1^1\lambda_3^5 & = 0. \end{array}\right.$$

   Assuming that $\lambda_1^1 = \lambda_2^7 = \lambda_3^7 = 1$, we deduce $\lambda_1^0 = \lambda_2^5 = \lambda_3^5$, in contradiction with the choice of θ_2 et θ_3 since they are linearly independent modulo V_4.

3. $gr(\mathfrak{g}) \sim \mathfrak{n}_6^{10}$. $\mathfrak{g}$ is isomorphic to the algebra $\mathfrak{n}_6^{10}$, which is the algebra of Example 91 with $\delta = 1$ admitting a complex structure.

Corollary 93 *Let $\mathfrak{g}$ be a $2n$-dimensional real Lie algebra with $n \geq 4$. If $\mathfrak{g}$ is provided with a complex structure then its characteristic sequence $s(\mathfrak{g})$ satisfies $s(\mathfrak{g}) \leq (2n-2, 1, 1)$.*

9.5 Lie Algebra $\mathfrak{n}_6^{10}$

The mentioned paper [110] listed the 6-dimensional nilpotent Lie algebras endowed with a complex structure, we verify that $\mathfrak{n}_6^{10}$ is the only quasi-filiform Lie algebra in this classification. Our aim is now to write precisely all the complex structures on this algebra.

We say that two complex structures J_1 and J_2 on a real Lie algebra $\mathfrak{g}$ are equivalent if there exists an automorphism $\sigma \in Aut(\mathfrak{g})$ such that $\sigma J_1 = J_2 \sigma$.

Proposition 77 *The algebra $\mathfrak{n}_6^{10}$ has only two non-equivalent complex structures.*

This result has been proved by Magnin [82]. If we consider the commutation relations of the basis $\{X_0, X_1, \ldots, X_5\}$:

$$\begin{cases} [X_0, X_i] = X_{i+1}, \quad i = 1, 2, 3, \\ [X_1, X_2] = X_5, \\ [X_1, X_5] = X_4, \end{cases}$$

the non-equivalent complex structures can be expressed by the matrix

$$J(\zeta) \sim \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \zeta & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & \zeta & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \tag{9.4}$$

where $\zeta = \pm 1$.

The study of Kähler structures on a nilmanifold was initiated by [24] and completed by [11]. We can quickly look at this result for this special case.

In order to compute all the symplectic structures of the algebra $\mathfrak{n}_6^{10}$, we consider a skew-symmetric 2-form $\omega = \sum_{0 \leq i < j \leq 5} \lambda_{i,j} \omega_i \wedge \omega_j$ with $\{\omega_0, \omega_1, \ldots, \omega_5\}$ the dual of the preceding basis and we impose Conditions (9.1). We deduce that the symplectic structures of $\mathfrak{n}_6^{10}$ are given by:

$$\begin{aligned} \omega =& \lambda_{0,1}\omega_0 \wedge \omega_1 + \lambda_{0,2}\omega_0 \wedge \omega_2 + \lambda_{0,3}\omega_0 \wedge \omega_3 + \lambda_{0,4}\omega_0 \wedge \omega_4\omega_0 \wedge \omega_4 + \lambda_{0,5}\omega_0 \wedge \omega_5 \\ &+ \lambda_{1,2}\omega_1 \wedge \omega_2 + \lambda_{0,5}\omega_1 \wedge \omega_3 + \lambda_{1,4}\omega_1 \wedge \omega_4 + \lambda_{1,5}\omega_1 \wedge \omega_5 - \lambda_{1,4}\omega_2 \wedge \omega_3 + \lambda_{0,4}\omega_2 \wedge \omega_5 \end{aligned}$$

with

$$\lambda_{0,3}\lambda_{0,4}\lambda_{1,4} + \lambda_{0,4}^2\lambda_{0,5} + \lambda_{0,4}\lambda_{1,4}\lambda_{1,5} - \lambda_{0,5}\lambda_{1,4}^2 \neq 0. \tag{9.5}$$

A symplectic form ω of a Lie algebra $\mathfrak{g}$ is a Kähler structure if there exists a complex structure J which is compatible with ω, that is,

$$\omega(JX, JY) = \omega(X, Y) \quad \forall X, Y \in \mathfrak{g}.$$

Note that if a symplectic structure ω is compatible with a complex structure J_1 and J_1 is equivalent to another complex structure J_2 then ω is also compatible with J_2. Thus, to determine the Kähler structures of $\mathfrak{n}_6^{10}$ is enough to compute the symplectic forms compatible with the complex structures $J(\pm 1)$ of (9.4). We obtain the extra conditions

$$\begin{cases} \lambda_{0,2} = \lambda_{0,4} = \lambda_{0,5} = \lambda_{1,2} = \lambda_{1,4} = 0, \\ \lambda_{0,3} = \lambda_{1,5}, \end{cases} \tag{9.6}$$

which is in contradiction with (9.5). We conclude that there is no Kähler structure on $\mathfrak{n}_6^{10}$.

Chapter 10

Affine structures on nilpotent Lie algebras

An affine structure on a Lie algebra corresponds to a left invariant affine structure on a connected Lie group whose Lie algebra is isomorphic to the given one. This left invariant affine structure is given by an affine connection ∇ whose curvature and torsion tensors are null. As this connection operator ∇ is left invariant, it determines a bilinear operator on the Lie algebra which corresponds to a Vinberg product, also called left-symmetric product. Then the problem of existence of a left invariant affine structure on a connected Lie group is equivalent to the problem of existence of a Vinberg product on a Lie algebra such that the Lie bracket obtained by skew-symmetrization of the Vinberg product is the Lie bracket of the initial Lie algebra. Recall that a Vinberg algebra is a G_3-associative algebra. From Benoist's work, we know that nilpotent Lie algebras with no affine structure exist. Thus the problem of characterizing the class of nilpotent Lie algebras provided with an affine structure is open. A trivial example of a nilpotent Lie algebra with an affine structure is the abelian Lie algebras. In this case, an interesting geometrical problem is to classify all non equivalent affine structures; each of these structures describes an affine geometry on a real vector space. Recently, the works of Kim [68] and Dekimpe-Ongenae [25] precise the number of non equivalent invariant complete affine structures on an abelian Lie group (complete means that the geodesics are globally defined). In this chapter we give the classification of affine structures on the 3-dimensional abelian Lie algebra, which corresponds to the complete or not invariant affine structures on $\mathbb{R}^3$. The Nagano-Yagi-Goldmann theorem states that on the torus $\mathbb{T}^2$, every affine (or projective) structure is invariant or is constructed on the basis of some Goldmann rings [35]. An interesting problem that arises is to study the invariant affine structure on the torus $\mathbb{T}^2$. The presented classification permits to describe all the affine structures, complete or not, on the torus $\mathbb{T}^2$ and $\mathbb{T}^3$.

The second part is devoted to the existence problem of an affine structure on nilpotent Lie algebras. We first prove that every filiform non characteristically nilpotent Lie algebra admits an affine structure. Then we investigate the existence problem for contact Lie algebras. It is known that any nilpotent Lie algebras provided with a symplectic form admits an affine structure. In this part, we give a necessary condition for a nilpotent $(2p+1)$-dimensional nilpotent Lie algebra with a contact form (that is, a linear form α satisfying $\alpha \wedge (d\alpha)^p \neq 0$) to have an affine structure. We finish this chapter by giving a tensor product process to construct explicit affine Lie algebras of high dimensions.

10.1 Recalls : Affine structures on Lie groups and Lie algebras

The Lie group $Aff(\mathbb{R}^n)$ is the group of affine transformations of $\mathbb{R}^n$. It is constituted of matrices

$$\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}$$

with $A \in GL(n,\mathbb{R})$, $b \in \mathbb{R}^n$. It acts on the real affine space $\widetilde{\mathbb{R}}^n$ by

$$\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v \\ 1 \end{pmatrix} = \begin{pmatrix} Av+b \\ 1 \end{pmatrix}$$

where $(v,1)^t \in \widetilde{\mathbb{R}}^{n+1}$.

Its Lie algebra, noted $aff(\mathbb{R}^n)$, is the linear algebra

$$aff(\mathbb{R}^n) = \left\{ \begin{pmatrix} A & b \\ 0 & 0 \end{pmatrix}; A \in gl(n,\mathbb{R}), b \in \mathbb{R}^n \right\}$$

Definition 94 *An affine structure on a Lie algebra $\mathfrak{g}$ is a morphism*

$$\Psi : \mathfrak{g} \to aff(\mathbb{R}^n)$$

of Lie algebras.

Remark. Let us consider an affine representation of a Lie group G, that is, an homomorphism $\varphi : G \to Aff(\mathbb{R}^n)$. For every $g \in G$, $\varphi(g)$ is an affine transformation on the affine space $\widetilde{\mathbb{R}}^{n+1}$. This representation induces an affine structure on the Lie algebra $\mathfrak{g}$ of G.

Proposition 78 *The Lie algebra $\mathfrak{g}$ is provided with an affine structure if and only if the underlying vector space $A(\mathfrak{g})$ is a left symmetric algebra, that is, there exists a bilinear map*

$$\begin{array}{ccc} A(\mathfrak{g}) \times A(\mathfrak{g}) & \to & A(\mathfrak{g}) \\ (X,Y) & \mapsto & X \cdot Y \end{array}$$

satisfying

1) $X \cdot (Y \cdot Z) - Y \cdot (X \cdot Z) = (X \cdot Y) \cdot Z - (Y \cdot X) \cdot Z$

2) $X \cdot Y - Y \cdot X = [X,Y]$

for every $X, Y, Z \in A(\mathfrak{g})$

If Ψ is a morphism giving an affine structure on $\mathfrak{g}$, then the left symmetric product on $A(\mathfrak{g})$ is defined as this:

$$\forall X \in \mathfrak{g}, \Psi(X) = \begin{pmatrix} A(X) & b(X) \\ 0 & 0 \end{pmatrix}$$

and we put

$$X \cdot Y = Y = b^{-1}(A(X) \cdot b(X))$$

where $b : A(\mathfrak{g}) \to \mathbb{R}^n$ is supposed to be bijective.

The fact that Ψ is a representation implies that $X \cdot Y$ is a left symmetric product. Conversely, if $X \cdot Y$ is a left symmetric product on $A(\mathfrak{g})$, and if L_X indicates the left representation : $L_X(Y) = X \cdot Y$, then the map

$$X \to \begin{pmatrix} L_X & X \\ 0 & 0 \end{pmatrix}$$

defines an affine structure on the Lie algebra $\mathfrak{g}$.

If Ψ is an affine structure on $\mathfrak{g}$, it defines a representation

$$\Psi(X) = \begin{pmatrix} A(X) & b(X) \\ 0 & 0 \end{pmatrix}$$

and a left symmetric product $X \cdot Y$. This last induces an affine representation

$$\begin{pmatrix} L_X & X \\ 0 & 0 \end{pmatrix}$$

which is equivalent to Ψ (and equal if $b = Id$).

Definition 95 *An affine structure on $\mathfrak{g}$ is called complete if the endomorphism*

$$\begin{array}{cccc} \theta_X : & A(\mathfrak{g}) & \rightarrow & A(\mathfrak{g}) \\ & Y & \mapsto & Y + Y \cdot X \end{array}$$

is bijective for every $X \in A(\mathfrak{g})$.

This is equivalent to one of the following properties:

a)

$$\begin{array}{cccc} R_X : & A(\mathfrak{g}) & \rightarrow & A(\mathfrak{g}) \\ & Y & \mapsto & Y \cdot X \end{array}$$

is nilpotent for all $X \in A(\mathfrak{g})$

b) $tr(R_X) = 0$ for all $X \in A(\mathfrak{g})$.

Remarks

1. On the 3-dimensional Heisenberg algebra the affine structure associated to the following representation

$$\begin{pmatrix} a(x_1+x_2) & a(x_1+x_2) & 0 & x_1 \\ a(x_1+x_2) & a(x_1+x_2) & 0 & x_2 \\ \alpha x_1 + (\beta-1)x_2 & \beta x_1 + (\alpha+1)x_2 & 0 & x_3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

is non complete. Such a structure will be studied latter.

2. The complete affine structure on $\mathfrak{g}$ corresponds to the simply-transitive affine action of the connected corresponding Lie group G.

10.2 Affine structure on abelian Lie algebras

10.2.1 Link with commutative and associative algebras

Let $\mathfrak{g}$ be a real abelian Lie algebra. If $\mathfrak{g}$ is provided with an affine structure then the left symmetric algebra $A(\mathfrak{g})$ is a commutative and associative real algebra. In fact, we have

$$X \cdot Y - Y \cdot X = 0 = [X, Y]$$

and

$$X \cdot (Y \cdot Z) - Y \cdot (X \cdot Z) = ([X, Y] \cdot Z) = 0.$$

This gives

$$X \cdot (Z \cdot Y) = (X \cdot Z) \cdot Y$$

and $A(\mathfrak{g})$ is associative.

Let Ψ_1 and Ψ_2 be two affine structures on $\mathfrak{g}$. They are affinely equivalent if and only if the corresponding commutative and associative algebras are isomorphic. Thus the classification of affine structures on abelian Lie algebras corresponds to the classification of commutative and associative (unitary or not) algebras. If the affine structure is complete, the endomorphisms R_X are nilpotent. As $A(\mathfrak{g})$ is also commutative, it is a nilpotent associative commutative algebra. The classification of complete affine structures on abelian Lie algebras corresponds to the classification of nilpotent associative algebras. In this frame, we can cite the works of Gabriel [?] and Mazzola [95] who study the varieties of unitary associative complex laws and give their classification for dimensions less than 5.

10.2.2 Affine structures on the 2-dimensional abelian Lie algebra

Classification of commutative associative 2-dimensional real algebras

• Suppose that the algebra A is unitary. Then its law is isomorphic to

$$\begin{cases} \mu_1(e_1,e_1)=e_1 \\ \mu_1(e_1,e_2)=e_2 \\ \mu_1(e_2,e_1)=e_2 \\ \mu_1(e_2,e_2)=e_2 \end{cases} ; \quad \begin{cases} \mu_2(e_1,e_1)=e_1 \\ \mu_2(e_1,e_2)=e_2 \\ \mu_2(e_2,e_1)=e_2 \\ \mu_2(e_2,e_2)=0 \end{cases} ; \quad \begin{cases} \mu_3(e_1,e_1)=e_1 \\ \mu_3(e_1,e_2)=e_2 \\ \mu_3(e_2,e_1)=e_2 \\ \mu_3(e_2,e_2)=-e_1 \end{cases}$$

• Suppose that A is nilpotent (and not unitary). The law is isomorphic to

$$\mu_4(e_1,e_1)=e_2 \quad ; \quad \mu_5=0$$

• Suppose that A is non nilpotent and non unitary. Then its law is isomorphic to

$$\mu_6(e_1,e_1)=e_1.$$

Description of the affine structures

Proposition 79 *There are 6 affinely non-equivalent affine structures on the 2-dimensional abelian Lie algebra.*

In the following table we give the affine structures on the 2-dimensional Lie algebra, the corresponding action and precise the completeness or not of these structures.

		affine structure	affine action	complete
A_1		$\begin{pmatrix} a & 0 & a \\ b & a+b & b \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} e^a & 0 & e^a-1 \\ e^a(e^b-1) & e^ae^b & e^a(e^b-1) \\ 0 & 0 & 1 \end{pmatrix}$	NO
A_2		$\begin{pmatrix} a & 0 & a \\ b & a & b \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} e^a & 0 & e^a-1 \\ be^a & e^a & be^a \\ 0 & 0 & 1 \end{pmatrix}$	NO
A_3		$\begin{pmatrix} a & -b & a \\ b & a & b \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} e^a\cos b & -e^a\sin b & 1-e^a\cos b \\ e^a\sin b & e^a\cos b & e^a\sin b \\ 0 & 0 & 1 \end{pmatrix}$	NO
A_4		$\begin{pmatrix} 0 & 0 & a \\ a & 0 & b \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & a \\ a & 1 & \frac{a^2}{2}+b \\ 0 & 0 & 1 \end{pmatrix}$	YES
A_5		$\begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$	YES
A_6		$\begin{pmatrix} a & 0 & a \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} e^a & 0 & e^a-1 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$	NO

On the group of affine transformations

To each affine structure A_i corresponds a flat affine connection without torsion ∇^i on the abelian Lie group G.

The set of all affine transformations of (G,∇^i) is a Lie group, noted $Aff(G,\nabla^i)$. Its Lie algebra is the set of complete affine vector fields ([3]), that is, complete vector field satisfying

$$\left[X,\nabla^i_Y Z\right]=\nabla^i_{[X,Y]}Z+\nabla^i_Y\left[X,Z\right]$$

If ∇^i is complete (in our case $i = 4, 5$) then the Lie algebra of $Aff(G, \nabla^i)$ is the Lie algebra of affine vector fields $aff(G, \nabla^i)$ and $Aff(G, \nabla^i)$ acts transitively on G.

Let us consider the corresponding affine action to ∇_i described in the previous section. The translation part defines an open set $U_i \subset \mathbb{R}^2$. For the complete case we obtain $U_i = \mathbb{R}^2$. For the non complete case we have

$$\begin{array}{rcl} U_1 & = & U_2 = U_6 = \left\{(x,y) \in \mathbb{R}^2 \,/\, x > -1\right\} \\ U_3 & = & \left\{(x,y) \in \mathbb{R}^2 \,,\, (x,y) \neq (1,0)\right\} \end{array}$$

If ϕ is an affine transformation which leaves U_i invariant, for $i = 1, 2, 6$ the matrix of ϕ is seen to have the following form

$$\begin{pmatrix} a & 0 & e^t - 1 \\ b & c & u \\ 0 & 0 & 1 \end{pmatrix}.$$

The group $Aff(G, \nabla^i)$ is the maximal group included in the semi group generated by the previous matrices.

Then the group $Aff(G, \nabla^i) = B_2 \times \mathbb{R}^2$, where B_2 is the subgroup of $GL(2, \mathbb{R})$ constituted of triangular matrices.

10.2.3 Affine structures on the 3-dimensional abelian Lie algebra

Classification of 3-dimensional commutative associative real algebras

Let us begin by describing the classification of real associative commutative algebras. Let $\mathfrak{a}$ be a 3-dimensional (not necessarily unitary) real commutative associative algebra.

If $\mathfrak{a}$ is simple, then $\mathfrak{a}$ is, following Wedderburn's theorem, isomorphic to $(M_1(\mathbb{R}))^3$, $M_1(\mathbb{R}) \oplus M_1(\mathbb{C})$, where $M_n(D)$ is a matrix algebra on a division algebra on $\mathbb{R}$, that is, $D = \mathbb{R}$ or $\mathbb{C}$. This gives the following algebras

$$\left\{\begin{array}{ll} \mu_1(e_1, e_i) = e_i & i = 1, 2, 3 \\ \mu_1(e_i, e_1) = e_i & i = 1, 2, 3 \\ \mu_1(e_2, e_2) = e_2 & \\ \mu_1(e_3, e_3) = e_3 & \end{array}\right. , \qquad \left\{\begin{array}{ll} \mu_2(e_1, e_i) = e_i & i = 1, 2, 3 \\ \mu_2(e_i, e_1) = e_i & i = 1, 2, 3 \\ \mu_2(e_2, e_2) = e_2 & \\ \mu_2(e_3, e_3) = e_2 - e_1 & \end{array}\right.$$

If $\mathfrak{a}$ is not simple, then $\mathfrak{a} = J(\mathfrak{a}) \oplus \mathfrak{s}$, where $\mathfrak{s}$ is simple and $J(\mathfrak{a})$ is the Jacobson radical of $\mathfrak{a}$. If $\mathfrak{s} = (M_1(\mathbb{R}))^2$, we obtain

$$\left\{\begin{array}{ll} \mu_3(e_1, e_i) = e_i & i = 1, 2, 3 \\ \mu_3(e_i, e_1) = e_i & i = 1, 2, 3 \\ \mu_3(e_2, e_2) = e_2 & \end{array}\right.$$

If $\mathfrak{s} = M_1(\mathbb{C})$

$$\left\{\begin{array}{ll} \mu_4(e_1, e_i) = e_i & i = 1, 2, 3 \\ \mu_4(e_i, e_1) = e_i & i = 1, 2, 3 \\ \mu_4(e_3, e_3) = e_2 & \end{array}\right.$$

where $J(\mathfrak{a})$ is not abelian or

$$\left\{\begin{array}{ll} \mu_5(e_1, e_i) = e_i & i = 1, 2, 3 \\ \mu_5(e_i, e_1) = e_i & i = 1, 2, 3 \end{array}\right.$$

where $J(\mathfrak{a})$ is abelian.

Suppose that $\mathfrak{a}$ is not unitary. As the Levi decomposition also holds in this case, we have the following possibilities : $\mathfrak{a} \simeq (M_1(\mathbb{R}))^2 \oplus J(\mathfrak{a})$ or $M_1(\mathbb{C}) \oplus J(\mathfrak{a})$. This gives the following algebras

$$\left\{\begin{array}{l} \mu_6(e_1, e_1) = e_1 \\ \mu_6(e_2, e_2) = e_2 \end{array}\right. , \qquad \left\{\begin{array}{l} \mu_7(e_1, e_1) = e_1 \\ \mu_7(e_1, e_2) = e_2 \\ \mu_7(e_2, e_1) = e_2 \\ \mu_7(e_2, e_2) = -e_1 \end{array}\right.$$

$$\{\mu_8(e_1, e_1) = e_1 \qquad , \qquad \begin{cases} \mu_9(e_1, e_1) = e_1 \\ \mu_9(e_1, e_2) = e_2 \\ \mu_9(e_2, e_1) = e_2 \end{cases}$$

$$\begin{cases} \mu_{10}(e_1, e_1) = e_1 \\ \mu_{10}(e_2, e_2) = e_3 \end{cases}$$

If moreover $\mathfrak{a}$ is nilpotent, then it is isomorphic to one of the following algebras

$$\begin{cases} \mu_{11}(e_1, e_1) = e_2 \\ \mu_{11}(e_3, e_3) = e_2 \end{cases} \qquad , \qquad \begin{cases} \mu_{12}(e_1, e_1) = e_2 \\ \mu_{12}(e_3, e_3) = -e_2 \end{cases}$$

$$\{\mu_{13}(e_1, e_1) = e_2 \qquad , \qquad \begin{cases} \mu_{14}(e_1, e_1) = e_2 \\ \mu_{14}(e_1, e_2) = e_3 \\ \mu_{14}(e_2, e_1) = e_3 \end{cases}$$

$$\{\mu_{15}(e_i, e_j) = 0 \qquad i, j \in \{1, 2, 3\}$$

Theorem 96 *Every 3-dimensional real commutative associative Lie algebra $\mathfrak{a}$ is isomorphic to one of the algebras $\mathfrak{a}_i$, $i = 1, 2, \cdots, 15$.*

If $\mathfrak{a}$ is nilpotent, $\mathfrak{a}$ is isomorphic to $\mathfrak{a}_i$, $i = 11, \cdots, 15$.

Affine structures on $\mathbb{R}^3$

Theorem 97 *There exist 15 invariant affinely non-equivalent affine structures on the 3-dimensional abelian Lie algebra. They are given by :*

A_1	$(x,y,z) \to \begin{cases} e^a x + e^a - 1, \\ e^a\left(e^b - 1\right)x + e^{a+b}y + e^a(e^b - 1), \\ e^a\left(e^c - 1\right)x + e^{a+c}z + e^a\left(e^c - 1\right) \end{cases}$
A_2	$(x,y,z) \to \begin{cases} xe^a \cos c - ze^a \sin c + e^a \cos c - 1, \\ \left(-e^a \cos c + e^{a+b}\right)x + e^{a+b}y + ze^a \sin c - e^a \cos c + e^{a+b}, \\ xe^a \sin c + ze^a \cos c + e^a \sin c \end{cases}$
A_3	$(x,y,z) \to \begin{cases} e^a x + e^a - 1, \\ e^a\left(e^b - 1\right)x + e^{a+b}y + e^a\left(e^b - 1\right), \\ ce^a x + e^a z + ce^a \end{cases}$
A_4	$(x,y,z) \to \begin{cases} e^a x - 1 + e^a, \\ (b + \frac{c^2}{2})e^a x + e^a y + ce^a z + (b + \frac{c^2}{2})e^a, \\ ce^a x + e^a z + ce^a \end{cases}$
A_5	$(x,y,z) \to \begin{cases} e^a x - 1 + e^a, \\ be^a x + e^a y + be^a, \\ ce^a x + e^a z + ce^a \end{cases}$
A_6	$(x,y,z) \to \begin{cases} e^a x - 1 + e^a, \\ e^b y - 1 + e^b, \\ z + c \end{cases}$
A_7	$(x,y,z) \to \begin{cases} xe^a \cos b - ye^a \sin b - 1 + e^a \cos b, \\ xe^a \sin b + ye^a \cos b + e^a \sin b, \\ z + c \end{cases}$
A_8	$(x,y,z) \to \begin{cases} e^a x - 1 + e^a, \\ y + b, \\ z + c \end{cases}$
A_9	$(x,y,z) \to \begin{cases} e^a x - 1 + e^a, \\ be^a x + e^a y + be^a, \\ z + c \end{cases}$
A_{10}	$(x,y,z) \to \begin{cases} e^a x - 1 + e^a, \\ y + b, \\ by + z + \frac{b^2}{2}c \end{cases}$
A_{11}	$(x,y,z) \to \begin{cases} x + a, \\ ax + y + cz + b + \frac{1}{2}(a^2 + c^2), \\ z + c \end{cases}$
A_{12}	$(x,y,z) \to \begin{cases} x + a, \\ ax + y - cz + b + \frac{1}{2}(a^2 - c^2), \\ z + c \end{cases}$

A_{13}	$(x,y,z) \to \begin{cases} x+a, \\ ax+y+b+\frac{a^2}{2}, \\ z+c \end{cases}$
A_{14}	$(x,y,z) \to \begin{cases} x+a, \\ ax+y+b+\frac{a^2}{2}, \\ (b+\frac{a^2}{2})x+ay+z+\frac{a^3}{6}+ab+c \end{cases}$
A_{15}	$(x,y,z) \to \begin{cases} x+a, \\ y+b, \\ z+c \end{cases}$

where $a,b,c \in \mathbb{R}$. Only the structures $A_i,\ i=11,..,15$ are complete.

10.3 Rigid affine structures

10.3.1 Definition

Let us consider a fixed basis $\{e_1,\cdots,e_n\}$ of the vector space $\mathbb{R}^n$. An associative law on $\mathbb{R}^n$ is given by a bilinear map

$$\mu : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$$

satisfying $\mu(e_i,\mu(e_j,e_k)) = \mu(\mu(e_i,e_j),e_k)$. If we put $\mu(e_i,e_j) = \sum C_{ij}^k e_k$, then the structural constants C_{ij}^k satisfy

$$(1) \quad \sum_l C_{il}^s C_{jk}^l - C_{ij}^l C_{lk}^s = 0, \qquad s = 1,\cdots,n.$$

Moreover if μ is commutative, we have

$$(2) \quad C_{ij}^k = C_{ji}^k$$

Thus the set of associative laws on $\mathbb{R}^n$ is identified to the real algebraic set embedded in $\mathbb{R}^{n^3}$, defined by the polynomial equations (1) and (2). We note this set $\mathcal{A}^c(n)$.

The law μ is unitary if there exists an $e \in \mathbb{R}^n$ such that $\mu(e,x) = x$. The set of unitary laws of $\mathcal{A}^c(n)$ is noted by $\mathcal{A}_1^c(n)$.

The linear group $GL(n,\mathbb{R})$ acts on $\mathcal{A}^c(n)$:

$$\begin{array}{ccc} GL(n,\mathbb{R}) \times \mathcal{A}^c(n) & \to & \mathcal{A}^c(n) \\ (f,\mu) & \mapsto & \mu_f \end{array}$$

where $\mu_f(e_i,e_j) = f^{-1}\mu(f(e_i),f(e_j))$.

We note by $\theta(\mu)$ the orbit of μ by this action. The orbit is isomorphic to the homogeneous space $\frac{GL(n,\mathbb{R})}{G_\mu}$, where $G_\mu = \{f \in GL(n,\mathbb{R}) \quad / \quad \mu_f = \mu\}$. The topology of $\mathcal{A}^c(n)$ is the induced topology of $\mathbb{R}^{n^3}$.

Definition 98 *The law $\mu \in \mathcal{A}^c(n)$ is rigid if $\theta(\mu)$ is open in $\mathcal{A}^c(n)$.*

Let μ be a real associative algebra and let us note by $\mu_{\mathbb{C}}$ the corresponding complex associative algebra. If μ is rigid in $\mathcal{A}^c(n)$ then either $\mu_{\mathbb{C}}$ is rigid in the scheme Ass_n of complex associative law, or $\mu_{\mathbb{C}}$ admits a deformation $\widetilde{\mu}_{\mathbb{C}}$ which is never the complexification of a real associative algebra.

This topological approach of the variety of associative algebras allows to introduce the notion of rigidity on the affine structures.

Definition 99 *An affine structure Ψ on an abelian Lie algebra $\mathfrak{g}$ is called rigid if the corresponding associative algebra $A(\mathfrak{g})$ is rigid in $\mathcal{A}^c(n)$.*

10.3.2 Cohomological approach

It is well known that a sufficient condition for an associative algebra $\mathfrak{a}$ to be rigid in $\mathcal{A}(n)$ is $H^2(\mathfrak{a},\mathfrak{a})=0$, where $H^*(\mathfrak{a},\mathfrak{a})$ is the Hochschild cohomology of $\mathfrak{a}$. Suppose that $\mathfrak{a}$ is commutative. If μ is the bilinear map defining the product of $\mathfrak{a}$, then $\mathfrak{a}$ is rigid if every deformation $\mu_t = \mu + \sum t^i\varphi_i$ of μ is isomorphic to μ. By the commutativity of μ we can assume that every φ_i is a symmetric bilinear map. Then considering only the symmetric cochains φ, we can define the commutative Harrison cohomology of μ as follows :

$$Z_s^2(\mu,\mu) = \{\varphi \in Sym(\mathbb{R}^3 \times \mathbb{R}^3, \mathbb{R}^3) \quad | \quad \delta_\mu\varphi = 0\}$$

where

$$\delta_\mu\varphi(x,y,z) = \mu(x,\varphi(y,z)) - \varphi(\mu(x,y),z) - \mu(\varphi(x,y),z) + \varphi(x,\mu(y,z)).$$

We can note that for every $f \in End(\mathbb{R}^3)$ the coboundary $\delta_\mu f \in Sym(\mathbb{R}^3 \times \mathbb{R}^3, \mathbb{R}^3)$ and then $\delta_\mu f \in Z_s^2(\mu,\mu)$.

Proposition 80 *If $H_s^2(\mu,\mu) = \frac{Z_s^2(\mu,\mu)}{B_s^2(\mu,\mu)} = \{0\}$, the corresponding affine structure on $\mathbb{R}^3$ is rigid.*

10.3.3 Rigid affine structures on the 2-dimensional abelian Lie algebra

Lemma 16 *Every infinitesimal deformation of an unitary associative algebra in $\mathcal{A}^c(n)$ is unitary.*

The proof is based on the study of perturbations of idempotent elements made in [11]. Let X be an idempotent element in an associative algebra of law μ. The operators

$$l_X : Y \longrightarrow XY$$

and

$$r_X : Y \longrightarrow YX$$

are simultaneously diagonalizable and the eigenvalues are respectively $(1,\cdots,1,0,\cdots,0)$ and $(1,1,1,\cdots,1,0,\cdots,0)$.∎ This set of eigenvalues is called bisystem associated to X. If X corresponds to the identity, the bisystem is $\{(1,\cdots,1)\,(1,\cdots,1)\}$.

If μ' is a perturbation of μ, there exists in μ' an idempotent element X' such that $b(X') = b(X)$. Consider the perturbation of the identity in μ'. As the bisystem corresponds to $\{\{1,\cdots,1\},\{1,\cdots,1\}\}$, we can conclude that X' is the identity of μ'.

Consequence. The set of unitary associative algebra is open in $\mathcal{A}^c(n)$.

Let us note by $\theta(\mu)$ the orbit of the law μ in $\mathcal{A}^c(n)$. From the previous classification, we see that

$$\mu_i \in \overline{\theta(\mu_1)}$$

for $i = 2,5,6$. Then we have

Theorem 100 *The affine structures A_1 and A_3 on the 2-dimensional abelian Lie algebra are rigid. The other structures can be deformed into A_1 or A_3.*

Consequence. Not any complete affine structure is rigid.

10.3.4 Rigid affine structures on $\mathbb{R}^3$

We refer to the classification of affine structures on 3-dimensional abelain Lie algebra given in the previous section. As the laws μ_1 and μ_2 are semi-simple associative, their second cohomological group is trivial. These structures are rigid.

Consider the remaining laws μ_i. We can easily compute the linear space $H_s^2(\mu,\mu)$ and the results are in the following tables :

Unitary case :

laws	$dim H_s^2(\mu,\mu)$	basis of $H_s^2(\mu,\mu)$
μ_3	1	$\varphi(e_3,e_3)=e_2-e_1$
μ_4	2	$\left\{\begin{array}{l}\varphi_1(e_2,e_2)=e_2\\ \varphi_2(e_2,e_3)=e_3.\end{array}\right.$
μ_5	4	$\left\{\begin{array}{l}\varphi_1(e_2,e_2)=e_2\\ \varphi_2(e_2,e_3)=e_3.\end{array}\right. ;\left\{\begin{array}{l}\varphi_3(e_3,e_3)=e_2\\ \varphi_4(e_3,e_3)=e_3.\end{array}\right.$

Non unitary case :

laws	$dim H_s^2$	basis of $H_s^2(\mu,\mu)$
μ_6	1	$\varphi_1(e_3,e_3)=e_3$
μ_7	1	$\varphi_1(e_3,e_3)=e_3$
μ_8	6	$\left\{\begin{array}{l}\varphi_1(e_2,e_2)=e_2\\ \varphi_2(e_2,e_2)=e_3.\end{array}\right. ;\left\{\begin{array}{l}\varphi_3(e_3,e_3)=e_2\\ \varphi_4(e_3,e_3)=e_3.\end{array}\right. ;\left\{\begin{array}{l}\varphi_5(e_2,e_3)=e_2\\ \varphi_6(e_3,e_3)=e_3.\end{array}\right.$
μ_9	3	$\left\{\begin{array}{l}\varphi_1(e_2,e_2)=e_1\\ \varphi_2(e_2,e_3)=e_2.\end{array}\right. ; \quad \varphi_3(e_3,e_3)=e_3$
μ_{10}	1	$\varphi_1(e_2,e_3)=e_2, \varphi_1(e_3,e_3)=e_3.$

Nilpotent and complete case :

laws	$dimH_s^2$	basis of $H_s^2(\mu,\mu)$
μ_{11}	3	$\begin{cases} \varphi_1(e_1,e_3)=e_1 \\ \varphi_1(e_2,e_3)=e_2. \end{cases}$; $\begin{cases} \varphi_2(e_1,e_3)=e_3 \\ \varphi_2(e_3,e_3)=e_1. \end{cases}$; $\{ \varphi_3(e_3,e_3)=e_3.$
μ_{12}	4	$\varphi_1(e_1,e_2)=e_2.$ $\varphi_2(e_3,e_3)=e_3.$; $\begin{cases} \varphi_3(e_1,e_3)=e_3 \\ \varphi_3(e_3,e_3)=e_1. \end{cases}$; $\begin{cases} \varphi_4(e_1,e_1)=e_3 \\ \varphi_4(e_1,e_3)=e_1 \\ \varphi_4(e_2,e_3)=2e_2. \end{cases}$
μ_{13}	7	$\begin{cases} \varphi_1(e_1,e_2)=e_1 \\ \varphi_1(e_2,e_2)=e_2. \end{cases}$ $\varphi_2(e_1,e_2)=e_2.$; $\varphi_3(e_1,e_2)=e_3$ $\begin{cases} \varphi_4(e_1,e_3)=e_1 \\ \varphi_4(e_2,e_3)=e_2. \end{cases}$ $\varphi_5(e_1,e_3)=e_3.$ $\varphi_6(e_3,e_3)=e_2.$ $\varphi_7(e_3,e_3)=e_3.$
μ_{14}	3	$\begin{cases} \varphi_1(e_1,e_3)=e_1 \\ \varphi_1(e_2,e_2)=e_1 \\ \varphi_1(e_2,e_3)=e_2 \\ \varphi_1(e_3,e_3)=e_3. \end{cases}$; $\begin{cases} \varphi_2(e_1,e_3)=e_2 \\ \varphi_2(e_2,e_2)=e_2 \\ \varphi_2(e_2,e_3)=e_3. \end{cases}$; $\begin{cases} \varphi_3(e_1,e_3)=e_3 \\ \varphi_3(e_2,e_2)=e_3. \end{cases}$
μ_{15}	18	

In this box we suppose that $\varphi(e_i,e_j)=\varphi(e_j,e_i)$ and the non defined $\varphi(e_s,e_t)$ are equal to zero.

We will note by $\mu_i \rightarrow \mu_j$ when $\mu_i \in \overline{\mathcal{O}(\mu_j)}$. We obtain the following diagram

$$\begin{tikzcd}
 & \mu_{12} \arrow[dr] & & \mu_{14} \arrow[dl] & & \\
\mu_{11} \arrow[r] & \mu_{10} \arrow[r] & \mu_1 & \mu_6 \arrow[l] & \mu_8 \arrow[l] \arrow[r] & \mu_9 \arrow[d] \\
\mu_{13} \arrow[u] & & \mu_4 \arrow[u] & \mu_5 \arrow[l] \arrow[r] & \mu_3 \arrow[d] & \mu_7 \arrow[l] \\
 & & & & \mu_2 &
\end{tikzcd}$$

Theorem 101 *There exist two rigid affine structures on the 3-dimensional abelian Lie algebra. They are the structures associated to the semi simple associative algebras μ_1 and μ_2.*

10.4 Invariant affine structures on $\mathbb{T}^2$ and $\mathbb{T}^3$

10.4.1 Invariant affine structure on $\mathbb{T}^2$

The compact abelian Lie group $\mathbb{T}^2$ is defined by $\mathbb{T}^2=\mathbb{R}^2/\mathbb{Z}^2$ identifying (x,y) with $(x+p,y+q)$, $(p,q)\in\mathbb{Z}^2$.

Proposition 81 *Only the structures A_4 and A_5 induce affine structures on the torus $\mathbb{T}^2$.*

Proof. It is easy to see that the affine action associated to A_1,A_2,A_3 and A_6 are incompatible with the lattice defined by $\mathbb{Z}^2$. Thus only complete structures provide affine structures on $\mathbb{T}^2$. For A_4 we obtain the following affine transformations

$$(x,y)\rightarrow \left(x+p,px+y+\left(q+p^2\right)\right)$$

This structure on $\mathbb{T}^2$ is not Euclidean. For A_5 the affine structure on $\mathbb{T}^2$ which is Euclidean corresponds to the transformations

$$(x, y) \to (x + p, y + q)$$

Remark 102 *In this proposition we find again, for the particular case of the torus, a classical result of Kuiper [9] giving the classification of affine structures on surfaces.*

10.4.2 Invariant affine structure on $\mathbb{T}^3$

We can see that the affine actions A_1 to A_{10} are compatible with the action of $\mathbb{Z}^3$ on $\mathbb{R}^3$ if the exponentials which appear in the analytic expressions of the affine transformations are equal to 1. This gives the identity for A_1 and A_3. As A_2 is incompatible with the action of $\mathbb{Z}^3$ for any values of the parameters a, b, c, the affine structures corresponding to the unitary cases are given by A_4 and A_5 for $a = 0$. This corresponds to the following affine structures on the torus $\mathbb{T}^3$:

$$(x, y, z) \in \mathbb{R}^3/\mathbb{Z}^3 \to \begin{cases} x \\ px + y + qz + p \\ qx + z + q \end{cases}$$

and

$$(x, y, z) \in \mathbb{R}^3/\mathbb{Z}^3 \to \begin{cases} x \\ px + y + p \\ qx + z + q \end{cases}$$

with $p, q \in \mathbb{Z}$.

For the actions A_6 to A_{10}, they induce affine actions on the torus if $a = b = 0$ for A_6 and $a = 0$ for A_7 to A_{10} but this appears as a particular case of A_{11}, A_{13} and A_{14}. Let us examine the complete and nilpotent cases; we find the following affine structure on $\mathbb{T}^3$:

$(x, y, z) \in \mathbb{R}^3/\mathbb{Z}^3 \to$	$\begin{cases} x + p, \\ px + y + rz + q, \\ z + r \end{cases}$
$(x, y, z) \in \mathbb{R}^3/\mathbb{Z}^3 \to$	$\begin{cases} x + p, \\ px + y - rz + q, \\ z + r \end{cases}$
$(x, y, z) \in \mathbb{R}^3/\mathbb{Z}^3 \to$	$\begin{cases} x + p, \\ px + y + q, \\ z + r \end{cases}$
$(x, y, z) \in \mathbb{R}^3/\mathbb{Z}^3 \to$	$\begin{cases} x + p, \\ px + y + q, \\ qx + py + z + r \end{cases}$
$(x, y, z) \in \mathbb{R}^3/\mathbb{Z}^3 \to$	$\begin{cases} x + p, \\ y + q, \\ z + r \end{cases}$

Theorem 103 *There exist 7 affine structures on the torus $\mathbb{T}^3$. They correspond to the following affine*

crystallographic subgroups of $Aff(\mathbb{R}^3)$:

$\Gamma_1 = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ p & 1 & q & p \\ q & 0 & 1 & q \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}$	$\Gamma_2 = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ p & 1 & 0 & p \\ q & 0 & 1 & q \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}$
$\Gamma_3 = \left\{ \begin{pmatrix} 1 & 0 & 0 & p \\ p & 1 & r & q \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}$	$\Gamma_4 = \left\{ \begin{pmatrix} 1 & 0 & 0 & p \\ p & 1 & -r & q \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}$
$\Gamma_5 = \left\{ \begin{pmatrix} 1 & 0 & 0 & p \\ p & 1 & 0 & q \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}$	$\Gamma_6 = \left\{ \begin{pmatrix} 1 & 0 & 0 & p \\ p & 1 & 0 & q \\ q & p & 1 & r \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}$
$\Gamma_7 = \left\{ \begin{pmatrix} 1 & 0 & 0 & p \\ 0 & 1 & 0 & q \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}$	

Remark 104 *The (complete) nilpotent and unimodular cases are completely classified in [35]*

10.5 Affine structures on nilpotent Lie algebras

In this section, we prove that every non characteristically nilpotent filiform algebra is provided with an affine structure. We generalize this result to the class of nilpotent algebras whose derived algebra admits non singular derivation. In a second part we describe some obstructions to lift a symplectic affine structure on a contact Lie algebra defined by a central extension. In a last section, we use the tensor product to define new affine structure.

10.5.1 The example of Benoist

The problem of existence of affine structures on nilpotent Lie algebras has been put by John Milnor. We know that every nilpotent Lie algebras of dimension less or equal to 7 admits an affine structure, every nilpotent Lie algebras provided with a symplectic form admits also an affine structure. But Benoist has proposed examples of 11-dimensional nilpotent Lie algebras which are not endowed with such a structure.

These Lie algebras are given by:

$$\begin{array}{ll}
[X_1,X_i]=X_{i+1} \qquad i=2,..,10 & \\
[X_2,X_4]=X_6 & \\
[X_2,X_6]=-5X_8+2X_9+2tX_{10} & \\
[X_2,X_8]=\frac{26}{5}X_{10}+\frac{28}{25}X_{11} & \\
[X_3,X_4]=3X_7-X_8-tX_9 & \\
[X_3,X_6]=-\frac{12}{5}X_9-\frac{1}{25}X_{10}+\frac{-448+1525t}{2000}X_{11} & \\
[X_3,X_8]=\frac{321}{80}X_{11} & \\
[X_4,X_6]=\frac{27}{5}X_{10}-\frac{24}{25}X_{11} & \\
[X_5,X_6]=\frac{1377}{80}X_{11} & t\in\mathbb{R} \\
[X_2,X_3]=X_5 & \\
[X_2,X_5]=-2X_7+X_8+tX_9 & \\
[X_2,X_7]=-\frac{13}{5}X_9+\frac{51}{25}X_{10}+\frac{448+2475t}{2000}X_{11} & \\
[X_2,X_9]=\frac{19}{16}X_{11} & \\
[X_3,X_5]=3X_8-X_9-tX_{10} & \\
[X_3,X_7]=\frac{-39}{5}X_{10}+\frac{23}{25}X_{11} & \\
[X_4,X_5]=\frac{27}{5}X_9-\frac{24}{25}X_{10}+\frac{448-3525t}{2000}X_{11} & \\
[X_4,X_7]=-\frac{189}{16}X_{11} &
\end{array}$$

Then, the determination of the class of nilpotent Lie algebras with an affine structure is now open.

10.5.2 Affine structures on non characteristically filiform algebra

Recall that a Lie algebra is called characteristically nilpotent is every derivation is nilpotent. Examining the counter-examples of Benoist and Burde, the following conjecture becomes natural:

Conjecture Every nilpotent Lie algebra which doest not admit affine structure is characteristically nilpotent.

Theorem 105 *Every filiform non characteristically nilpotent Lie algebra admits an affine structure*

Before to prove this theorem, we prove the following lemma:

Lemma 17 *Every Lie algebras admitting a derivation whose restriction to the derived subalgebra is regular, admits an affine structure.*

Proof of the lemma. In fact, such an algebra is necessary nilpotent. Let f be such a derivation. For every $X\in\mathfrak{g}$ we put

$$\nabla_X=f^{-1}\circ adX\circ f.$$

This operator is well defined because the image of the operator $adX\circ f$ is contained in the derived subalgebra. Then ∇ defines an affine structure on $\mathfrak{g}$.

Proof of the theorem. For a non characteristically nilpotent Lie algebra $\mathfrak{g}$, let us call rank of $\mathfrak{g}$ the dimension of a maximal exterior torus of derivations (a maximal abelian subalgebra of $Der(\mathfrak{g})$ of which elements are semi-simple derivations). We have the following results [53]:

1. If the Lie algebra $\mathfrak{g}$ is filiform, its rank $r(\mathfrak{g})$ satisfies

$$r(\mathfrak{g})\leq 2.$$

2.Every filiform Lie algebras of rank 2 is isomorphic to L_n or Q_n. where L_n and Q_n are the n-dimensional filiform Lie algebras defined by

$$L_n:\{[Y_1,Y_j]=Y_{1+j},\quad j=2,\cdots,n-1$$

$$Q_n=\left\{\begin{array}{l}[Y_1,Y_j]=Y_{1+j},\quad j=2,\cdots,n-1\\ [Y_i,Y_{n-i+1}]=(-1)^{i+1}Y_n,\quad i=2,\cdots,p\end{array}\right.\qquad n=2p.$$

For each Lie algebra, a maximal exterior torus is precisely determined.

If $\mathfrak{g} = L_n$, there exists a torus generated by the diagonal derivations :

$$f_1(Y_1) = 0, \quad f_1(Y_i) = Y_i, \quad 2 \leq i \leq n$$

$$f_2(Y_1) = Y_1, \quad f_2(Y_i) = iY_i, \quad 2 \leq i \leq n$$

the basis $\{Y_i\}$ being as above.

If $\mathfrak{g} = Q_n$, the basis $\{Y_i\}$ is not a basis of eigenvectors for a diagonalizable derivation. We can consider the new basis given by

$$Z_1 = Y_1 - Y_2, Z_2 = Y_2, \cdots , Z_n = Y_n$$

This basis satisfies

$$[Z_1, Z_j] = Z_{1+j}, \quad j = 2, \cdots , n-2, [Z_i, Z_{n-i+1}] = (-1)^{i+1} Z_n, \quad i = 2, \cdots , n/2$$

Then the diagonal derivations

$$f_1(Z_1) = 0, \quad f_1(Z_i) = Z_i, \quad 2 \leq i \leq n-1, \quad f_1(Z_n) = 2Z_n$$

$$f_2(Z_1) = Z_1, \quad f_2(Z_i) = (i-2)Z_i, \quad 2 \leq i \leq n-1, \quad f_2(Z_n) = (n-3)Z_n.$$

generates a maximal exterior torus of derivations.

3. Every filiform Lie algebra of rank 1 and dimension n is isomorphic to one of the following Lie algebras

i) $A_n^k(\lambda_1, \cdots , \lambda_{t-1}), t = \left[\frac{n-k+1}{2}\right], 2 \leq k \leq n-3$

$$\begin{cases} [Y_1, Y_i] = Y_{i+1}, \quad i = 2, \cdots , n-1 \\ [Y_i, Y_{i+1}] = \lambda_{i-1} Y_{2i+k-1} \quad , \quad 2 \leq i \leq t \\ [Y_i, Y_j] = a_{ij} Y_{i+j+k-2} \quad , \quad 2 \leq i \leq j \quad i+j+k-2 \leq n \end{cases}$$

ii) $B_n^k(\lambda_1, \cdots , \lambda_{t-1})$ $n = 2m$, $t = \left[\frac{n-k}{2}\right]$, $2 \leq k \leq n-3$

$$\begin{cases} [Y_1, Y_i] = Y_{i+1} \quad i = 2, \cdots , n-2 \\ [Y_i, Y_{n-i+1}] = (-1)_n^{i+1} Y \quad , \quad i = 2, \cdots , n-1 \\ [Y_i, Y_{i+1}] = \lambda_{i-1} Y_{2i+k-1} \quad , \quad i = 2, \cdots , t \\ [Y_i, Y_j] = a_{ij} Y_{i+j-k-2} \quad , \quad 2 \leq i, j \leq n-2, i+j+k-2 \leq n-2, \quad j \neq i+1 \end{cases}$$

iii) $C_n(\lambda_1, \cdots , \lambda_t)$, $n = 2m+2$, $t = m-1$

$$\begin{cases} [Y_1, Y_i] = Y_{i+1} \quad i = 2, \cdots , n-2 \\ [Y_i, Y_{n-i+1}] = (-1)_n^{i-1} Y_n \quad , \quad i = 2, \cdots m+1 \\ [Y_i, Y_{n-i-2k+1}] = (-1)^{i+1} \lambda_k Y_n \end{cases}$$

The non defined brackets are equal to zero. In this theorem, $[x]$ denotes the integer of x and $(\lambda_1, \cdots , \lambda_t)$ are non simultaneously vanishing parameters satisfying polynomial equations associated to the Jacobi conditions. Moreover, the constants a_{ij} satisfy

$$a_{ij} = a_{ij+1} + a_{i+1,j}$$

and $a_{ii+1} = \lambda_{i-1}$.

We can easily see that the filiform algebra L_n , Q_n or of type A^n or B^n admit regular derivations. Then they admits affine structure. Let us consider the case C^n. This algebra is of rank 1. The exterior torus of derivation is generated by

$$f(Y_1) = 0, f(Y_i) = Y_i, \quad i = 2, \cdots , n-1, \quad f(Y_n) = 2Y_n.$$

Thus every derivation is singular.

Lemma 18 *The restriction of the derivation f to the derived subalgebra $D(\mathfrak{g})$ is a regular derivation of $D(\mathfrak{g})$.*

Let us consider a vectorial endomorphism g of $\mathfrak{g}$ which leaves invariant $D(\mathfrak{g})$, and such that the restriction to $D(\mathfrak{g})$ satisfies $f \circ g = Id$. Then the bilinear map given by

$$\nabla_X = g \circ adX \circ f.$$

defines an affine structure on C^n. In fact,

$$\nabla_X(Y) - \nabla_Y(X) = g \circ adX \circ f(Y) - g \circ adY \circ f(X) = g(f[X,Y])$$

because f is a derivation. As $g = f^{-1}$ on the derived subalgebra, we can deduce

$$\nabla_X(Y) - \nabla_Y(X) = [X,Y].$$

In the some way

$$\nabla_X \nabla_Y(Z) - \nabla_Y \nabla_X(Z) = g[X,[Y,f(Z)]] - g[Y,[X,f(Z)]] = -g[f(Z),[X,Y]]$$

Then

$$\nabla_X \nabla_Y(Z) - \nabla_Y \nabla_X(Z) = \nabla_{[X,Y]}(Z)$$

This proves the theorem.

10.6 Faithful representations associated to an affine connection

10.6.1 Nilpotent representations

If $\mathfrak{g}$ is affine, then the corresponding connected Lie group G is an affine manifold such that every left translation is an affine isomorphism of G. In this case, the operator ∇ is nothing that the connection operator of the affine connection on G.

Let $\mathfrak{g}$ be an affine Lie algebra. Then the map

$$f : \mathfrak{g} \to End(\mathfrak{g})$$

defined by

$$f(X)(Y) = \nabla(X,Y)$$

is a linear representation (non faithful) of $\mathfrak{g}$ satisfying

$$f(X)(Y) - f(Y)(X) = [X,Y] \qquad (*)$$

Let ∇ be an affine connection on n-dimensional Lie algebra $\mathfrak{g}$. Let us consider the $(n+1)$-dimensional linear representation given by

$$\rho : \mathfrak{g} \to \mathfrak{g} \oplus \mathbb{R}$$

given by

$$\rho(X) : (Y,t) \mapsto (f_X(Y) + tX, 0)$$

It is easy to verify that ρ is a faithful representation of dimension $n+1$ if and only if $f_X(Y) = \nabla(X,Y)$ is an affine connection. We say that the representation ρ is nilpotent if the endomorphism $\rho(X)$ is nilpotent for every X in $\mathfrak{g}$.

Proposition 82 *Suppose that $\mathfrak{g}$ is a complex nilpotent Lie algebra and let ρ be a faithful representation of $\mathfrak{g}$. Then there exists a faithful nilpotent representation.*

Proof. Let us consider the $\mathfrak{g}$-module M associated to ρ. Then, as $\mathfrak{g}$ is nilpotent, M can be decomposed as

$$M = \oplus_{i=1}^{k} M_{\lambda_i}$$

where M_{λ_i} is a $\mathfrak{g}$-submodule, and the λ_i are linear forms on $\mathfrak{g}$. For all $X \in \mathfrak{g}$, the restriction of $\rho(X)$ to M_i as the following form

$$\begin{pmatrix} \lambda_i(X) & * & \cdots & * \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & \lambda_i(X) \end{pmatrix}$$

Let $\mathbb{C}_{\lambda_i}$ be the one dimensional $\mathfrak{g}$-module defined by

$$\mu : X \in \mathfrak{g} \to \mu(X) \in End\mathbb{C}$$

with

$$\mu(X)(a) = \lambda_i(X)a$$

The tensor product $M_{\lambda_i} \otimes \mathbb{C}_{-\lambda_i}$ is the $\mathfrak{g}$-module associated to

$$X \cdot (Y \otimes a) = \rho(X)(Y) \otimes a - Y \otimes \lambda_i(X)a$$

Then $\widetilde{M} = \oplus (M_{\lambda_i} \otimes C_{-\lambda_i})$ is a nilpotent $\mathfrak{g}$-module. Let us prove that $\widetilde{M}$ is faithful. Recall that a representation ρ of $\mathfrak{g}$ is faithful if and only if $\rho(Z) \neq 0$ for every $Z \neq 0 \in Z(\mathfrak{g})$. Consider $X \neq 0 \in Z(\mathfrak{g})$. If $\widetilde{\rho}(X) = 0$, the endomorphism $\rho(X)$ is diagonal. Suppose that $\mathfrak{g} \neq Z(\mathfrak{g})$ and let $\mathcal{C}^{k-1}(\mathfrak{g}) = Z(\mathfrak{g})$ where k is the index of nilpotence of $\mathfrak{g}$. Then

$$\exists\, (Y, Z) \in \left(\mathcal{C}^{k-2}(\mathfrak{g}), \mathfrak{g}\right) \,/\, [Y, Z] = X$$

The endomorphism $\rho(Y)\rho(Z) - \rho(Z)\rho(Y)$ is nilpotent and the eigenvalues of $\rho(X)$ are 0. Thus $\rho(X) = 0$ and ρ is not faithful. We can conclude that $\widetilde{\rho}(X) \neq 0$ and $\widetilde{\rho}$ is a faithful representation.

10.6.2 On the nilpotent affine connection

Let $\mathfrak{g}$ be a filiform affine Lie algebra of dimension n, and ρ be the $(n+1)$-dimensional associated faithful representation. Let $M = \mathfrak{g} \oplus \mathbb{C}$ be the corresponding complex $\mathfrak{g}$-module. As $\mathfrak{g}$ is filiform, its decomposition has the following form

1) $M = M_0$ and M is irreducible,
2) $M = M_0 \oplus M_\lambda$.

For a general faithful representation, let us call characteristic the ordered sequence of the dimensions of the irreducible submodules. In the filiform case we have $c(\rho) = (n+1)$ or $(n, 1)$. In fact, the filiformity of $\mathfrak{g}$ implies that exists an irreducible submodule of dimension greater than $n-1$. More generally, if the characteristic sequence of a nilpotent Lie algebra is equal to $(c_1, .., c_p, 1)$ (see [6]) then for every faithful representation ρ we have $c(\rho) = (d_1, .., d_q)$ with $d_1 \geq c_1$.

Theorem 106 *Let $\mathfrak{g}$ be the filiform Lie algebra L_n. There are faithful $\mathfrak{g}$-modules which are not nilpotent.*

Proof. Consider the following representation given by the matrices $\rho(X_i)$ where $\{X_1, .., X_n\}$ is a basis of $\mathfrak{g}$

$$\rho(X_1) = \begin{pmatrix} a & a & 0 & \cdots & \cdots & & & 0 & 1 \\ a & a & 0 & & & & & \vdots & 0 \\ 0 & 0 & 0 & & & & & 0 & 0 \\ \vdots & \ddots & \frac{1}{2} & \ddots & & & & \vdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & & \vdots & 0 \\ \vdots & & & \ddots & \frac{i-3}{i-2} & \ddots & & \vdots & 0 \\ 0 & 0 & & & \ddots & \ddots & \ddots & \vdots & 0 \\ \alpha & \beta & 0 & \cdots & \cdots & 0 & \frac{n-3}{n-2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\rho(X_2) = \begin{pmatrix} a & a & 0 & \cdots & \cdots & & \cdots & 0 & 0 \\ a & a & 0 & & & & & \vdots & 1 \\ -1 & 1 & 0 & & & & & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \ddots & & & & \vdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & & \vdots & 0 \\ \vdots & & & \ddots & \frac{1}{i-2} & \ddots & & \vdots & 0 \\ 0 & 0 & & & \ddots & \ddots & \ddots & \vdots & 0 \\ \beta & \alpha & 0 & \cdots & \cdots & \cdots & \frac{1}{n-2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and for $3 \leq j \leq n-1$ the endomorphisms $\rho(X_j)$ satisfy :

$$\begin{cases} \rho(X_j)(e_1) = -\dfrac{1}{j-1} e_{j+1} \\ \rho(X_j)(e_2) = \dfrac{1}{j-1} e_{j+1} \\ \rho(X_j)(e_3) = \dfrac{1}{j(j-1)} e_{j+2} \\ \cdots \\ \rho(X_j)(e_{i-j+1}) = \dfrac{(j-2)!(i-j-1)!}{(i-2)!} e_i, \quad i = j-2, \cdots, n \\ \rho(X_j)(e_{i-j+1}) = 0, \quad i = n+1, .., n+j-1 \\ \rho(X_j)(e_{n+1}) = e_j \end{cases}$$

and for $j = n$

$$\begin{cases} \rho(X_n)(e_i) = 0 \quad i = 1, \cdots, n \\ \rho(X_n)(e_{n+1}) = e_n \end{cases}$$

where $\{e_1, \cdots, e_n, e_{n+1}\}$ is the basis given by $e_i = (X_i, 0)$ and $e_{n+1} = (0, 1)$. We easily verify that these matrices describe a non nilpotent faithful representation.

10.6.3 Study of an associated connection

The previous representation is associated to an affine connection on the filiform Lie algebra L_n given by

$$\nabla_{X_i} = \rho(X_i) \mid_{\mathfrak{g}}$$

where $\mathfrak{g}$ designates the $n-$dimensional first factor of the $(n+1)-$ dimensional faithful module. This connection is complete if and only if the endomorphisms $R_X \in End(\mathfrak{g})$ define by

$$R_X(Y) = \nabla_Y(X)$$

are nilpotent for all $X \in \mathfrak{g}$ ([5]). But the matrix of R_{X_1} has the form :

$$\begin{pmatrix} a & a & 0 & \cdots & 0 & \cdots & 0 & 0 \\ a & a & & & \vdots & & \vdots & 0 \\ 0 & -1 & & & \vdots & & \vdots & 0 \\ 0 & 0 & -\frac{1}{2} & \cdots & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & 0 & \ddots & & \cdots & \vdots & 0 \\ 0 & 0 & \vdots & \ddots & -\frac{1}{j-1} & & \vdots & 0 \\ \alpha & \beta & \vdots & \cdots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{n-2} & 0 \end{pmatrix}$$

Its trace is $2a$ and for $a \neq 0$ it is not nilpotent. We have proved :

Theorem 107 *There exist affine connections on the filiform Lie algebra L_n which are non complete.*

Remark. The most simple example is on $dim3$ and concerns the Heisenberg algebra. We find a non nilpotent faithful representation associated to the non complete affine connection given by :

$$\nabla_{X_1} = \begin{pmatrix} a & a & 0 \\ a & a & 0 \\ \alpha & \beta & 0 \end{pmatrix}, \quad \nabla_{X_2} = \begin{pmatrix} a & a & 0 \\ a & a & 0 \\ \beta-1 & \alpha+1 & 0 \end{pmatrix}, \quad \nabla_{X_3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The affine representation is written

$$\begin{pmatrix} a(x_1+x_2) & a(x_1+x_2) & 0 & x_1 \\ a(x_1+x_2) & a(x_1+x_2) & 0 & x_2 \\ \alpha x_1 + (\beta-1)x_2 & \beta x_1 + (\alpha+1)x_2 & 0 & x_3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

10.7 Affine structures on nilpotent Lie algebras with a contact form

10.7.1 Nilpotent Lie algebras with a contact form

Definition 108 *Let $\mathfrak{g}$ be an $(2p+1)$-dimensional algebra. A contact form on $\mathfrak{g}$ is a linear form $\omega \neq 0$ of $\mathfrak{g}^*$ such that $\omega \wedge (dw)^p \neq 0$. In this case $(\mathfrak{g}, \omega)$ or $\mathfrak{g}$ is called a contact Lie algebra.*

Proposition 83 *[49] Let $\mathfrak{g}$ be a contact nilpotent Lie algebra. Then the center $Z(\mathfrak{g})$ is one-dimensional.*

Proof. If $\mathfrak{g}$ is $(2p+1)$-dimensional and equipped with a contact form ω, $\dim Z(\mathfrak{g}) \leq 1$. This follows the fact that if we suppose that $\omega(Z(\mathfrak{g})) = 0$ then

$$\forall X \in Z(\mathfrak{g}) \quad d\omega(X,Y) = -\omega\,[X,Y] = 0.$$

Thus there exists X such that $\omega(X) = 0$ and $X \lrcorner d\omega = 0$, where $\lrcorner$ denotes the inner product. The vector X belongs to the characteristic subspace and $\omega \wedge d\omega^p = 0$. Thus $\omega(Z(\mathfrak{g})) \neq 0$ which proves that $\dim Z(\mathfrak{g}) \leq 1$. If moreover the Lie algebra $\mathfrak{g}$ is nilpotent then $\dim Z(\mathfrak{g}) = 1$ as the center of a nilpotent Lie algebra is never zero. ∎

Corollary 109 *Let $\mathfrak{g}$ be a contact nilpotent Lie algebra. Then $\mathfrak{g}/Z(\mathfrak{g})$ is a symplectic Lie algebra.*

Thus any contact Lie algebra is a one-dimensional central extension of a symplectic Lie algebra:

$$0 \to V \to \mathfrak{g}_{2p+1} \to (\mathfrak{g}_{2p}, \theta) \to 0.$$

As any symplectic nilpotent Lie algebra can be equipped with an affine structure, we have

Proposition 84 *Any nilpotent contact Lie algebra is a one-dimensional central extension of an affine Lie algebra.*

10.7.2 Extension of symplectic affine structure

Let $(\mathfrak{g}_{2p}, \theta)$ be a symplectic nilpotent Lie algebra and ∇ the affine structure coming from this symplectic form, that is,

$$\nabla_X Y = f(X)Y$$

where $f(X)$ is the following endomorphism:

$$\forall X, Y, Z \in \mathfrak{g} \qquad \theta\left(f(X)(Y), Z\right) = -\theta(Y, [X, Z]).$$

Let $\widetilde{\mathfrak{g}}$ be the contact Lie algebra defined by the one-dimensional extension

$$0 \to V \to \mathfrak{g} \xrightarrow{\pi} \widetilde{\mathfrak{g}} \to 0.$$

The Lie algebra $\widetilde{\mathfrak{g}}$ identified with $\mathfrak{g} \oplus V$ has the following brackets

$$[(X, \alpha), (Y, \lambda)]_{\widetilde{\mathfrak{g}}} = \Big([X, Y]_{\mathfrak{g}}, \theta(X, Y)\Big)$$

Let $\widetilde{\nabla} : \widetilde{\mathfrak{g}} \otimes \widetilde{\mathfrak{g}} \to \widetilde{\mathfrak{g}}$ be an operator satisfying

$$(*) \begin{cases} \widetilde{\nabla}\left((X,0),(Y,0)\right) = \left(\nabla(X,Y), \varphi(X,Y)\right) \\ \widetilde{\nabla}\left((X,0),(0,\lambda)\right) = \widetilde{\nabla}\left((0,\lambda),(X,0)\right) \end{cases}$$

where φ is a bilinear map on $\mathfrak{g}$ such as

$$\varphi(X, Y) - \varphi(Y, X) = \theta(X, Y).$$

Lemma 19 *the operator $\widetilde{\nabla}$ satisfies the following identity:*

$$\widetilde{\nabla}\left((X,\alpha),(Y,\lambda)\right) - \widetilde{\nabla}\left((Y,\lambda),(X,\alpha)\right) = [(X,\alpha),(Y,\lambda)]_{\widetilde{\mathfrak{g}}}$$

Proof. We have for all $X, Y \in \mathfrak{g}$ and $\lambda, \mu \in \mathbb{K}$

$$\begin{aligned}
&\widetilde{\nabla}\left((X,\alpha),(Y,\lambda)\right) - \widetilde{\nabla}\left((Y,\lambda),(X,\alpha)\right) \\
&= \left(\nabla(X,Y), \varphi(X,Y)\right) + \lambda\widetilde{\nabla}\left((X,0),(0,1)\right) + \alpha\widetilde{\nabla}\left((0,1),(Y,0)\right) \\
&+ \alpha\lambda\widetilde{\nabla}\left((0,1),(0,1)\right) - \left(\nabla(Y,X), \varphi(Y,X)\right) - \alpha\widetilde{\nabla}\left((Y,0),(0,1)\right) \\
&- \lambda\widetilde{\nabla}\left((0,1),(X,0)\right) - \lambda\alpha\widetilde{\nabla}\left((0,1),(0,1)\right) \\
&= \Big([X,Y]_{\mathfrak{g}}, \theta(X,Y)\Big)
\end{aligned}$$

But

$$[(X, \alpha), (Y, \lambda)]_{\widetilde{\mathfrak{g}}} = \Big([X, Y]_{\mathfrak{g}}, \theta(X, Y)\Big),$$

which implies that

$$\widetilde{\nabla}\left((X,\alpha),(Y,\lambda)\right) - \widetilde{\nabla}\left((Y,\lambda),(X,\alpha)\right) = [(X,\alpha),(Y,\lambda)]_{\widetilde{\mathfrak{g}}}. \qquad \blacksquare$$

The operator $\widetilde{\nabla}$ is associated to a flat torsion-free connection on $\widetilde{\mathfrak{g}}$.

We can note that if $\widetilde{\nabla'}$ is another bilinear map on $\widetilde{\mathfrak{g}}$ such that $\pi^*\widetilde{\nabla'} = \nabla$, the nullity of the torsion of the linear connection associated to $\widetilde{\nabla'}$ implies that $\widetilde{\nabla'}$ satisfies the same conditions $(*)$. This justifies the choice of the conditions $(*)$.

As we want that $\widetilde{\nabla}$ defines an affine structure on $\mathfrak{g}$, we introduce, in order to study the curvature of the linear connection associated to $\widetilde{\nabla}$, the following application:

$$\begin{aligned}&C(\ (X,\alpha)\,,(Y,\lambda)\,,(Z,\rho))\\ &=\widetilde{\nabla}\left((X,\alpha)\,,\widetilde{\nabla}\left((Y,\lambda)\,,(Z,\rho)\right)\right)-\widetilde{\nabla}\left((Y,\lambda)\,,\widetilde{\nabla}\left((X,\alpha)\,,(Z,\rho)\right)\right)\\ &-\widetilde{\nabla}\left([(X,\alpha)\,,(Y,\lambda)]_{\widetilde{\mathfrak{g}}},(Z,\rho)\right).\end{aligned}$$

This gives:

$$\begin{aligned}&C((X,\alpha)\,,(Y,\lambda)\,,(Z,\rho))\\ &=\widetilde{\nabla}((X,\alpha)\,,(\nabla\,(Y,Z)\,,\varphi\,(Y,Z))+\rho\widetilde{\nabla}\,((Y,0)\,,(0,1))+\lambda\widetilde{\nabla}\,((0,1)\,,(Z,0))+\\ &\lambda\rho\widetilde{\nabla}\,((0,1)\,,(0,1)))-\widetilde{\nabla}((Y,\lambda)\,,(\nabla\,(X,Z)\,,\varphi\,(X,Z))+\rho\widetilde{\nabla}\,((X,0)\,,(0,1))+\\ &\alpha\widetilde{\nabla}\,((0,1)\,,(Z,0))+\alpha\rho\widetilde{\nabla}\,((0,1)\,,(0,1)))-\widetilde{\nabla}(([X,Y]_{\mu}\,,\theta(X,Y)),(Z,\rho))\end{aligned}$$

Lemma 20 *The operator $\widetilde{\nabla}$ satisfies:*

1)

$$\begin{aligned}&C((X,0),(Y,0),(Z,0))=(0,\varphi(X,\nabla(Y,Z))-\varphi(Y,\nabla(X,Z))-\varphi([X,Y]_{\mu}\,,Z))\\ &+\varphi(Y,Z)\widetilde{\nabla}((X,0),(0,1))-\varphi(X,Z)\widetilde{\nabla}((Y,0),(0,1))-\theta(X,Y)\widetilde{\nabla}((Z,0),(0,1))\end{aligned}$$

2)

$$\begin{aligned}&C((X,0),(0,1),(Y,0))=\widetilde{\nabla}\left((X,0)\,,\widetilde{\nabla}\left((Y,0)\,,(0,1)\right)\right)\\ &-\widetilde{\nabla}\left((\nabla(X,Y),0),(0,1)\right)-\varphi\,(X,Y)\,\widetilde{\nabla}\left((0,1)\,,(0,1)\right)\end{aligned}$$

3)

$$C((0,1)\,,(Y,0)\,,(0,1))=\widetilde{\nabla}\left((0,1)\,,\widetilde{\nabla}\left((Y,0)\,,(0,1)\right)\right)-\widetilde{\nabla}\left((Y,0)\,,\widetilde{\nabla}\left((0,1)\,,(0,1)\right)\right)$$

This follows directly when we develop the expressions.

Proposition 85 *If*

$$C((X,0),(0,1),(Y,0))=0$$

then $C(\ (X,0)\,,(Y,0)\,,(0,1))=0$.

In fact,

$$\begin{aligned}C((X,0),(Y,0),(0,1))&=\widetilde{\nabla}((X,0),\widetilde{\nabla}((Y,0),(0,1)))-\widetilde{\nabla}((Y,0),\widetilde{\nabla}((X,0),(0,1)))\\ &\quad-\widetilde{\nabla}(([X,Y]\,,\theta(X,Y)),(0,1))\\ &=\widetilde{\nabla}((\nabla(X,Y),0),(0,1))+\varphi(X,Y)\widetilde{\nabla}((0,1),(0,1))-\widetilde{\nabla}((\nabla(Y,X),0),(0,1))\\ &\quad-\varphi(Y,X)\widetilde{\nabla}((0,1),(0,1))-\widetilde{\nabla}(([X,Y]\,,\theta(X,Y)),(0,1))\\ &=\widetilde{\nabla}(([X,Y]\,,0),(0,1))+\theta(X,Y)\widetilde{\nabla}((0,1),(0,1))-\widetilde{\nabla}([X,Y]\,,0),(0,1))\\ &\quad-\theta(X,Y)\widetilde{\nabla}(0,1),(0,1))\\ &=0.\end{aligned}$$

Let us write some necessary conditions for the application C to be equal to zero. Let π be the canonical projection of $\widetilde{\mathfrak{g}}$ on $\mathfrak{g}$, that is:

$$\pi(X,\alpha)=X.$$

Let us identify $(X,0)$ with X which permits to consider $\mathfrak{g}$ as a vector subspace of $\widetilde{\mathfrak{g}}$. Let us denote V_X the vector defined by

$$V_X = \pi(\widetilde{\nabla}\left(\left(X,0\right),\left(0,1\right)\right).$$

If $C=0$ we have:

$$C(\left(X,0\right),\left(Y,0\right),\left(Z,0\right))=0$$

which implies that:

$$\varphi\left(Y,Z\right)V_X-\varphi\left(X,Z\right)V_Y-\theta\left(X,Y\right)V_Z=0$$

for all $X,Y,Z\in\mathfrak{g}$.

Remark. If $\varphi=0$, we have $\theta=0$, the extension is trivial and we came out of the symplectic case. The operator $\widetilde{\nabla}$ define by

$$\begin{array}{rcl}\widetilde{\nabla}\left(\left(X,0\right),\left(Y,0\right)\right) & = & \left(\nabla\left(X,Y\right),0\right)\\ \widetilde{\nabla}\left(\left(X,0\right),\left(0,\lambda\right)\right) & = & 0\\ \widetilde{\nabla}\left(\left(0,\mu\right),\left(0,\lambda\right)\right) & = & 0\end{array}$$

give an affine structure on $\widetilde{\mathfrak{g}}=\mathfrak{g}\oplus\mathbb{R}$ which is a direct sum of ideals

We must then suppose that $\varphi\neq 0$.

10.7.3 Case $\varphi=\frac{\theta}{2}$.

Then we have

$$\theta\left(\left[X,Y\right],Z\right)=-\theta\left(Y,\nabla(X,Z)\right)$$

and we deduce from the lemma the following relations:

$$\begin{array}{rl} & C(\left(X,0\right),\left(Y,0\right),\left(Z,0\right))\\ = & -\dfrac{1}{2}\left(0,\theta\left(\left[X,Y\right]_\mu,Z\right)\right)+\dfrac{1}{2}\theta\left(Y,Z\right)\widetilde{\nabla}\left(\left(X,0\right),\left(0,1\right)\right)\\ & -\dfrac{1}{2}\theta\left(X,Z\right)\widetilde{\nabla}\left(\left(Y,0\right),\left(0,1\right)\right)-\theta\left(X,Y\right)\widetilde{\nabla}\left((Z,0),(0,1)\right).\end{array}$$

For all $X\in\mathfrak{g}$ we define $a_X\in\mathbb{R}$ by

$$\widetilde{\nabla}((X,0),(0,1))=(V_X,a_X).$$

The nullity of the curvature tensor implies that

$$(**)\left\{\begin{array}{l}\frac{1}{2}\theta\left(Y,Z\right)V_X-\frac{1}{2}\theta\left(X,Z\right)V_Y-\theta\left(X,Y\right)V_Z=0\\ \theta\left(\left[X,Y\right]_\mu,Z\right)+\theta\left(Y,Z\right)a_X-\theta\left(X,Z\right)a_Y-2\theta\left(X,Y\right)a_Z=0\end{array}\right.$$

As θ is of maximal rank, for all $X\in\mathfrak{g}$, there exists $Y,Z\in\mathfrak{g}$ such as $\theta\left(X,Z\right)=0=\theta\left(X,Y\right)$ and $\theta\left(Y,Z\right)=1$. The first of the relations $(**)$ implies that $V_X=0$. Thus

$$\widetilde{\nabla}\left(\left(X,0\right),\left(0,1\right)\right)=\left(0,a_X\right)$$

for all $X\in\mathfrak{g}$. Then we have

$$\widetilde{\nabla}\left(\left(X,0\right),\widetilde{\nabla}\left(\left(Y,0\right),\left(0,1\right)\right)\right)=\widetilde{\nabla}\left(\left(X,0\right),\left(0,a_Y\right)\right)=a_Xa_Y\left(0,1\right)$$

and

$$C((X,0),(0,1),(Y,0))=0$$

implies that

$$a_X a_Y(0,1) = a_{\nabla(X,Y)}(0,1) + \frac{1}{2}\theta(X,Y)\widetilde{\nabla}((0,1),(0,1)).$$

Let us chose $X, Y \in \mathfrak{g}$ such that $\theta(X,Y) = 1$. We can deduce

$$\pi(\widetilde{\nabla}((0,1),(0,1))) = 0$$

and then

$$\widetilde{\nabla}((0,1),(0,1)) = (0,a), \qquad a \in \mathbb{R}.$$

Moreover

$$\begin{aligned} a_X a_Y &= a_{\nabla(X,Y)} + \frac{1}{2}\theta(X,Y)a \\ a_X a_Y &= a_{\nabla(Y,X)} - \frac{1}{2}\theta(X,Y)a \end{aligned}$$

which gives

$$\begin{aligned} \theta(X,Y)a &= -a_{\nabla(X,Y)} + a_{\nabla(X,Y)} \\ &= -a_{[X,Y]}. \end{aligned}$$

Let us denote the linear form α de $\mathfrak{g}^*$ defined by $\alpha(X) = a_X$. If $a \neq 0$, then $\theta(X,Y) = \frac{1}{a}d\alpha(X,Y)$. The symplectic cocycle θ is then exact. But on any nilpotent Lie algebra, the class of linear forms is odd ([G]). We deduce that it can not exist exact symplectic form on $\mathfrak{g}$ and the previous equality can not be true. If $a = 0$,

$$\widetilde{\nabla}((0,1),(0,1)) = 0$$

and

$$a_{[X,Y]} = 0.$$

Then the application

$$\alpha : \mathfrak{g} \to \mathbb{R}$$

given by $\alpha(X) = a_X$ defines a one-dimensional linear representation.

Proposition 86 *The application $\widetilde{\nabla}$ on the contact Lie algebra $\widetilde{\mathfrak{g}}$ given by*

$$\begin{cases} \widetilde{\nabla}((X,0),(Y,0)) = (\nabla(X,Y), 1/2\theta(X,Y)) \\ \widetilde{\nabla}((X,0),(0,\lambda)) = \widetilde{\nabla}((0,\lambda),(X,0)) \end{cases}$$

where ∇ is the affine structure on $\mathfrak{g} = \frac{\widetilde{\mathfrak{g}}}{Z(\widetilde{\mathfrak{g}})}$ associated to the symplectic structure θ is an affine structure if there exits a one-dimensional representation of $\mathfrak{g}$ such that

$$\theta\left([X,Y]_\mu, Z\right) + \theta(Y,Z)a_X - \theta(X,Z)a_Y - 2\theta(X,Y)a_Z = 0$$

for all $X, Y, Z \in \mathfrak{g}$.

10.7.4 General case

We still suppose

$$\theta([X,Y],Z) = -\theta(Y, \nabla(X,Z))$$

We saw that the nullity of the curvature of the connection associated to the bilinear application $\widetilde{\nabla}$ implies:

$$\varphi(Y,Z)V_X - \varphi(X,Z)V_Y - \theta(X,Y)V_Z = 0$$

for all $X, Y, Z \in \mathfrak{g}$.

Let us suppose that we have $X \in \mathfrak{g}$ such that $V_X \neq 0$. Let us take a vector X satisfying this property. The orthogonal space for θ of the space $\mathbb{R}\{X\}$ is of codimension 1. Let $Y \in \mathbb{R}\{X\}^{\perp}$. Then $\theta(X,Y)=0$ and

$$\varphi(Y,Z)\,V_X = \varphi(X,Z)\,V_Y, \quad Z \in \mathfrak{g}$$

If we can find a vector $Z \in \mathfrak{g}$ such as $\varphi(Y,Z) \neq 0$, we have that $V_Y \neq 0$ and in this case the non zero vectors V_X and V_Y are colinear. Let us assume that

$$V_X = \lambda_{X,Y} V_Y$$

then

$$\lambda_{X,Y}\varphi(Y,Z) = \varphi(X,Z)$$

for all $Z \in \mathfrak{g}$. Let us consider a basis $(X_1, \cdots, X_{2m})$ of $\mathfrak{g}$ in which the matrix of θ is reduced to the following form:

$$\begin{pmatrix} \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & & 0 \\ & \ddots & \\ 0 & & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \end{pmatrix}$$

This shows that the matrix of φ is at least of rank 2 because $\varphi(X,Y) - \varphi(Y,X) = \theta(X,Y)$. For instance, if $\mathfrak{g}$ is 4-dimensional, the matrix of φ would be of the following form:

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \alpha_2 - 1 & \alpha_6 & \alpha_7 & \alpha_8 \\ \alpha_3 & \alpha_7 & \alpha_{11} & \alpha_{12} \\ \alpha_4 & \alpha_8 & \alpha_{12} - 1 & \alpha_{16} \end{pmatrix}$$

We can suppose that $\varphi(X_1, \cdot) \neq 0$ (in other case, there would exist a vector X_i such that $\varphi(X_i, Z) \neq 0$ since φ is of non zero rank and it is sufficient to take a basis adapted to θ with X_i as first vector). Let Y belong to vector space $\{X_3, \cdots, X_{2n}\}$; we have that $\theta(X_1, Y) = 0$

$$\varphi(X_1, Z) = \lambda_{X_1,Y}\varphi(Y,Z) \qquad \text{for all } Z \in \mathfrak{g}$$

and

$$\lambda_{X_1,Y} \neq 0$$

Therefore the $(2m-2)$ last columns of the matrix of φ (associated to the chosen basis) are proportional to the first one. But we also have $\theta(X_2, X_3) = 0$ and

$$\varphi(X_2, Z) = \lambda_{X_2,X_3}\varphi(X_3, Z)$$

Similarly we show that $\theta(X_1, X_3) = 0$ and

$$\varphi(X_1, Z) = \lambda_{X_1,X_3}\varphi(X_3, Z) \qquad \text{for all } Z \in \mathfrak{g}.$$

We deduce that $\varphi(X_2, Z) = \lambda\varphi(X_1, Z)$, which implies that φ is of rank 1. This is impossible and then we have $V_X = 0$.

Proposition 87 *Let $\widetilde{\mathfrak{g}}$ be a contact nilpotent Lie algebra. If the affine structure ∇ which is defined by a symplectic cocycle on $\bar{\mathfrak{g}} = \frac{\widetilde{\mathfrak{g}}}{Z(\widetilde{\mathfrak{g}})}$ can be extended to an affine structure $\widetilde{\nabla}$ on $\widetilde{\mathfrak{g}}$, we have:*

$$\pi(\widetilde{\nabla}(X,T)) = 0$$

for all $X \in \mathfrak{g}$ and $T \in Z(\widetilde{\mathfrak{g}})$.

We have that for all vector X in $\mathfrak{g}$, $V_X = 0$ and $\widetilde{\nabla}\left((X,0),(0,1)\right) = (0, a_X)$. Then the equality

$$C((X,0),(Y,0),(Z,0)) = 0$$

implies that

$$\begin{aligned}&\varphi\left(X, \nabla\left(Y,Z\right)\right) - \varphi\left(Y, \nabla\left(X,Z\right)\right) - \varphi\left(\left[X,Y\right]_{\mu}, Z\right)\\ &= -a_X\varphi\left(Y,Z\right) + a_Y\varphi\left(X,Z\right) + a_Z\theta\left(X,Y\right)\end{aligned}$$

Similarly $C((X,0),(0,1),(Y,0)) = 0$ implies that

$$\widetilde{\nabla}\left((X,0),(0,a_Y)\right) - \left(0, a_{\nabla(X,Y)}\right) - \varphi\left(X,Y\right)\widetilde{\nabla}\left((0,1),(0,1)\right) = 0.$$

This gives the following equation

$$\varphi\left(X,Y\right)\widetilde{\nabla}\left((0,1),(0,1)\right) = (a_Y a_X - a_{\nabla(X,Y)})(0,1)$$

and

$$\varphi\left(Y,X\right)\widetilde{\nabla}\left((0,1),(0,1)\right) = (a_Y a_X - a_{\nabla(Y,X)})(0,1)$$

if we permute the vectors X and Y. We combine this two equations to obtain:

$$\begin{aligned}\theta(X,Y)\widetilde{\nabla}\left((0,1),(0,1)\right) &= (a_{\nabla(Y,X)} - a_{\nabla(X,Y)})(0,1)\\ &= a_{[X,Y]}(0,1).\end{aligned}$$

This shows in particular that $\widetilde{\nabla}\left((0,1),(0,1)\right) = \rho(0,1)$ and

$$\rho\theta(X,Y) = a_{[X,Y]}.$$

Finally $C((0,1),(Y,0),(0,1)) = 0$ implies

$$a_Y\widetilde{\nabla}\left((0,1),(0,1)\right) = \widetilde{\nabla}\left((Y,0), \widetilde{\nabla}\left((0,1),(0,1)\right)\right)$$

thus

$$a_Y\widetilde{\nabla}\left((0,1),(0,1)\right) = \rho\left(0,a_Y\right).$$

This last equation is already satisfied.

Then let us suppose $\rho \neq 0$. In this case $a_{[X,Y]} \neq 0$ when $\theta(X,Y) \neq 0$. Let us take X in $Z(\mathfrak{g})$. As θ is of maximal rank, there is one Y such that $\theta(X,Y) \neq 0$. But $[X,Y] = 0$ implies $a_{[X,Y]} = 0$. This leads to contradiction.

Conclusion . As $\rho = 0$ we have that $\widetilde{\nabla}\left((0,1),(0,1)\right) = 0$. Then $a_{[X,Y]} = 0$ and the application $\alpha : \mathfrak{g} \rightarrow \mathbb{R}$ defined by $\alpha(X) = a_X$ gives an one-dimensional linear representation of $\mathfrak{g}$. We deduce

Theorem 110 *Let $\alpha : \mathfrak{g} \rightarrow \mathbb{R}$ be a one-dimensional linear representation of $\mathfrak{g}$.*

When α is the trivial representation, $\widetilde{\nabla}$ is an affine structure if and only if

1) $\widetilde{\nabla}\left(U,(0,1)\right) = 0$ for all $U \in \widetilde{\mathfrak{g}}$.

2) φ satisfies $\varphi\left(X, \nabla\left(Y,Z\right)\right) - \varphi\left(Y, \nabla\left(X,Z\right)\right) - \varphi\left(\left[X,Y\right]_{\mu}, Z\right) = 0$, i.e. if it is a 2-cocycle for cohomology of the Vinberg algebra associated to ∇ with values in a trivial module.

When α is a non-trivial representation, $\widetilde{\nabla}$ is an affine structure if and only if

1) $\widetilde{\nabla}\left((0,1),(0,1)\right) = 0$, $\widetilde{\nabla}\left((X,0),(0,1)\right) = 0$ for all $X \in Ker\alpha$.

2) $\varphi\left(X, \nabla\left(Y,Z\right)\right) - \varphi\left(Y, \nabla\left(X,Z\right)\right) - \varphi\left(\left[X,Y\right]_{\mu}, Z\right) = \alpha(Z)\theta(X,Y)$ for all $X, Y \in Ker(\alpha)$.

10.8 Current affine structures on nilpotent or solvable Lie algebras

In the first part, we have introduced the notion of associated current operad. Recall that any affine structure on a Lie algebra is determinate by a Vinberg algebra. If we consider the operad of Vinberg algebras, the associated current operad is the dual operad denoted by $\mathcal{P}erm$. A $\mathcal{P}erm$-algebra is an associative algebra satisfying

$$abc = bac$$

for any a, b, c in this algebra. This result permits to construct interesting classes of Vinberg algebras and to give new examples of Lie algebras provided with affine structure. For example suppose $dimA = dimB = 2$. Then A is isomorphic to one of the following algebras :

1. A is commutative and isomorphic to

$$A_1 : \begin{cases} X_1 \cdot X_1 = X_1 \\ X_1 \cdot X_2 = X_2 \cdot X_1 = X_2 \\ X_2 \cdot X_2 = X_2 \end{cases} \qquad A_2 : \begin{cases} X_1 \cdot X_1 = X_1 \\ X_1 \cdot X_2 = X_2 \cdot X_1 = X_2 \\ X_2 \cdot X_2 = 0 \end{cases}$$

$$A_3 : \begin{cases} X_1 \cdot X_1 = X_1 \\ X_1 \cdot X_2 = X_2 \cdot X_1 = X_2 \\ X_2 \cdot X_2 = -X_1 \end{cases} \qquad A_4 : \{ X_1 \cdot X_1 = X_2$$

$$A_5 : \{ \ X_1 \cdot X_1 = X_1 \qquad A_6 : \{ \ X_i \cdot X_j = 0$$

2. A is non commutative and isomorphic to

$$A_7 : \begin{cases} X_1 \cdot X_1 = \frac{b^2+2e}{e} X_1 - b\frac{b^2+e}{e^2} X_2 \\ X_1 \cdot X_2 = bX_1 - \frac{b^2-e}{e} X_2 \\ X_2 \cdot X_1 = bX_1 - \frac{b^2}{e} X_2 \\ X_2 \cdot X_2 = eX_1 - bX_2 \end{cases} \qquad A_8 : \begin{cases} X_1 \cdot X_1 = aX_1 + cX_2 \\ X_1 \cdot X_2 = X_2 \\ X_2 \cdot X_1 = X_2.X_2 = 0 \end{cases}$$

$$A_9 : \begin{cases} X_1 \cdot X_1 = aX_1 \\ X_1 \cdot X_2 = (a+1)X_2 \\ X_2 \cdot X_1 = aX_2 \\ X_2 \cdot X_2 = 0 \end{cases}$$

In this case the Lie algebra associated to A is the two-dimensional solvable abelian Lie algebra.

Let us classify the $Vinb^!$ algebra of dimension 2.

1. B is commutative and isomorphic to A_i, $i = 1, .., 7$.
2. B is non commutative and isomorphic to

$$B_7 : \begin{cases} e_1 \cdot e_1 = e_1 \\ e_1 \cdot e_2 = e_2 \\ e_2 \cdot e_1 = e_2 \cdot e_2 = 0 \end{cases}$$

If A and B are commutative, the corresponding Lie algebra is the 4-dimensional abelian Lie algebra.

Suppose A is commutative and B is not commutative. In this case the bracket of the Lie algebra associated to $A \otimes B$ satisfy

$$[X_i \otimes e_j, X_k \otimes e_l] = X_i X_k \otimes e_j e_l - X_k X_i \otimes e_l e_j = X_i X_k \otimes [e_j, e_l]$$

with $[e_1, e_2] = e_2$.

When we put $f_{ij} = X_i \otimes e_j$, we obtain the following list of Lie algebras, $\mathfrak{g}_{i7}$,underlying to the Vinberg algebra $A_i \otimes B_7$:

g_{17}	$[f_{11}, f_{12}] = f_{12}, \quad [f_{11}, f_{22}] = f_{22}, \quad [f_{12}, f_{21}] = -f_{22}, \quad [f_{21}, f_{22}] = f_{22}$
g_{27}	$[f_{11}, f_{12}] = f_{12}, \quad [f_{11}, f_{22}] = f_{22}, \quad [f_{12}, f_{21}] = -f_{22}$
g_{37}	$[f_{11}, f_{12}] = f_{12}, \quad [f_{11}, f_{22}] = f_{22}, \quad [f_{12}, f_{21}] = -f_{22}, \quad [f_{21}, f_{22}] = -f_{12}$
g_{47}	$[f_{11}, f_{12}] = f_{22}$
g_{57}	$[f_{11}, f_{12}] = f_{12}$
g_{67}	*abelian*

Likewise, if A is a non commutative Vinberg algebra and B a commutative $Vinb^!$. algebra then the bracket of the corresponding Lie algebra satisfies

$$[X_i \otimes e_j, X_k \otimes e_l] = X_iX_k \otimes e_je_l - X_kX_i \otimes e_le_j = [X_i, X_k] \otimes e_je_l$$

For the algebras $A_i \otimes B_j$, $i = 7, 8, 9$, let us note that the corresponding Lie algebras $\mathfrak{g}_{ij}$ satisfy $\mathfrak{g}_{7j} = \mathfrak{g}_{8j} = \mathfrak{g}_{9j}$ for $j = 1, \cdots, 7$. Using the same previous notations we obtain the following Lie algebras :

g_{i1}	$[f_{11}, f_{21}] = f_{21}, \quad [f_{11}, f_{22}] = f_{22}, \quad [f_{12}, f_{21}] = f_{22}, \quad [f_{12}, f_{22}] = f_{22}$
g_{i2}	$[f_{11}, f_{21}] = f_{21}, \quad [f_{11}, f_{22}] = f_{22}, \quad [f_{12}, f_{21}] = f_{22}$
g_{i3}	$[f_{11}, f_{21}] = f_{22}, \quad [f_{11}, f_{22}] = f_{22}, \quad [f_{12}, f_{21}] = f_{22}, \quad [f_{12}, f_{22}] = -f_{21}$
g_{i4}	$[f_{11}, f_{21}] = f_{22}$
g_{i5}	$[f_{11}, f_{21}] = f_{21}$
g_{i6}	*abelian*

At last, let us look the case $A = A_i$, $i = 7, 8, 9$ and $B = B_7$. We obtain :

g_{77} : $[f_{11}, f_{12}] = \frac{b^2+2e}{e} f_{12} - b\frac{b^2+e}{e^2} f_{22}, \quad [f_{11}, f_{21}] = f_{21},$
$[f_{11}, f_{22}] = bf_{12} - \frac{b^2-e}{e} f_{22}$
$[f_{12}, f_{21}] = -bf_{12} + \frac{b^2}{e} f_{22}, \quad [f_{21}, f_{22}] = ef_{12} - bf_{22}$

g_{87} : $[f_{11}, f_{12}] = af_{12}, \quad [f_{11}, f_{21}] = f_{21}, \quad [f_{11}, f_{22}] = (a+1) f_{22},$
$[f_{12}, f_{21}] = -af_{22}.$

g_{97} : $[f_{11}, f_{12}] = af_{12} + cf_{22}, \quad [f_{11}, f_{21}] = f_{21}, \quad [f_{11}, f_{22}] = f_{22}$

Comparing with the classification of 4-dimensional real Lie algebras presented in [**?**], we obtain :

Theorem 111 *The following solvable Lie algebras have an affine structure of tensorial type :*

$\mathfrak{g}_{3,2}(1) \oplus \mathbb{R}, \quad \mathfrak{g}_{4,5}(1, a), \quad \mathfrak{g}_{4,6}(1), \quad \mathfrak{g}_{4,9}(\alpha), \quad \mathfrak{g}_{4,10}, \quad \mathfrak{g}_2 \oplus \mathfrak{g}_2, \quad \mathfrak{g}_{4,1}$
$\mathfrak{g}_{4,2}, \quad \mathfrak{g}_2 \oplus \mathbb{R}^2, \quad \mathfrak{g}_{3,1}(1) \oplus \mathbb{R}, \quad \mathbb{R}^4.$

Chapter 11

RIEMANNIAN Γ-SYMMETRIC SPACES

A Γ-symmetric space is a reductive homogeneous space $M = G/H$ provided at each of its points with a finite abelian group of "symmetries" isomorphic to Γ. In case of $\Gamma = \mathbb{Z}_2^p$, the Lie algebra $\mathfrak{g}$ of G is graded by Γ and this grading permits to construct again the symmetries of M. An adapted Riemannian metric will be a tensor metric for which the symmetries are isometries. We give in case of $p = 2$ the classification of compact $\mathbb{Z}_2^2$-symmetric spaces corresponding to G compact simple. In particular we find again the oriented flag manifold. For this class of nonsymmetric spaces and for the space $SO(2m)/Sp(m)$ we describe all these metrics. In particular we prove that in the definite positive case, these metrics are not, in general, naturally reductive. We class also the Lorentzian $\mathbb{Z}_2^2$-symmetric metrics.

11.1 Riemannian reductive homogeneous spaces

Let $M = G/H$ be a homogeneous space where G is a connected Lie group which acts effectively on M. It is a reductive space if the Lie algebra $\mathfrak{g}$ of G can be decomposed into a direct sum of vector spaces of the Lie algebra $\mathfrak{h}$ of H and an $ad(H)$-invariant subspace $\mathfrak{m}$:

$$\begin{cases} \mathfrak{g} = \mathfrak{h} + \mathfrak{m} \\ ad(H)\mathfrak{m} \subset \mathfrak{m}. \end{cases}$$

If H is connected, and we will assume it in all the following, the second condition is equivalent to

$$[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}.$$

We denote by ∇_M (or simply ∇) the canonical G-invariant connection on M. Recall that its torsion tensor T_{∇_M} and curvature tensor R_{∇_M} satisfy:

$$\begin{cases} T_{\nabla_M}(X,Y)_0 = [X,Y]_{\mathfrak{m}}, \quad X, Y \in \mathfrak{m} \quad (0 \text{ denotes the class of } 1_G \text{ in } G/H), \\ (R_{\nabla_M}(X,Y)Z)_0 = -[[X,Y],Z], \quad X,Y,Z \in \mathfrak{m}, \\ \nabla_M T_{\nabla_M} = \nabla_M R_{\nabla_M} = 0. \end{cases}$$

We denote by $\widetilde{\nabla}_M$ the natural (complete) torsion-free G-invariant connection on M. It admits the same geodesics as ∇_M and it is defined by

$$\widetilde{\nabla}_M(X)(Y) = \frac{1}{2}[X,Y]_{\mathfrak{m}}, \quad X, Y \in \mathfrak{m}.$$

Let g be a G-invariant indefinite Riemannian metric on the reductive homogeneous space $M = G/H$. It is completely determinate by an $ad(H)$-invariant non degenerate symmetric bilinear form B on $\mathfrak{m}$, the correspondence is given by

$$B(X,Y) = g(X,Y)_0, \quad X, Y \in \mathfrak{m}.$$

Recall that g is positive definite if and only if B is positive definite. We denote by ∇_g the corresponding Riemannian connection. It coincides with the natural connection $\widetilde{\nabla}_M$ if and only if we have

$$\forall X, Y, Z \in \mathfrak{m}, \quad B(X, [Z, Y]_{\mathfrak{m}}) + B([Z, X]_{\mathfrak{m}}, Y) = 0.$$

In this case the Riemannian reductive homogeneous space M is said to be naturally reductive.

For example, if M is a symmetric space, that is, the decomposition of $\mathfrak{g}$ satisfies $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ with

$$\begin{cases} [\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h} \\ [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m} \\ [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h} \end{cases}$$

then the canonical and the natural connections coincide. Moreover if ∇_g is a Riemannian symmetric connection, that is, the symmetries of M are isometries, we have

$$\nabla_M = \widetilde{\nabla}_M = \nabla_g$$

and M is naturally reductive.

11.2 Riemannian Γ-symmetric spaces

11.2.1 Γ-symmetric spaces

Let Γ be a finite abelian group. A Γ-symmetric space is a triple (G, H, Γ_G) where G is a connected Lie group, H a closed subgroup of G and Γ_G an abelian finite subgroup of the group of automorphisms of G isomorphic to Γ:

$$\Gamma_G = \{\rho_\gamma \in Aut(G), \ \gamma \in \Gamma\}$$

such that H lies between G_Γ, the closed subgroup of G consisting of all elements left fixed by the automorphisms of Γ_G, and the identity component of G_Γ. The elements of Γ_G satisfy:

$$\begin{cases} \rho_{\gamma_1} \circ \rho_{\gamma_2} = \rho_{\gamma_1\gamma_2}, \quad \forall \gamma_1, \gamma_2 \in \Gamma \\ \rho_e = Id \quad \text{where } e \text{ is the unit of } \Gamma \\ (\rho_\gamma(g) = g \quad \forall \gamma \in \Gamma) \Longleftrightarrow g \in H. \end{cases}$$

We also suppose that H does not contain any proper normal subgroup of G.

Given a Γ-symmetric space (G, H, Γ_G), we construct for each point x of $M = G/H$ a subgroup Γ_x of *Diff*(M), the group of diffeomorphisms of M, isomorphic to Γ and which has x as an isolated fixed point. We denote by $\bar{g}$ the class of $g \in G$ in M and by e the identity of G. We consider

$$\Gamma_{\bar{e}} = \{s_{(\gamma,\bar{e})} \in Diff(M), \ \gamma \in \Gamma\},$$

with $s_{(\gamma,\bar{e})}(\bar{g}) = \overline{\rho_\gamma(g)}$. For another point $x = \overline{g_0}$ of M we have

$$\Gamma_x = \{s_{(\gamma,x)} \in Diff(M), \ \gamma \in \Gamma\},$$

with $s_{(\gamma,\bar{g_0})}(y) = g_0(s_{(\gamma,\bar{e})})(g_0^{-1}y)$. All these subgroups Γ_x of *Diff*(M) are isomorphic to Γ. The elements of Γ_x will be called the symmetries of M at the point x or more generally the symmetries of M.

Remark. If (G, H, Γ_G) is a Γ-symmetric space, we will also say that the homogeneous space $M = G/H$ is Γ-symmetric.

11.2.2 Γ-grading of the Lie algebra $\mathfrak{g}$ of G

Let (G, H, Γ_G) be a Γ-symmetric space. Let $\mathfrak{g}$ (resp. $\mathfrak{h}$) be the Lie algebra of G (resp. H). Each automorphism ρ_γ of G induces an automorphism τ_γ of $\mathfrak{g}$ and the set

$$\Gamma_{\mathfrak{g}} = \{\tau_\gamma \in Aut(\mathfrak{g}), \gamma \in \Gamma\}$$

is a finite group isomorphic to Γ such that $(\forall \gamma \in \Gamma, \quad \tau_\gamma(X) = X\ ) \Leftrightarrow X \in \mathfrak{h}$. As each of the linear morphism τ_γ is diagonalizable, the relation $\tau_\gamma \circ \tau_{\gamma'} = \tau_{\gamma'} \circ \tau_\gamma$ implies that $\mathfrak{g}$ is a vectorial direct sum of root spaces $\mathfrak{g}_\chi, \chi \in \Gamma^*$ where Γ^* denotes the dual group of Γ. As Γ is abelian, Γ^* is isomorphic to Γ and we can identify Γ and Γ^*. We deduce that $\mathfrak{g}$ is Γ-graded, $\mathfrak{g} = \underset{\gamma\in\Gamma}{\oplus} \mathfrak{g}_\gamma$ with $\mathfrak{g}_e = \mathfrak{h}$ where e is the identity of Γ.

Conversely, we prove in [8] that every Γ-graded Lie algebra $\mathfrak{g} = \underset{\gamma\in\Gamma}{\oplus} \mathfrak{g}_\gamma$ determines a Γ-symmetric space (G, H, Γ) where G (resp. H) is a connected Lie group associated to $\mathfrak{g}$ (resp. $\mathfrak{g}_e$).

Theorem 112 *Every Γ-symmetric space is a reductive homogeneous space.*

In fact, if $M = G/H$ is a Γ-symmetric space, the Lie algebra $\mathfrak{g}$ of G is Γ-graded, that is, $\mathfrak{g} = \underset{\gamma\in\Gamma}{\oplus} \mathfrak{g}_\gamma$. Then we put $\mathfrak{h} = \mathfrak{g}_e$ and $\mathfrak{m} = \underset{\gamma\in\Gamma,\gamma\neq e}{\oplus} \mathfrak{g}_\gamma$. We have $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$ and $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$. If we assume that H is connected, this implies that $M = G/H$ is a reductive space

11.2.3 Riemannian and Indefinite Riemannian Γ-symmetric spaces

Let $M = G/H$ be a Γ-symmetric space. Recall that for all $x \in M, \Gamma_x = \{s(\gamma, x), \gamma \in \Gamma\}$ is a subgroup of $Diff(M)$ isomorphic to Γ. Let $S(\gamma, x)$ be the tangent map $T_x s(\gamma, x)$. Thus $S(\gamma, x) \in GL(T_xM)$ and $\{S(\gamma, x), \gamma \in \Gamma\}$ is a finite subgroup of $GL(T_xM)$ isomorphic to Γ.

Definition 113 *A Riemannian (resp. Indefinite Riemannian) metric on M is called a Γ-symmetric Riemannian (resp. Indefinite Riemannian) metric if for all $x \in M$, the linear symmetries $S(\gamma, x)$, $\gamma \in \Gamma$ are isometries.*

Remark. If $\Gamma = \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$, we find the classical notion of Riemannian symmetric space again. When the Riemannian metric g is positive definite, the natural and canonical connections coincide, and they are also equal to the Riemannian connection associated to g. All these spaces are naturally reductive. In general, this property is false for the Riemannian Γ-symmetric spaces, when Γ is not isomorphic to $\mathbb{Z}_2$. We will see in the following section examples of $\mathbb{Z}_2^2$-Riemannian spaces which are not naturally reductive.

11.2.4 Irreducible Riemannian Γ-symmetric spaces

Let (G, H, Γ_G) be a Γ-symmetric space. Since G/H is a reducible homogeneous space with an $ad(H)$-invariant decomposition $\mathfrak{g} = \mathfrak{g}_e \oplus \mathfrak{m}$, the Lie algebra of the holonomy group of ∇ is spanned by the endomorphisms of $\mathfrak{m}$ given by $R(X, Y)_0$ for all $X, Y \in \mathfrak{m}$. Recall that $(R(X, Y)Z)_0 = -[[X, Y]_{\mathfrak{h}}, Z]$ for all $X, Y, Z \in \mathfrak{m}$. In particular we have $R(X, Y)_0 = 0$ as soon as $X \in \mathfrak{g}_\gamma, Y \in \mathfrak{g}_{\gamma'}$ with $\gamma, \gamma' \neq e$. For example, if $\Gamma = \mathbb{Z}_2^2$ then $\mathfrak{g} = \mathfrak{g}_e \oplus \mathfrak{g}_a \oplus \mathfrak{g}_b \oplus \mathfrak{g}_c$ and $R(\mathfrak{g}_a, \mathfrak{g}_b)_0 = R(\mathfrak{g}_a, \mathfrak{g}_c)_0 = R(\mathfrak{g}_b, \mathfrak{g}_c)_0 = 0$.

Lemma 21 *Let $\mathfrak{g}$ be a simple Lie algebra $\mathbb{Z}_2^2$-graded. Then*

$$[\mathfrak{g}_a, \mathfrak{g}_a] \oplus [\mathfrak{g}_b, \mathfrak{g}_b] \oplus [\mathfrak{g}_c, \mathfrak{g}_c] = \mathfrak{g}_e.$$

Proof. Let U denote $[\mathfrak{g}_a, \mathfrak{g}_a] \oplus [\mathfrak{g}_b, \mathfrak{g}_b] \oplus [\mathfrak{g}_c, \mathfrak{g}_c]$. Then $I = U \oplus \mathfrak{g}_a \oplus \mathfrak{g}_b \oplus \mathfrak{g}_c$ is an ideal of $\mathfrak{g}$. In fact, $X \in I$ is decomposed as $X_U + X_a + X_b + X_c$. The main point is to prove that $[X_U, Y]$ is in I for any $Y \in \mathfrak{g}_e$. But X_U is decomposed as $[X_a, Y_a] + [X_b, Y_b] + [X_c, Y_c]$. The Jacobi identity shows that $[[X_a, Y_a], Y] \in [\mathfrak{g}_a, \mathfrak{g}_a]$ and it is similar for the other components. Then I is an ideal of $\mathfrak{g}$ which is simple, and hence $U = \mathfrak{g}_e$.

One should note that in any case, as soon as Γ is not $\mathbb{Z}_2$, the representation $ad\, \mathfrak{g}_e$ is not irreducible on $\mathfrak{m}$. In fact, each component $\mathfrak{g}_\gamma$ is an invariant subspace of $\mathfrak{m}$.

Definition 114 *The representation* $ad\,\mathfrak{g}_e$ *on* $\mathfrak{m}$ *is called* Γ-irreducible *if* $\mathfrak{m}$ *cannot be written* $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ *with* $\mathfrak{g}_e \oplus m_1$ *and* $\mathfrak{g}_e \oplus m_2$ Γ*-graded Lie algebras.*

Example. Let $\mathfrak{g}_1$ be a simple Lie algebra and $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_1$. Let $\sigma_1, \sigma_2, \sigma_3$ be the automorphisms of $\mathfrak{g}$ given by

$$\begin{cases} \sigma_1(X_1, X_2, X_3, X_4) = (X_2, X_1, X_3, X_4), \\ \sigma_2(X_1, X_2, X_3, X_4) = (X_1, X_2, X_4, X_3), \\ \sigma_3 = \sigma_1 \circ \sigma_2. \end{cases}$$

They define a $(\mathbb{Z}_2^2)$-graduation on $\mathfrak{g}$ and we have $\mathfrak{g}_e = \{(X, X, Y, Y)\}$, $\mathfrak{g}_a = \{(0, 0, Y, -Y)\}$, $\mathfrak{g}_b = \{(X, -X, 0, 0)\}$ and $\mathfrak{g}_c = \{(0, 0, 0, 0)\}$ with $X, Y \in \mathfrak{g}_1$. In particular, $\mathfrak{g}_a$ is isomorphic to $\mathfrak{g}_1$, hence $[\mathfrak{g}_e, \mathfrak{g}_a] = \mathfrak{g}_a$ and, since $\mathfrak{g}_1$ is simple, we cannot have $\mathfrak{g}_a = \mathfrak{g}_a^1 + \mathfrak{g}_a^2$ with $[\mathfrak{g}_e, \mathfrak{g}_a^i] = \mathfrak{g}_a^i$ for $i = 1, 2$. Then $\mathfrak{g}$ is $(\mathbb{Z}_2^2)$-graded and this decomposition is $(\mathbb{Z}_2^2)$-irreducible.

Suppose now that $\mathfrak{g}$ is a simple Lie algebra. Let K be the Killing-Cartan form of $\mathfrak{g}$. It is invariant by all automorphisms of $\mathfrak{g}$. In particular

$$K(\tau_\gamma X, \tau_\gamma Y) = K(X, Y)$$

for any $\tau_\gamma \in \check{\Gamma}$. If $X \in \mathfrak{g}_\alpha$ and $Y \in \mathfrak{g}_\beta$, $\alpha \neq \beta$ there exists $\gamma \in \Gamma$ such that $\tau_\gamma X = \lambda(\alpha, \gamma) X$ and $\tau_\gamma Y = \lambda(\beta, \gamma) Y$ with $\lambda(\alpha, \gamma)\lambda(\beta, \gamma) \neq 1$. Thus $K(X, Y) = 0$ and the homogeneous components $\mathfrak{g}_\gamma$ are pairwise orthogonal with respect to K. Moreover $K_\gamma = K|_{\mathfrak{g}_\gamma}$ is a nondegenerate bilinear form. Since $\mathfrak{g}$ is a simple Lie algebra, there exists an $ad\,\mathfrak{g}_e$-invariant inner product $\tilde{B}$ on $\mathfrak{g}$ such that the restriction $B = \tilde{B}|_\mathfrak{m}$ to $\mathfrak{m}$ defines a Riemannian Γ-symmetric structure on G/H. This means that $\tilde{B}(\mathfrak{g}_\gamma, \mathfrak{g}_{\gamma'}) = 0$ for $\gamma \neq \gamma' \in \Gamma$. We consider an orthogonal basis of $\tilde{B}$. For each $X \in \mathfrak{g}_e$, $ad\,X$ is expressed by a skew-symmetric matrix $(a_{ij}(X))$ and $K(X, X) = \sum_{i,j} a_{ij}(X) a_{ji}(X) < 0$. This implies that K is negative-definite on $\mathfrak{g}_e$.

Let K_γ and B_γ be the restrictions of K and B to the homogeneous component $\mathfrak{g}_\gamma$. Let $\beta \in \mathfrak{m}^*$ be such that

$$K_\gamma(X, Y) = B_\gamma(\beta_\gamma(X), Y)$$

for all $X, Y \in \mathfrak{g}_\gamma$ and $\beta_\gamma = \beta|_{\mathfrak{g}_\gamma}$. Since B_γ is nondegenerate on $\mathfrak{g}_\gamma$, the eigenvalues of β_γ are real and non-zero. The eigenspaces $\mathfrak{g}_\gamma^1, \cdots, \mathfrak{g}_\gamma^p$ of β_γ are pairwise orthogonal with respect to B_γ and K_γ. But for every $Z \in \mathfrak{g}_e$ we have

$$K_\gamma([Z, X], Y) = K_\gamma(X, [Z, Y]) = B_\gamma(\beta_\gamma(X), [Z, Y])$$

and hence $B_\gamma(\beta_\gamma[Z, X], Y) = B_\gamma([Z, \beta_\gamma(X)], Y)$ for every $Y \in \mathfrak{g}_\gamma$ and $\beta_\gamma[Z, X] = [Z, \beta_\gamma(X)]$, implying that $\beta_\gamma \circ ad\,Z = ad\,Z \circ \beta_\gamma$ for any $Z \in \mathfrak{g}_e$. This yields $[\mathfrak{g}_e, \mathfrak{g}_\gamma^i] \subset \mathfrak{g}_\gamma^i$.

Now we examine the particular case of $\Gamma = \mathbb{Z}_2^2$. The eigenvalues of the involutive automorphisms τ_γ being real, the Lie algebra $\mathfrak{g}$ admits a real Γ-decomposition $\mathfrak{g} = \sum_{\gamma \in \mathbb{Z}_2^2} \mathfrak{g}_\gamma$. Then we can assume that $\mathfrak{g}$ is a real Lie algebra. Now if $i \neq j$ we have

$$K_\gamma([\mathfrak{g}_\gamma^i, \mathfrak{g}_\gamma^j], [\mathfrak{g}_\gamma^i, \mathfrak{g}_\gamma^j]) \subset K([\mathfrak{g}_\gamma^i, \mathfrak{g}_\gamma^j], \mathfrak{g}_e) \subset (\mathfrak{g}_\gamma^i, \mathfrak{g}_\gamma^j) = 0$$

and we have, for $i \neq j$:

$$[\mathfrak{g}_\gamma^i, \mathfrak{g}_\gamma^j] = \{0\}$$

Let $\{e, a, b, c\}$ be the elements of $\mathbb{Z}_2^2$ with $a^2 = b^2 = c^2 = e$ and $ab = c$. Each component $\mathfrak{g}_\gamma$, $\gamma \neq e$, satisfies $[\mathfrak{g}_\gamma, \mathfrak{g}_\gamma] \subset \mathfrak{g}_e$ and $\mathfrak{g}_e \oplus \mathfrak{g}_\gamma$ is a symmetric Lie algebra. Endowed with the inner product $\tilde{B}$, the Lie algebra $\mathfrak{g}_e \oplus \mathfrak{g}_\gamma$ is an orthogonal symmetric Lie algebra. The Killing-Cartan form is not degenerate on $\mathfrak{g}_e \oplus \mathfrak{g}_\gamma$. Then $\mathfrak{g}_e \oplus \mathfrak{g}_\gamma$ is semi-simple. It is a direct sum of orthogonal symmetric Lie algebras of the following two kinds:

i) $\mathfrak{g} = \mathfrak{g}' + \mathfrak{g}'$with $\mathfrak{g}'$ simple
ii) $\mathfrak{g}$ is simple.

The first case has been studied above and the representation is $(\mathbb{Z}_2^2)$-irreducible. In the second case $ad\,[\mathfrak{g}_\gamma, \mathfrak{g}_\gamma]$ is irreducible in $\mathfrak{g}_\gamma$ and the representation is $(\mathbb{Z}_2^2)$-irreducible on $\mathfrak{m}$.

11.3 Classification of compact simple $\mathbb{Z}_2^2$ symmetric spaces

The paper [8] is, in a large part, devoted to the classification of $\mathbb{Z}_2^2$-symmetric spaces G/H in case of $\mathfrak{g}$ is simple of classical type. Recently, in [71], the exceptional case has been developed. Combining both results, we have a complete classification of $\mathbb{Z}_2^2$-symmetric space G/H when $\mathfrak{g}$ is simple complex. From these classifications, we can easily deduce the classification of compact $\mathbb{Z}_2^2$-symmetric spaces when $\mathfrak{g}$ is compact and real simple. Following the terminology of [12] and [8], if $(G, H, \mathbb{Z}_2^2)$ is a $\mathbb{Z}_2^2$-symmetric space, the corresponding pair $(\mathfrak{g}, \mathfrak{h})$ of Lie algebras is called local symmetric space. From the local classification, it is very easy to exhibit the classification of the $\mathbb{Z}_2^2$-symmetric spaces when G and H are connected and G simply connected. In the following table, we give the list of local $\mathbb{Z}_2^2$-symmetric spaces when $\mathfrak{g}$ is real simple of compact type and non exceptional.

$(\mathfrak{g}, \mathfrak{h})$
$(su(2n), su(n))$
$(su(k_1+k_2), su(k_1) \oplus su(k_2) \oplus \mathbb{C})$
$(su(k_1+k_2+k_3), su(k_1) \oplus su(k_2) \oplus su(k_3) \oplus \mathbb{C}^2)$
$(su(k_1+k_2+k_3+k_4), su(k_1) \oplus su(k_2) \oplus su(k_3) \oplus su(k_4) \oplus \mathbb{C}^3)$
$(su(n), so(n))$
$(su(2m), sp(m))$
$(su(k_1+k_2), so(k_1) \oplus so(k_2))$
$(su2(k_1+k_2), sp(2k_1) \oplus sp(2k_2))$
$(so(k_1+k_2+k_3), so(k_1) \oplus so(k_2) \oplus so(k_3)$
$(so(k_1+k_2+k_3+k_4), so(k_1) \oplus so(k_2) \oplus so(k_3) \oplus so(k_4))$
$(so(4m), sp(2m))$
$(so(2m), so(m))$
$(so(8), su(3) \oplus su(1))$
$(sp(k_1+k_2+k_3+k_4), sp(k_1) \oplus sp(k_2) \oplus sp(k_3) \oplus sp(k_4))$
$(sp(4m), sp(2m))$
$(sp(2m), so(m))$

11.4 On the classification of Riemannian compact $\mathbb{Z}_2^2$-symmetric spaces

Let $(M = G/H, g)$ be a Riemannian $\mathbb{Z}_2^2$-symmetric space. We assume that the local pair is in the previous table. In this case, M is compact. To each local pair, we have to classify, up to an isometry, the corresponding $ad(H)$-invariant bilinear form B. With regard to the symmetric case, the computation is more complicated because we obtain, not only the Killing Cartan metric, but also a large class of definite positive or only non degenerate invariant bilinear forms. In the work, we are interested by two classes

- The $\mathbb{Z}_2^2$-symmetric flag manifolds. These spaces have been very well studied and our approach permits to look these metrics with the symmetric point of view.
- The homogeneous space $SO(4m)/Sp(2m)$ because it has no equivalent in the symmetric case.

11.4.1 $\mathbb{Z}_2^2$-symmetric metrics on flag manifolds

Let $M = SO(2l+1)/SO(r_1) \times SO(r_2) \times SO(r_3) \times SO(r_4)$ be an oriented flag manifold (with $r_1 r_2 r_3 \neq 0$. This manifold is a $\mathbb{Z}_2^2$-symmetric space and the grading of the Lie algebra $so(2l+1)$ is given by

$$\mathfrak{g}_e = \begin{pmatrix} X_1 & 0 & 0 & 0 \\ 0 & X_2 & 0 & 0 \\ 0 & 0 & X_3 & 0 \\ 0 & 0 & 0 & X_4 \\ & & & \end{pmatrix}, \ \mathfrak{g}_a = \begin{pmatrix} 0 & A_1 & 0 & 0 \\ -{}^tA_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_2 \\ 0 & 0 & -{}^tA_2 & 0 \\ & & & \end{pmatrix}$$

$$\mathfrak{g}_b = \begin{pmatrix} 0 & 0 & B_1 & 0 \\ 0 & 0 & 0 & B_2 \\ -{}^tB_1 & 0 & 0 & 0 \\ 0 & -{}^tB_2 & 0 & 0 \end{pmatrix}, \ \mathfrak{g}_c = \begin{pmatrix} 0 & 0 & 0 & C_1 \\ 0 & 0 & C_2 & 0 \\ 0 & -{}^tC_2 & 0 & 0 \\ -{}^tC_1 & 0 & 0 & 0 \end{pmatrix}.$$

where A_1 (resp. B_1, C_1, C_2, A_2, X_i) is a matrix of order (r_1, r_2) (resp. $(r_1, r_3), (r_1, r_4), (r_2, r_3), (r_2, r_4), (r_3, r_4)$ and (r_i, r_i)). Let B be a $\mathfrak{g}_e$-invariant inner product on $\mathfrak{g}$. By hypothesis $B(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ as soon as $\alpha \neq \beta$ in $\mathbb{Z}_2^2$. This shows that B is written as $B = B_{\mathfrak{g}_e} + B_{\mathfrak{g}_a} + B_{\mathfrak{g}_b} + B_{\mathfrak{g}_c}$ where $B_{\mathfrak{g}_x}$ is an inner product on $\mathfrak{g}_x$. We denote by $\{\alpha_{ij}^1, \alpha_{ij}^2, \beta_{ij}^1, \beta_{ij}^2, \gamma_{ij}^1, \gamma_{ij}^2\}$ the dual basis of the elementary basis of $A_1 \oplus A_2 \oplus B_1 \oplus B_2 \oplus C_1 \oplus C_2$ given by the elementary matrices. A direct computation conduces to the following result:

Proposition 88 *Every ad($\mathfrak{g}_e$)-invariant inner product on $\mathfrak{m} = \mathfrak{g}_a \oplus \mathfrak{g}_b \oplus \mathfrak{g}_c$ is given by the formula*

$$B = t_{A_1}\Sigma(\alpha_{ij}^1)^2 + t_{A_2}\Sigma(\alpha_{ij}^2)^2 + t_{B_1}\Sigma(\beta_{ij}^1)^2 + t_{B_2}\Sigma(\beta_{ij}^2)^2 + t_{C_1}\Sigma(\gamma_{ij}^1)^2 + t_{C_2}\Sigma(\gamma_{ij}^2)^2$$

with $t_{A_1}, t_{A_2}, t_{B_1}, t_{B_2}, t_{C_1}, t_{C_2}$ not zero.

Consequences.

1. Such a bilinear form defines a naturally reductive structure on M if and only if

$$t_{A_1} = t_{A_2} = t_{B_1} = t_{B_2} = t_{C_1} = t_{C_2} = \lambda > 0.$$

2. On Lorentzian $\mathbb{Z}_2^2$-symmetric structure.

Definition 115 *Let (G, H, Γ_G) be a Γ-symmetric space, g a semi-Riemannian metric of signature $(1, n-1)$ where $n = dim\, M$ and B the corresponding $ad\mathfrak{g}_e$-invariant symmetric bilinear form on $\mathfrak{m}$. Then $M = G/H$ is called a* Γ-symmetric Lorentzian space *if the homogeneous components of $\mathfrak{m}$ are pairwise orthogonal with respect to B.*

From the classification of $ad\mathfrak{g}_e$-invariant forms on $so(2l+1)$ given in Proposition 88, the $(\mathbb{Z}_2^2)$-symmetric space $SO(2l+1)/SO(r_1) \times \cdots \times SO(r_4)$ is Lorentzian if and only if there exists one homogeneous component of $\mathfrak{m}$ of dimension 1. For example if we consider the $(\mathbb{Z}_2^2)$-symmetric space $SO(5)/SO(2) \times SO(2) \times SO(1)$ the homogeneous components are of dimension 2 and every semi-Riemannian metric is of signature $(2p, 8-2p)$ and cannot be a Lorentzian metric. So $SO(5)/SO(2) \times SO(2) \times SO(1)$ cannot be Lorentzian (as a $\mathbb{Z}_2^2$-symmetric space). Nevertheless one may consider the grading of $so(5)$ given by

$$\begin{pmatrix} 0 & a_1 & b_1 & b_2 & b_3 \\ -a_1 & 0 & c_1 & c_2 & c_3 \\ -b_1 & -c_1 & 0 & x_1 & x_2 \\ -b_2 & -c_2 & -x_1 & 0 & x_3 \\ -b_3 & -c_3 & -x_2 & -x_3 & 0 \end{pmatrix}$$

where $\mathfrak{g}_e$ is parameterized by x_1, x_2, x_3, $\mathfrak{g}_a$ by a_1, $\mathfrak{g}_b$ by b_1, b_2, b_3 and $\mathfrak{g}_c$ by c_1, c_2, c_3. Let us denote by $\{X_1, X_2, X_3, A_1, B_1, B_2, B_3, C_1, C_2, C_3\}$ the corresponding graded basis. Here $\mathfrak{g}_e$ is isomorphic to $so(3) \oplus so(1) \oplus so(1)$ and we obtain the $\mathbb{Z}_2^2$-symmetric homogeneous space

$$SO(5)/SO(3) \times SO(1) \times SO(1) = SO(5)/SO(3).$$

Every nondegenerate symmetric bilinear form on $so(5)$ invariant by $g_e = so(3)$ is written

$$q = t(\omega_1^2 + \omega_2^2 + \omega_3^2) + u\alpha_1^2 + v(\beta_1^2 + \beta_2^2 + \beta_3^2) + w(\gamma_1^2 + \gamma_2^2 + \gamma_3^2)$$

where$\{\omega_i, \alpha_1, \beta_i, \gamma_i\}$ is the dual basis of the basis $\{X_i, A_1, B_i, C_i\}$. In particular, we obtain:

11.4.2 The $\mathbb{Z}_2^2$-Riemannian symmetric space $SO(2m)/Sp(m)$

We first define the grading of $so(2m)$. Consider the matrices

$$S_m = \begin{pmatrix} 0 & I_m \\ -I_n & 0 \end{pmatrix},\ X_a = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, X_b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, X_c = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The linear maps on $so(2m)$ given by $\tau_\alpha(M) = J_\alpha^{-1} M J_\alpha,\ \alpha = a, b, c$ where $J_a = S_m \otimes X_a,\ J_b = S_m \otimes X_b,\ J_c = S_m \otimes X_c$ are involutive automorphisms of $so(2m)$ which pairwise commute. Thus $\{Id, \tau_a, \tau_b, \tau_c\}$ is a finite subgroup of $Aut(so(2m))$ isomorphic to $\mathbb{Z}_2^2$. We deduce the $\mathbb{Z}_2^2$-grading

$$so(2m) = \mathfrak{g}_e \oplus \mathfrak{g}_a \oplus \mathfrak{g}_b \oplus \mathfrak{g}_c$$

where

$$\mathfrak{g}_e = \left\{ \left(\begin{array}{cc|cc} A_1 & B_1 & A_2 & B_2 \\ -B_1 & A_1 & B_2 & -A_2 \\ \hline -{}^tA_2 & -{}^tB_2 & A_1 & B_1 \\ -{}^tB_2 & {}^tA_2 & -B_1 & A_1 \end{array} \right) \text{ with } \begin{array}{cc} {}^tA_1 = -A_1, & {}^tB_1 = B_1 \\ {}^tA_2 = A_2, & {}^tB_2 = B_2 \end{array} \right\}$$

$$\mathfrak{g}_a = \left\{ \left(\begin{array}{cc|cc} X_a & Y_a & Z_a & T_a \\ Y_a & -X_a & -T_a & Z_a \\ \hline -{}^tZ_a & {}^tT_a & -X_a & -Y_a \\ -{}^tT_a & -{}^tZ_a & -Y_a & X_a \end{array} \right) \text{ with } \begin{array}{cc} {}^tX_a = -X_a, & {}^tY_a = -Y_a \\ {}^tZ_a = -Z_a, & {}^tT_a = T_a \end{array} \right\}$$

$$\mathfrak{g}_b = \left\{ \left(\begin{array}{cc|cc} X_b & Y_b & Z_b & T_b \\ -Y_b & X_b & T_b & Z_b \\ \hline -{}^tZ_b & -{}^tT_b & -X_b & -Y_b \\ -{}^tT_b & -{}^tZ_b & Y_b & -X_b \end{array} \right) \text{ with } \begin{array}{cc} {}^tX_b = -X_b, & {}^tY_b = Y_b \\ {}^tZ_b = -Z_b, & {}^tT_b = -T_b \end{array} \right\}$$

$$\mathfrak{g}_c = \left\{ \left(\begin{array}{cc|cc} X_c & Y_c & Z_c & T_c \\ Y_c & -X_c & -T_c & Z_c \\ \hline -{}^tZ_c & {}^tT_c & X_c & Y_c \\ -{}^tT_c & -{}^tZ_c & Y_c & -X_c \end{array} \right) \text{ with } \begin{array}{cc} {}^tX_c = -X_c, & {}^tY_c = -Y_c \\ {}^tZ_c = Z_c, & {}^tT_c = -T_c \end{array} \right\}$$

The subalgebra $\mathfrak{g}_e$ is isomorphic to $sp(m)$ and from [8] , every $\mathbb{Z}_2^2$-grading of $so(2m)$ such that $\mathfrak{g}_e$ is isomorphic to $sp(m)$ is equivalent to the previous one.'The symmetries of the $\mathbb{Z}_2^2$-symmetric space $SO(2m)/Sp(m)$ at any point x can be described as soon as we know the expression of the symmetries at the point $\bar{1}$, the class on the quotient $SO(2m)/Sp(m)$ of 1, the unit of the group $SO(2m)$. Moreover we have $s_{\gamma,\bar{1}}(\overline{A}) = \overline{(\rho_\gamma(A))}$ with $\rho_\alpha(A) = J_\alpha^{-1} A J_\alpha,\ \alpha = a, b, c$. Let g be a Riemannian $\mathbb{Z}_2^2$-symmetric metric on $SO(2m)/Sp(m)$. The corresponding bilinear form B on $\mathfrak{g}_a \oplus \mathfrak{g}_b \oplus \mathfrak{g}_c$ is $\mathfrak{g}_e$ -invariant and the linear spaces $\mathfrak{g}_a$, $\mathfrak{g}_b$, $\mathfrak{g}_c$ are orthogonal. Then B writes $B = B_a + B_b + B_c$ where B_γ is a nondegenerate bilinear form on $\mathfrak{g}_\gamma$ for $\gamma = a, b, c$ such that the kernel contains $\oplus_{\gamma' \neq \gamma} \mathfrak{g}_{\gamma'}$. Let $\{X_{\gamma,ij}, Y_{\gamma,ij}, Z_{\gamma,ij}, T_{\gamma,ij}\}$ the basis of $\mathfrak{g}_\gamma$ given by elementary matrices which generate $(X_\gamma, Y_\gamma, Z_\gamma, T_\gamma)$ and $\{\alpha_{\gamma,ij}, \beta_{\gamma,ij}, \gamma_{\gamma,ij}, \delta_{\gamma,ij}\}$ its dual basis.

Proposition 89 *Every Riemannian (indefinite) $\mathbb{Z}_2^2$-symmetric on $SO(2m)/Sp(m)$ is defined from the bilinear form B whose quadratic form is written $q_B = q_{\mathfrak{g}_a} + q_{\mathfrak{g}_b} + q_{\mathfrak{g}_b}$ with*

$$\left\{ \begin{array}{l} q_{\mathfrak{g}_a} = \lambda_1^a \left(\sum (\alpha_{a,ij}^2 + \beta_{a,ij}^2 + \gamma_{a,ij}^2) + \sum_{i \neq j} \delta_{a,ij}^2 \right) + \lambda_2^a (\delta_{a,ii}^2) + (\lambda_2^a - \frac{\lambda_1^a}{2})(\sum_{i<j} (\delta_{a,ii}\delta_{a,jj}) \\ q_{\mathfrak{g}_b} = \lambda_1^b (\sum (\alpha_{b,ij}^2 + \gamma_{ij}^2) + \delta_{b,ij}^2 + \sum_{i \neq j} \beta_{b,ij}^2) + \lambda_2^b (\beta_{b,ii}^2) + (\lambda_2^b - \frac{\lambda_1^b}{2})(\sum_{i<j} (\beta_{b,ii}\beta_{b,jj}) \\ q_{\mathfrak{g}_c} = \lambda_1^c (\sum (\beta_{c,ij}^2 + \gamma_{c,ij}^2) + \delta_{c,ij}^2 + \sum_{i \neq j} \alpha_{c,ij}^2) + \lambda_2^c (\alpha_{c,ii}^2) + (\lambda_2^c - \frac{\lambda_1^c}{2})(\sum_{i<j} (\alpha_{c,ii}\alpha_{c,jj}) \end{array} \right.$$

Let $\gamma \in \{a, b, c\}$. The eigenvalues of $q_{\mathfrak{g}_\gamma}$ are

$$\mu_{1,\gamma} = \lambda_1^\gamma,\quad \mu_{2,\gamma} = \lambda_2^\gamma/2 + \lambda_1^\gamma/4,\quad \mu_{3,\gamma} = \lambda_2^\gamma \frac{r+1}{2} - \lambda_1^\gamma \frac{r-1}{4},$$

where $r = \frac{m^2+m-2}{m^2+m+2}$ is the order of symmetric matrices T_a, Y_b, Z_c. These roots are respectively of multiplicities $dim\mathfrak{g}_\gamma - r, r-1, 1$. We deduce

Theorem 116 *Every definite positive Riemannian $\mathbb{Z}_2^2$-symmetric metric on $SO(2m)/Sp(m)$ is given from the bilinear form B whose quadratic associated form*

$$q_B = q_{\mathfrak{g}_a}(\lambda_1^a, \lambda_2^a) + q_{\mathfrak{g}_b}(\lambda_1^b, \lambda_2^b) + q_{\mathfrak{g}_b}(\lambda_1^b, \lambda_2^b)$$

satisfies $\lambda_1^\gamma > 0$ and $\lambda_2^\gamma > \lambda_1^\gamma \frac{m^2+m-2}{2(m^2+m+2})$ for all $\gamma \in \{a, b, c\}$. Such a metric is naturally reductive if and only if

$$\lambda_1^a = \lambda_1^b = \lambda_1^c = 2\lambda_2^a = 2\lambda_2^b = 2\lambda_2^c.$$

For the lorentzian case, we have

Theorem 117 *Every lorentzian $\mathbb{Z}_2^2$-symmetric metric on $SO(2m)/Sp(m)$ is given from the bilinear form B whose quadratic associated form*

$$q_B = q_{\mathfrak{g}_a}(\lambda_1^a, \lambda_2^a) + q_{\mathfrak{g}_b}(\lambda_1^b, \lambda_2^b) + q_{\mathfrak{g}_b}(\lambda_1^b, \lambda_2^b)$$

satisfies

$$\begin{cases} \forall\gamma \in \{a, b, c\}, \ \lambda_1^\gamma > 0, \\ \exists\gamma_0 \in \{a, b, c\} \text{ such that } -\lambda_1^{\gamma_0}/2 < \lambda_2^{\gamma_0} < \lambda_1^{\gamma_0} \frac{r-1}{2(r+1)}, \\ \forall\gamma \neq \gamma_0, \quad \lambda_2^\gamma > \lambda_1^\gamma \frac{r-1}{2(r+1)}. \end{cases}$$

Part III

Annexe: Deformations of algebras with coefficients in a valuation ring

Chapter 12

Deformations with coefficients in a valuation ring

We develop the notion of deformations using a valuation ring as ring of coefficients. This permits to consider in particular the classical Gerstenhaber deformations of associative or Lie algebras as infinitesimal deformations and to solve the equation of deformations in a polynomial frame. We consider also the deformations of the enveloping algebra of a rigid Lie algebra and we define valued deformations for some classes of non associative algebras.

12.1 Valued deformations of Lie algebras

12.1.1 Rings of valuation

We recall briefly the classical notion of ring of valuation. Let $\mathbb{F}$ be a (commutative) field and A a subring of $\mathbb{F}$. We say that A is a ring of valuation of $\mathbb{F}$ if A is a local integral domain satisfying:

$$\text{If } x \in \mathbb{F} - A, \quad \text{then} \quad x^{-1} \in \mathfrak{m}.$$

where $\mathfrak{m}$ is the maximal ideal of A.

A ring A is called ring of valuation if it is a ring of valuation of its field of fractions.

Examples: Let $\mathbb{K}$ be a commutative field of characteristic 0. The ring of formal series $\mathbb{K}[[t]]$ is a valuation ring. On other hand the ring $\mathbb{K}[[t_1, t_2]]$ of two (or more) indeterminates is not a valuation ring.

12.1.2 Versal deformations of Fialowski [31]

Let $\mathfrak{g}$ be a $\mathbb{K}$-Lie algebra and A an unitary commutative local $\mathbb{K}$-algebra. The tensor product $\mathfrak{g} \otimes A$ is naturally endowed with a Lie algebra structure:

$$[X \otimes a, Y \otimes b] = [X, Y] \otimes ab.$$

If $\epsilon : A \longrightarrow \mathbb{K}$, is an unitary augmentation with kernel the maximal ideal $\mathfrak{m}$, a deformation λ of $\mathfrak{g}$ with base A is a Lie algebra structure on $\mathfrak{g} \otimes A$ with bracket $[\,,\,]_\lambda$ such that

$$id \otimes \epsilon : \mathfrak{g} \otimes A \longrightarrow \mathfrak{g} \otimes \mathbb{K}$$

is a Lie algebra homomorphism. In this case the bracket $[\,,\,]_\lambda$ satisfies

$$[X \otimes 1, Y \otimes 1]_\lambda = [X, Y] \otimes 1 + \sum Z_i \otimes a_i,$$

where $a_i \in A$ and $X, Y, Z_i \in \mathfrak{g}$. Such a deformation is called infinitesimal if the maximal ideal $\mathfrak{m}$ satisfies $\mathfrak{m}^2 = 0$. An interesting example is described in [F]. If we consider the commutative algebra $A = \mathbb{K} \oplus (H^2(\mathfrak{g}, \mathfrak{g}))^*$ (where * denotes the dual as vector space) such that $\dim(H^2) \leq \infty$, the deformation with base A is an infinitesimal deformation (which plays the role of an universal deformation).

12.1.3 Valued deformations of Lie algebra

Let $\mathfrak{g}$ be a $\mathbb{K}$-Lie algebra and A a commutative $\mathbb{K}$-algebra of valuation. Then $\mathfrak{g} \otimes A$ is a $\mathbb{K}$-Lie algebra. We can consider this Lie algebra as an A-Lie algebra. We denote this last by $\mathfrak{g}_A$. If $\dim_{\mathbb{K}}(\mathfrak{g})$ is finite then

$$\dim_A(\mathfrak{g}_A) = \dim_{\mathbb{K}}(\mathfrak{g}).$$

Since the valued ring A is also a $\mathbb{K}$-algebra, we have a natural embedding of the $\mathbb{K}$-vector space $\mathfrak{g}$ into the free A-module $\mathfrak{g}_A$. Without loss of generality we can consider that this embedding is the identity map.

Definition 118 *Let $\mathfrak{g}$ be a $\mathbb{K}$-Lie algebra and A a commutative $\mathbb{K}$-algebra of valuation such that the residual field $\frac{A}{\mathfrak{m}}$ is isomorphic to $\mathbb{K}$ (or to a subfield of $\mathbb{K}$). A valued deformation of $\mathfrak{g}$ with base A is a A-Lie algebra $\mathfrak{g}'_A$ such that the underlying A-module of $\mathfrak{g}'_A$ is $\mathfrak{g}_A$ and that*

$$[X, Y]_{\mathfrak{g}'_A} - [X, Y]_{\mathfrak{g}_A}$$

is in the $\mathfrak{m}$-quasi-module $\mathfrak{g} \otimes \mathfrak{m}$ where $\mathfrak{m}$ is the maximal ideal of A.

The classical notion of deformation studied by Gerstenhaber ([41]) is a valued deformation. In this case $A = \mathbb{K}[[t]]$ and the residual field of A is isomorphic to $\mathbb{K}$. Likewise a versal deformation is a valued deformation. The algebra A is in this case the finite dimensional $\mathbb{K}$-vector space $\mathbb{K} \oplus (H^2(\mathfrak{g}, \mathfrak{g}))^*$ where H^2 denotes the second Chevalley cohomology group of $\mathfrak{g}$. The algebra law is given by

$$(\alpha_1, h_1) \cdot (\alpha_2, h_2) = (\alpha_1 \cdot \alpha_2, \alpha_1 \cdot h_2 + \alpha_2 \cdot h_1).$$

It is a local field with maximal ideal $\{0\} \oplus (H^2)^*$. It is also a valuation field because we can endow this algebra with a field structure, the inverse of (α, h) being $((\alpha)^{-1}, -(\alpha)^{-2}h)$.

12.2 Decomposition of valued deformations

In this section we show that every valued deformation can be decomposed in a finite sum (and not as a series) with pairwise comparable infinitesimal coefficients (that is, in $\mathfrak{m}$). The interest of this decomposition is to avoid the classical problems of convergence.

12.2.1 Decomposition in $\mathfrak{m} \times \mathfrak{m}$

Let A be a valuation ring satisfying the conditions of Definition 118. Let us denote by $\mathcal{F}_A$ the field of fractions of A and $\mathfrak{m}^2$ the cartesian product $\mathfrak{m} \times \mathfrak{m}$. Let $(a_1, a_2) \in \mathfrak{m}^2$ with $a_i \neq 0$ for $i = 1, 2$.

i) Suppose that $a_1 \cdot a_2^{-1} \in A$ and $a_2 \cdot a_1^{-1} \in A$. Consider $\alpha = \pi(a_1 \cdot a_2^{-1})$ where π is the canonical projection on $\frac{A}{\mathfrak{m}}$. Clearly, there exists a global section $s : \mathbb{K} \to A$ which permits to identify α with $s(\alpha)$ in A. Then

$$a_1 \cdot a_2^{-1} = \alpha + a_3,$$

with $a_3 \in \mathfrak{m}$. Then if $a_3 \neq 0$,

$$(a_1, a_2) = (a_2(\alpha + a_3), a_2) = a_2(\alpha, 1) + a_2 a_3(0, 1).$$

If $\alpha \neq 0$ we can also write

$$(a_1, a_2) = aV_1 + abV_2,$$

with $a, b \in \mathfrak{m}$ and V_1, V_2 linearly independent in $\mathbb{K}^2$. If $\alpha = 0$ then $a_1 \cdot a_2^{-1} \in \mathfrak{m}$ and $a_1 = a_2 a_3$. We have

$$(a_1, a_2) = (a_2 a_3, a_2) = ab(1,0) + a(0,1).$$

So in this case, $V_1 = (0,1)$ and $V_2 = (1,0)$. If $a_3 = 0$ then

$$a_1 a_2^{-1} = \alpha$$

and

$$(a_1, a_2) = a_2(\alpha, 1) = aV_1.$$

This correspond to the previous decomposition but with $b = 0$.

ii) If $a_1 \cdot a_2^{-1} \in \mathcal{F}_A - A$, then $a_2 \cdot a_1^{-1} \in \mathfrak{m}$. We put in this case $a_2 \cdot a_1^{-1} = a_3$ and we have

$$(a_1, a_2) = (a_1, a_1 \cdot a_3) = a_1(1, a_3) = a_1(1,0) + a_1 a_3(0,1)$$

with $a_3 \in \mathfrak{m}$. Then, in this case the point (a_1, a_2) admits the following decomposition :

$$(a_1, a_2) = aV_1 + abV_2$$

with $a, b \in \mathfrak{m}$ and V_1, V_2 linearly independent in $\mathbb{K}^2$. Note that this case corresponds to the previous one but with $\alpha \neq 0$.

iii) If $a_1 \cdot a_2^{-1} \in A$ and $a_2 \cdot a_1^{-1} \notin A$ then as $a_2 \cdot a_1^{-1} \in \mathcal{F}_A - A$, $a_1 a_2^{-1} \in \mathfrak{m}$ and we find again the case i) with $\alpha = 0$. Then we have proved

Proposition 90 *For every point $(a_1, a_2) \in \mathfrak{m}^2$, there exist linearly independent vectors V_1 and V_2 in the $\mathbb{K}$-vector space $\mathbb{K}^2$ such that*

$$(a_1, a_2) = aV_1 + abV_2,$$

for some $a, b \in \mathfrak{m}$.

Such decomposition est called of length 2 if $b \neq 0$. If not it is called of length 1.

12.2.2 Decomposition in $\mathfrak{m}^k$

Suppose that A is valuation ring satisfying the hypothesis of Definition 118. Arguing as before, we can conclude

Theorem 119 *For every $(a_1, a_2, \cdots, a_k) \in \mathfrak{m}^k$ there exist h ($h \leq k$) independent vectors $V_1, V_2, \cdots, V_h$ whose components are in $\mathbb{K}$ and elements $b_1, b_2, \cdots, b_h \in \mathfrak{m}$ such that*

$$(a_1, a_2, \cdots, a_k) = b_1 V_1 + b_1 b_2 V_2 + \cdots + b_1 b_2 \cdots b_h V_h.$$

The parameter h which appears in this theorem is called the length of the decomposition. This parameter can be different from k. It corresponds to the dimension of the smallest $\mathbb{K}$-vector space V such that $(a_1, a_2, \cdots, a_k) \in V \otimes \mathfrak{m}$.

If the coordinates a_i of the vector $(a_1, a_2, \cdots, a_k)$ are in A and not necessarily in its maximal ideal, then writing $a_i = \alpha_i + a_i'$ with $\alpha_i \in \mathbb{K}$ and $a_i' \in \mathfrak{m}$, we decompose

$$(a_1, a_2, \cdots, a_k) = (\alpha_1, \alpha_2, \cdots, \alpha_k) + (a_1', a_2', \cdots, a_k')$$

and we can apply Theorem 119 to the vector $(a_1', a_2', \cdots, a_k')$.

12.2.3 Uniqueness of the decomposition

Let us begin by a technical lemma.

Lemma 22 *Let V and W be two vectors with components in the valuation ring A. There exist V_0 and W_0 with components in $\mathbb{K}$ such that $V = V_0 + V_0'$ and $W = W_0 + W_0'$ and the components of V_0' and W_0' are in the maximal ideal $\mathfrak{m}$. Moreover if the vectors V_0 and W_0 are linearly independent then V and W are also independent.*

Proof. The decomposition of the two vectors V and W is evident. It remains to prove that the independence of the vectors V_0 and W_0 implies those of V and W. Let V, W be two vectors with components in A such that $\pi(V) = V_0$ and $\pi(W) = W_0$ are independent. Let us suppose that

$$xV + yW = 0$$

with $x, y \in A$. One of the coefficients xy^{-1} or yx^{-1} is not in $\mathfrak{m}$. Let us suppose that $xy^{-1} \notin \mathfrak{m}$. If $xy^{-1} \notin A$ then $x^{-1}y \in \mathfrak{m}$. Then $xV + yW = 0$ is equivalent to $V + x^{-1}yW = 0$. This implies that $\pi(V) = 0$ and this is impossible. Then $xy^{-1} \in A - \mathfrak{m}$. Thus if there exists a linear relation between V and W, there exists a linear relation with coefficients in $A - \mathfrak{m}$. We can suppose that $xV + yW = 0$ with $x, y \in A - \mathfrak{m}$. Since $V = V_0 + V_0'$, $W = W_0 + W_0'$ we have

$$\pi(xV + yW) = \pi(x)V_0 + \pi(y)W_0 = 0.$$

Thus $\pi(x) = \pi(y) = 0$. This is impossible and the vectors V and W are independent as soon as V_0 and W_0 are independent vectors. $\square$

Let $(a_1, a_2, \cdots, a_k) = b_1V_1 + b_1b_2V_2 + \cdots + b_1b_2\cdots b_hV_h$ and $(a_1, a_2, \cdots, a_k) = c_1W_1 + c_1c_2W_2 + \cdots + c_1c_2\cdots c_sW_s$ be two decompositions of the vector $(a_1, a_2, \cdots, a_k)$. Let us compare the coefficients b_1 and c_1. By hypothesis $b_1c_1^{-1}$ is in A or the inverse is in $\mathfrak{m}$. Then we can suppose that $b_1c_1^{-1} \in A$. Since the residual field is a subfield of $\mathbb{K}$, there exists $\alpha \in \frac{A}{\mathfrak{m}}$ and $c_1 \in \mathfrak{m}$ such that

$$b_1c_1^{-1} = \alpha + b_{11}$$

thus $b_1 = \alpha c_1 + b_{11}c_1$. Replacing this term in the decompositions we obtain

$$\begin{aligned}&(\alpha c_1 + b_{11}c_1)V_1 + (\alpha c_1 + b_{11}c_1)b_2V_2 + \cdots + (\alpha c_1 + b_{11}c_1)b_2\cdots b_hV_h\\ &= c_1W_1 + c_1c_2W_2 + \cdots + c_1c_2\cdots c_sW_s.\end{aligned}$$

Simplifying by c_1, this expression is written

$$\alpha V_1 + m_1 = W_1 + m_2$$

where m_1, m_2 are vectors with coefficients $\in \mathfrak{m}$. From Lemma 1, if V_1 and W_1 are linearly independent, as its coefficients are in the residual field, the vectors $\alpha V_1 + m_1$ and $W_1 + m_2$ would be also linearly independent ($\alpha \neq 0$). Thus $W_1 = \alpha V_1$. One deduces

$$b_1V_1 + b_1b_2V_2 + \cdots + b_1b_2\cdots b_hV_h = c_1(\alpha V_1) + c_1b_{11}V_1 + c_1b_{12}V_2 + \cdots + c_1b_{12}b_3\cdots b_hV_h,$$

with $b_{12} = b_2(\alpha + b_{11})$. Then

$$b_{11}V_1 + b_{11}b_{12}V_2 + \cdots + b_{11}b_{12}b_3\cdots b_hV_h = c_2W_2 + \cdots + c_2\cdots c_sW_s.$$

Continuing this process by induction we deduce the following result

Theorem 120 *Let be $b_1V_1 + b_1b_2V_2 + \cdots + b_1b_2\cdots b_hV_h$ and $c_1W_1 + c_1c_2W_2 + \cdots + c_1c_2\cdots c_sW_s$ two decompositions of the vector $(a_1, a_2, \cdots, a_k)$. Then*

i. $h = s$,

ii. The flag generated by the ordered free family $(V_1, V_2, \cdots, V_h)$ is equal to the flag generated by the ordered free family $(W_1, W_2, \cdots, W_h)$, that is, $\forall i \in 1, \cdots, h$

$$\{V_1, \cdots, V_i\} = \{W_1, \cdots, W_i\}$$

where $\{U_j\}$ is the linear space generated by the vectors U_j.

12.2.4 Geometrical interpretation of this decomposition

Let A be an $\mathbb{R}$ algebra of valuation. Consider a differential curve γ in $\mathbb{R}^3$. We can embed γ in a differential curve

$$\Gamma : \mathbb{R} \otimes A \to \mathbb{R}^3 \otimes A.$$

Let $t = t_0 \otimes 1 + 1 \otimes \varepsilon$ an parameter infinitely close to t_0, that is, $\varepsilon \in \mathfrak{m}$. If M corresponds to the point of Γ of parameter t and M_0 those of t_0, then the coordinates of the point $M - M_0$ in the affine space $\mathbb{R}^3 \otimes A$ are in $\mathbb{R} \otimes \mathfrak{m}$. In the flag associated to the decomposition of $M - M_0$ we can consider a direct orthonormal frame (V_1, V_2, V_3). It is the Serret-Frenet frame to γ at the point M_0.

12.3 Decomposition of a valued deformation of a Lie algebra

12.3.1 Valued deformation of Lie algebras

Let $\mathfrak{g}'_A$ be a valued deformation with base A of the $\mathbb{K}$-Lie algebra $\mathfrak{g}$. By definition, for every X and Y in $\mathfrak{g}$ we have $[X, Y]_{\mathfrak{g}'_A} - [X, Y]_{\mathfrak{g}_A} \in \mathfrak{g} \otimes \mathfrak{m}$. Suppose that $\mathfrak{g}$ is finite dimensional and let $\{X_1, \cdots, X_n\}$ be a basis of $\mathfrak{g}$. In this case

$$[X_i, X_j]_{\mathfrak{g}'_A} - [X_i, X_j]_{\mathfrak{g}_A} = \sum_k C_{ij}^k X_k,$$

with $C_{ij}^k \in \mathfrak{m}$. Using the decomposition of the vector of $\mathfrak{m}^{n^2(n-1)/2}$ with for components C_{ij}^k , we deduce that

$$\begin{array}{rcl} [X_i, X_j]_{\mathfrak{g}'_A} - [X_i, X_j]_{\mathfrak{g}_A} & = & a_{ij}(1)\phi_1(X_i, X_j) + a_{ij}(1)a_{ij}(2)\phi_2(X_i, X_j) \\ & & + \cdots + a_{ij}(1)a_{ij}(2)\cdots a_{ij}(l)\phi_l(X_i, X_j), \end{array}$$

where $a_{ij}(s) \in \mathfrak{m}$ and $\phi_1, \cdots, \phi_l$ are linearly independent. The index l depends on i and j. Let k be the supremum of indices l when $1 \leq i,\ j \leq n$. Then we have

$$\begin{array}{rcl} [X, Y]_{\mathfrak{g}'_A} - [X, Y]_{\mathfrak{g}_A} & = & \varepsilon_1(X, Y)\phi_1(X, Y) + \varepsilon_1(X, Y)\varepsilon_2(X, Y)\phi_2(X, Y) \\ & & + \cdots + \varepsilon_1(X, Y)\varepsilon_2(X, Y)\cdots\varepsilon_k(X, Y)\phi_k(X, Y) \end{array}$$

where the bilinear maps ε_i have values in $\mathfrak{m}$ and linear maps $\phi_i : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ are linearly independent.

If $\mathfrak{g}$ is infinite dimensional with a countable basis $\{X_n\}_{n\in\mathbb{N}}$ then the $\mathbb{K}$-vector space of linear map $T_2^1 = \{\phi : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}\}$ also admits a countable basis.

Theorem 121 *If $\mu_{\mathfrak{g}'_A}$ (resp. $\mu_{\mathfrak{g}_A}$) is the law of the Lie algebra $\mathfrak{g}'_A$ (resp. $\mathfrak{g}_A$) then*

$$\mu_{\mathfrak{g}'_A} - \mu_{\mathfrak{g}_A} = \sum_{i \in I} \varepsilon_1\varepsilon_2\cdots\varepsilon_i\phi_i$$

where I is a finite set of indices, $\varepsilon_i : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{m}$ are linear maps and ϕ_i's are linearly independent maps in T_2^1.

12.3.2 Equations of valued deformations

We will prove that the classical equations of deformation given by Gerstenhaber are still valid in the general framework of valued deformations. Nevertheless we can prove that the infinite system described by Gerstenhaber and which gives the conditions to obtain a deformation, can be reduced to a system of finite rank. Let

$$\mu_{\mathfrak{g}'_A} - \mu_{\mathfrak{g}_A} = \sum_{i \in I} \varepsilon_1\varepsilon_2\cdots\varepsilon_i\phi_i$$

be a valued deformation of μ (the bracket of $\mathfrak{g}$). Then $\mu_{\mathfrak{g}'_A}$ satisfies the Jacobi equations. Following Gerstenhaber we consider the Chevalley-Eilenberg graded differential complex $\mathcal{C}(\mathfrak{g}, \mathfrak{g})$ and the product $\circ$ defined by

$$(g_q \circ f_p)(X_1, \cdots, X_{p+q}) = \sum (-1)^{\varepsilon(\sigma)} g_q(f_p(X_{\sigma(1)}, \cdots, X_{\sigma(p)}), X_{\sigma(p+1)}, \cdots, X_{\sigma(q)})$$

where σ is a permutation of $1, \cdots, p+q$ such that $\sigma(1) < \cdots < \sigma(p)$ and $\sigma(p+1) < \cdots < \sigma(p+q)$ (it is a (p,q)-shuffle); $g_q \in \mathcal{C}^q(\mathfrak{g},\mathfrak{g})$ and $f_p \in \mathcal{C}^p(\mathfrak{g},\mathfrak{g})$. As $\mu_{\mathfrak{g}'_A}$ satisfies the Jacobi identities, $\mu_{\mathfrak{g}'_A} \circ \mu_{\mathfrak{g}'_A} = 0$. This gives

$$(\mu_{\mathfrak{g}_A} + \sum_{i\in I} \varepsilon_1\varepsilon_2\cdots\varepsilon_i\phi_i) \circ (\mu_{\mathfrak{g}_A} + \sum_{i\in I} \varepsilon_1\varepsilon_2\cdots\varepsilon_i\phi_i) = 0.$$

Since $\mu_{\mathfrak{g}_A} \circ \mu_{\mathfrak{g}_A} = 0$, this equation becomes :

$$\varepsilon_1(\mu_{\mathfrak{g}_A} \circ \phi_1 + \phi_1 \circ \mu_{\mathfrak{g}_A}) + \varepsilon_1 U = 0$$

where U is in $\mathcal{C}^3(\mathfrak{g},\mathfrak{g}) \otimes \mathfrak{m}$. If we simplify by ε_1 which is supposed to be non zero (if not the deformation is trivial), we obtain

$$(\mu_{\mathfrak{g}_A} \circ \phi_1 + \phi_1 \circ \mu_{\mathfrak{g}_A})(X,Y,Z) + U(X,Y,Z) = 0$$

for all $X, Y, Z \in \mathfrak{g}$. As $U(X,Y,Z)$ is in the module $\mathfrak{g} \otimes \mathfrak{m}$ and the first part in $\mathfrak{g} \otimes A$, each one of these vectors is null. Then

$$(\mu_{\mathfrak{g}_A} \circ \phi_1 + \phi_1 \circ \mu_{\mathfrak{g}_A})(X,Y,Z) = 0.$$

Proposition 91 *For every valued deformation with base A of the $\mathbb{K}$-Lie algebra $\mathfrak{g}$, the first term ϕ appearing in the associated decomposition is a 2-cochain of the Chevalley-Eilenberg cohomology of $\mathfrak{g}$ belonging to $Z^2(\mathfrak{g},\mathfrak{g})$.*

We thus rediscover the classical result of Gerstenhaber but in the broader context of valued deformations and not only for the valued deformation of basis the ring of formal series.

In order to describe the properties of other terms of the Jacobi equations, we use the super-bracket of Gerstenhaber which endows the space of Chevalley-Eilenberg cochains $\mathcal{C}(\mathfrak{g},\mathfrak{g})$ with a Lie superalgebra structure. When $\phi_i \in \mathcal{C}^2(\mathfrak{g},\mathfrak{g})$, it is defines by

$$[\phi_i, \phi_j] = \phi_i \circ \phi_j + \phi_j \circ \phi_i$$

and $[\phi_i, \phi_j] \in \mathcal{C}^3(\mathfrak{g},\mathfrak{g})$.

Lemma 23 *Let us suppose that $I = \{1, \cdots, k\}$. If*

$$\mu_{\mathfrak{g}'_A} = \mu_{\mathfrak{g}_A} + \sum_{i\in I} \varepsilon_1\varepsilon_2\cdots\varepsilon_i\phi_i$$

is a valued deformation of μ, then the 3-cochains $[\phi_i, \phi_j]$ and $[\mu, \phi_i]$, $1 \leq i,j \leq k-1$, generate a linear subspace V of $\mathcal{C}^3(\mathfrak{g},\mathfrak{g})$ of dimension less or equal to $k(k-1)/2$. Moreover, the 3-cochains $[\phi_i, \phi_j]$, $1 \leq i,j \leq k-1$, form a system of generators of this space.

Proof. Let V be the subspace of $\mathcal{C}^3(\mathfrak{g},\mathfrak{g})$ generated by $[\phi_i, \phi_j]$ and $[\mu, \phi_i]$. If ω is a linear form on V of which kernel contains the vectors $[\phi_i, \phi_j]$ for $1 \leq i,j \leq (k-1)$, then the equation (1) gives

$$\varepsilon_1\varepsilon_2\cdots\varepsilon_k\omega([\phi_1,\phi_k]) + \varepsilon_1\varepsilon_2^2\cdots\varepsilon_k\omega([\phi_2,\phi_k]) + \cdots + \varepsilon_1\varepsilon_2^2\cdots\varepsilon_k^2\omega([\phi_k,\phi_k]) + \varepsilon_2\omega([\mu,\phi_2])$$
$$+\varepsilon_2\varepsilon_3\omega([\mu,\phi_3])\cdots + \varepsilon_2\varepsilon_3\cdots\varepsilon_k\omega([\mu,\phi_k]) = 0.$$

Since the coefficients which appear in this equation are each one in one $\mathfrak{m}^p$, we have necessarily

$$\omega([\phi_1,\phi_k]) = \cdots = \omega([\phi_k,\phi_k]) = \omega([\mu,\phi_2]) = \cdots = \omega([\mu,\phi_k]) = 0$$

and this for every linear form ω of which kernel contains V. This proves the lemma.

From this lemma and using the descending sequence

$$\mathfrak{m} \supset \mathfrak{m}^{(2)} \supset \cdots \supset \mathfrak{m}^{(p)} \cdots$$

where $\mathfrak{m}^{(p)}$ is the ideal generated by the products $a_1a_2\cdots a_p,\ a_i \in \mathfrak{m}$ of length p, we obtain :

Proposition 92 *If*

$$\mu_{\mathfrak{g}'_A} = \mu_{\mathfrak{g}_A} + \sum_{i \in I} \varepsilon_1 \varepsilon_2 \cdots \varepsilon_i \phi_i$$

is a valued deformation of μ, then we have the following linear system:

$$\begin{cases} \delta\phi_2 = a_{11}^2[\phi_1, \phi_1], \\ \delta\phi_3 = a_{12}^3[\phi_1, \phi_2] + a_{22}^3[\phi_1, \phi_1], \\ \cdots \\ \delta\phi_k = \sum_{1 \leq i \leq j \leq k-1} a_{ij}^k[\phi_i, \phi_j], \\ [\phi_1, \phi_k] = \sum_{1 \leq i \leq j \leq k-1} b_{ij}^1[\phi_i, \phi_j], \\ \cdots \\ [\phi_{k-1}, \phi_k] = \sum_{1 \leq i \leq j \leq k-1} b_{ij}^{k-1}[\phi_i, \phi_j], \end{cases}$$

where $\delta\phi_i = [\mu, \phi_i]$ is the coboundary operator of the Chevalley cohomology of the Lie algebra $\mathfrak{g}$.

Let us suppose that the dimension of V is the maximum $k(k-1)/2$. In this case we have no other relations between the generators of V and the previous linear system is complete, that is, the equation of deformations does not give other relations than the relations of this system. The following result shows that, in this case, such deformation is isomorphic, as Lie algebra laws,to a "polynomial" valued deformation.

Proposition 93 *Let be $\mu_{\mathfrak{g}'_A}$ a valued deformation of μ such that*

$$\mu_{\mathfrak{g}'_A} = \mu_{\mathfrak{g}_A} + \sum_{i=1,\cdots,k} \varepsilon_1 \varepsilon_2 \cdots \varepsilon_i \phi_i$$

and $\dim V = k(k-1)/2$. Then there exists an automorphism of $\mathbb{K}^n \otimes \mathfrak{m}$ of the form $f = Id \otimes P_k(\varepsilon)$ with $P_k(X) \in \mathbb{K}^k[X]$ satisfying $P_k(0) = 1$ and $\varepsilon \in \mathfrak{m}$ such that the valued deformation $\mu_{\mathfrak{g}''_A}$ defined by

$$\mu_{\mathfrak{g}''_A}(X, Y) = h^{-1}(\mu_{\mathfrak{g}'_A}(h(X), h(Y)))$$

is of the form

$$\mu_{\mathfrak{g}_A"} = \mu_{\mathfrak{g}_A} + \sum_{i=1,\cdots,k} \varepsilon^i \varphi_i,$$

where $\varphi_i = \sum_{j \leq i} \phi_j$.

Proof. Considering the Jacobi equation

$$[\mu_{\mathfrak{g}'_A}, \mu_{\mathfrak{g}'_A}] = 0$$

and writting that $dimV = k(k-1)/2$, we deduce that there exist polynomials $P_i(X) \in \mathbb{K}[X]$ of degree i such that

$$\varepsilon_i = a_i \varepsilon_k \frac{P_{k-i}(\varepsilon_k)}{P_{k-i+1}(\varepsilon_k)},$$

with $a_i \in \mathbb{K}$. Then we have

$$\mu_{\mathfrak{g}'_A} = \mu_{\mathfrak{g}_A} + \sum_{i=1,\cdots,k} a_1 a_2 \cdots a_i (\varepsilon_k)^i \frac{P_{k-i}(\varepsilon_k)}{P_k(\varepsilon_k)} \phi_i.$$

Thus

$$P_k(\varepsilon_k)\mu_{\mathfrak{g}'_A} = P_k(\varepsilon_k)\mu_{\mathfrak{g}_A} + \sum_{i=1,\cdots,k} a_1 a_2 \cdots a_i (\varepsilon_k)^i P_{k-i}(\varepsilon_k)\phi_i.$$

If we write this expression according to the increasing powers we obtain the announced expression. □

Let us note that, for such a deformation we have

$$\begin{cases} \delta\varphi_2 + [\varphi_1, \varphi_1] = 0, \\ \delta\varphi_3 + [\varphi_1, \varphi_2] = 0, \\ \cdots \\ \delta\varphi_k + \sum_{i+j=k}[\varphi_i, \varphi_j] = 0, \\ \sum_{i+j=k+s}[\varphi_i, \varphi_j] = 0. \end{cases}$$

12.3.3 Particular case: one-parameter deformations of Lie algebras

In this section the valuation ring A is $\mathbb{K}[[t]]$. Its maximal ideal is $t\mathbb{K}[[t]]$ and the residual field is $\mathbb{K}$. Let $\mathfrak{g}$ be a $\mathbb{K}$- Lie algebra. Consider $\mathfrak{g} \otimes A$ as an A-algebra and let be $\mathfrak{g}'_A$ a valued deformation of $\mathfrak{g}$. The bracket $[-,-]_t$ of this Lie algebra satisfies

$$[X,Y]_t = [X,Y] + \sum t^i \phi_i(X,Y).$$

Considered as a valued deformation with base $\mathbb{K}[[t]]$, this bracket can be written

$$[X,Y]_t = [X,Y] + \sum_{i=1}^{i=k} c_1(t)\cdots c_i(t)\psi_i(X,Y),$$

where $(\psi_1, \cdots, \psi_k)$ are linearly independent and $c_i(t) \in t\mathbb{C}[[t]]$. As $\phi_1 = \psi_1$, this bilinear map belongs to $Z^2(\mathfrak{g},\mathfrak{g})$ and we find again the classical result of Gerstenhaber. Let V be the $\mathbb{K}$-vector space generated by $[\phi_i, \phi_j]$ and $[\mu, \phi_i]$, $i, j = 1, \cdots, k-1$, μ being the law multiplication of $\mathfrak{g}$. If $\dim V = k(k-1)/2$ we will say that one-parameter deformation $[-,-]_t$ is of maximal rank.

Proposition 94 *Let*

$$[X,Y]_t = [X,Y] + \sum t^i \phi_i(X,Y)$$

be a one-parameter deformation of $\mathfrak{g}$. If its rank is maximal then this deformation is equivalent to a polynomial deformation

$$[X,Y]'_t = [X,Y] + \sum_{i=1,\cdots,k} t^i \varphi_i,$$

with $\varphi_i = \sum_{j=1,\cdots,i} a_{ij}\psi_j$.

Corollary 122 *Every one-parameter deformation of maximal rank is equivalent to a local non valued deformation with base the local algebra $\mathbb{K}[t]$.*

Recall that the algebra $\mathbb{K}[t]$ is not an algebra of valuation. But every local ring is dominated by a valuation ring. Then this corollary can be interpreted as saying that every deformation in the local algebra $\mathbb{C}[t]$ of polynomials with coefficients in $\mathbb{C}$ is equivalent to a "classical"-Gerstenhaber deformation with maximal rank.

12.4 Deformations of the enveloping algebra of a rigid Lie algebra

12.4.1 Valued deformation of associative algebras

Let us recall that the category of $\mathbb{K}$-associative algebras is a monoidal category.

Definition 123 *Let $\mathfrak{a}$ be a $\mathbb{K}$-associative algebra and A an $\mathbb{K}$-algebra of valuation of such that the residual field $\frac{A}{\mathfrak{m}}$ is isomorphic to $\mathbb{K}$ (or to a subfield $\mathbb{K}'$ of $\mathbb{K}$). A valued deformation of $\mathfrak{a}$ with base A is an A-associative algebra $\mathfrak{a}'_A$ such that the underlying A-module of $\mathfrak{a}'_A$ is $\mathfrak{a}_A$ and that*

$$(X \cdot Y)_{\mathfrak{a}'_A} - (X \cdot Y)_{\mathfrak{a}_A}$$

belongs to the $\mathfrak{m}$-quasi-module $\mathfrak{a} \otimes \mathfrak{m}$ where $\mathfrak{m}$ is the maximal ideal of A.

The classical one-parameter deformation is a valued deformation. As in the Lie algebra case we can develop the decomposition of a valued deformation. It is sufficient to change the Lie bracket by the associative product and the Chevalley cohomology by the Hochschild cohomology.

The most important example concerning valued deformations of associative algebras is those of the associative algebra of smooth fonctions of a manifold. But we will be interested here by associative algebras that are the enveloping algebras of Lie algebras. More precisely, what can we say about the valued deformations of the enveloping algebra of a rigid Lie algebra?

12.4.2 Complex rigid Lie algebras

In this section we suppose that $\mathbb{K} = \mathbb{C}$. Let $\mathcal{L}_n$ be the algebraic variety of structure constants of n-dimensional complex Lie algebra laws. The basis of $\mathbb{C}^n$ being fixed, we can identify a law with its structure constants. Let us consider the action of the linear group $Gl(n, \mathbb{C})$ on $\mathcal{L}_n$:

$$\mu'(X, Y) = f^{-1}\mu(f(X), f(Y)).$$

We denote by $\mathcal{O}(\mu)$ the orbit of μ.

Definition 124 *The law $\mu \in \mathcal{L}_n$ is called rigid if $\mathcal{O}(\mu)$ is Zariski-open in $\mathcal{L}_n$.*

Let $\mathfrak{g}$ be a n-dimensional complex Lie algebra with product μ and $\mathfrak{g}_A$ a valued deformation with base A. As before $\mathcal{F}_A$ is the field of fractions of A.

Definition 125 *Let A be a valued $\mathbb{C}$-algebra. We say that $\mathfrak{g}$ is A-rigid if for every valued deformation $\mathfrak{g}'_A$ of $\mathfrak{g}_A$ there exists a $\mathcal{F}_A$-linear isomorphism between $\mathfrak{g}'_A$ and $\mathfrak{g}_A$.*

Let $\mu_{\mathfrak{g}'_A}$ be a valued deformation of $\mu_{\mathfrak{g}_A}$. If we write $\mu_{\mathfrak{g}'_A} - \mu_{\mathfrak{g}_A} = \phi$, then $\phi(X, Y) \in \mathfrak{g} \otimes \mathfrak{m}$ for all $X, Y \in \mathfrak{g} \otimes A$. If $\mu_{\mathfrak{g}_A}$ is rigid, there exits $f \in Gl_n(\mathfrak{g} \otimes \mathcal{F}_{\mathcal{A}})$ such that

$$f^{-1}(\mu_{\mathfrak{g}'_A}(f(X), f(Y))) = \mu_{\mathfrak{g}_A}(X, Y).$$

Thus

$$\mu_{\mathfrak{g}_A}(f(X), f(Y)) - f(\mu_{\mathfrak{g}_A}(X, Y)) = \phi(f(X), f(Y)).$$

As $\mathfrak{g}_A$ is invariant by f, $\phi(f(X), f(Y)) \in \mathfrak{g} \otimes \mathfrak{m}$. So we can decompose f as $f = f_1 + f_2$ with $f_1 \in Aut(\mathfrak{g}_A)$ and $f_2 : \mathfrak{g}_A \to \mathfrak{g} \otimes \mathfrak{m}$. Let f' be $f' = f \circ f_1^{-1}$. Then

$$f'^{-1}(\mu_{\mathfrak{g}'_A}(f'(X), f'(Y))) = \mu_{\mathfrak{g}_A}(X, Y)$$

and $f' = Id + h$ with $h : \mathfrak{g}_A \to \mathfrak{g} \otimes \mathfrak{m}$. Thus we have proved

Lemma 24 *If $\mu_{\mathfrak{g}_A}$ is A-rigid for every valued deformation $\mu_{\mathfrak{g}'_A}$ there exits $f \in Gl_n(\mathfrak{g} \otimes \mathcal{F}_{\mathcal{A}})$ of the type $f = Id + h$ with $h : \mathfrak{g}_A \to \mathfrak{g} \otimes \mathfrak{m}$ such that*

$$f^{-1}(\mu_{\mathfrak{g}'_A}(f(X), f(Y))) = \mu_{\mathfrak{g}_A}(X, Y),$$

for every $X, Y \in \mathfrak{g}_A$.

Remark. If $f = Id + h$ then $f^{-1} = Id + k$. Since $\mathfrak{g}_A$ is invariant by f, the linear map k satisfies $k : \mathfrak{g}_A \to \mathfrak{g} \otimes \mathfrak{m}$.

Theorem 126 *If the residual field of the valued ring is isomorphic to $\mathbb{C}$ then the notions of A-rigidity and of rigidity are equivalent.*

Proof. Let us suppose that for every valued algebra with residual field $\mathbb{C}$, the Lie algebra $\mathfrak{g}$ is A-rigid. We will consider the following special valued algebra: let $\mathbb{C}^*$ be non standard extension of $\mathbb{C}$ in the Robinson sense ([Ro]). If $\mathbb{C}_l$ is the subring of non-infinitely large elements of $\mathbb{C}^*$ then the subring $\mathfrak{m}$ of infinitesimals is the maximal ideal of $\mathbb{C}_l$ and $\mathbb{C}_l$ is a valued ring. Let us consider $A = \mathbb{C}_l$. In this case we have a natural embedding of the variety of A-Lie algebras in the variety of $\mathbb{C}$-Lie algebras. Up this embedding (called the transfer principle in the Robinson theory), the set of A-deformations of $\mathfrak{g}_A$ is an infinitesimal neighborhood of $\mathfrak{g}$ contained in the orbit of $\mathfrak{g}$. Then $\mathfrak{g}$ is rigid. $\square$

Examples. If $A = \mathbb{C}[[t]]$ then $\mathbb{K}' = \mathbb{C}$ and we find again the classical approach to the rigidity. We have another example, considering a non standard extension $\mathbb{C}^*$ of $\mathbb{C}$. In this context the notion of rigidity has been developed in [A.G] (such a deformation is called perturbation). This work has allowed to classify complex finite dimensional rigid Lie algebras up the dimension eight.

12.4.3 Deformation of the enveloping algebra of a Lie algebra

Let $\mathfrak{g}$ be a finite dimensional $\mathbb{K}$-Lie algebra and $\mathcal{U}(\mathfrak{g})$ its enveloping algebra. In this section we consider a particular valued deformation of $\mathcal{U}(\mathfrak{g})$ corresponding to the valued algebra $\mathbb{K}[[t]]$. In [103], the following result is proved:

Proposition 95 *If $\mathfrak{g}$ is not rigid then $\mathcal{U}(\mathfrak{g})$ is not $\mathbb{K}[[t]]$-rigid.*

Recall that if the Hochschild cohomology $H^*(\mathcal{U}(\mathfrak{g}),\mathcal{U}(\mathfrak{g}))$ of $\mathcal{U}(\mathfrak{g})$ satisfies

$$H^2(\mathcal{U}(\mathfrak{g}),\mathcal{U}(\mathfrak{g})) = 0,$$

then $\mathcal{U}(\mathfrak{g})$ is $\mathbb{K}[[t]]$-rigid. By the Cartan-Eilenberg theorem, we have that

$$H^n(\mathcal{U}(\mathfrak{g}),\mathcal{U}(\mathfrak{g})) = H^n(\mathfrak{g},\mathcal{U}(\mathfrak{g})).$$

Theorem 127 *[103] Let $\mathfrak{g}$ be a rigid Lie algebra. If $H^2(\mathfrak{g},\mathbb{C}) \neq 0$, then $\mathcal{U}(\mathfrak{g})$ is not $\mathbb{K}[[t]]$-rigid.*

From [18] and [5] every solvable complex Lie algebra decomposes as $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{n}$ where $\mathfrak{n}$ is the niladical of $\mathfrak{g}$ and $\mathfrak{t}$ a maximal exterior torus of derivations in the Malcev sense. Recall that the rank of $\mathfrak{g}$ is the dimension of $\mathfrak{t}$. A direct consequence of Petit's theorem is that for every complex rigid Lie algebra of rank equal or greater than 2 its envelopping algebra is not rigid.

Theorem 128 *Let $\mathfrak{g}$ be a complex finite dimensional rigid Lie algebra of rank 1. Then*

$$\dim\ H^2(\mathfrak{g},\mathbb{C}) = 0$$

if and only if 0 is not a root of the nilradical $\mathfrak{n}$.

Proof. Suppose first that 0 is not a root of $\mathfrak{n}$, that is, for every $X \neq 0 \in \mathfrak{t}$, 0 is not an eigenvalue of the semisimple operator adX. Let θ be in $Z^2(\mathfrak{g},\mathbb{C})$. Let $(X, Y_i)_{i=1,\cdots,n-1}$ a basis of $\mathfrak{n}$ adapted to the decomposition $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{n}$. In particular we have

$$[X, Y_i] = \lambda_i Y_i,$$

with $\lambda_i \in \mathbb{N}^*$ for all $i = 1, \cdots, n-1$ ([A.G]). As $d\theta = 0$ we have for all $i, j = 1, \cdots, n-1$

$$d\theta(X, Y_i, Y_j) = \theta(X, [Y_i, Y_j]) + \theta(Y_i, [Y_j, X]) + \theta(Y_j, [X, Y_i]) = 0,$$

for all $1 \leq i, j \leq k-1$, and this gives

$$(\lambda_i + \lambda_j)\theta(Y_i, Y_j) = \theta(X, [Y_i, Y_j]). \qquad (*)$$

If $(\lambda_i + \lambda_j)$ is not a root, then $[Y_i, Y_j] = 0$ and this implies that $\theta(Y_i, Y_j) = 0$. If not, $(\lambda_i + \lambda_j) = \lambda_k$ is a root. Let us note $Y_k^1, \cdots, Y_k^{n_k}$ the eigenvectors of the chosen basis corresponding to the root λ_k. We have

$$[Y_i, Y_j] = \sum_{s=1}^{n_k} a_{ij}^s(k) Y_k^s.$$

Let us consider the dual basis $\{\omega_0, \omega_1, \cdots, \omega_{n-1}\}$ of $\{X, Y_1, \cdots, Y_{n-1}\}$. We have

$$d\omega_k^s = \lambda_k \omega_0 \wedge \omega_k^s + \sum_{l,m} a_{lm}^s(k)\omega_l \wedge \omega_m,$$

where the pairs (l, m) are such that $\lambda_l + \lambda_m = \lambda_k$. Then we deduce from (*)

$$\sum a_{ij}^s(k)\theta(X, Y_k^s) - \lambda_k \theta(Y_i, Y_j) = 0.$$

Let us fix λ_k. If we write

$$\begin{array}{rl}\theta = & \sum_{l,m,\lambda_l+\lambda_m=\lambda_k} b_{lm}(k)\omega_l \wedge \omega_m + \sum_{r,s,\lambda_r+\lambda_s\neq\lambda_k} c_{rs}(k)\omega_r \wedge \omega_s \\ & + \sum_k \sum_{s=1}^{n_k} \beta_k^s \omega_0 \wedge \omega_k^s\end{array}$$

then, for every pair (i,j) such that $\lambda_i + \lambda_j = \lambda_k$, (*) gives

$$-\lambda_k b_{ij}(k) + \sum_{s=1}^{n_k} a_{ij}^s(k)\beta_k^s = 0$$

and

$$b_{ij}(k) = \sum_{s=1}^{n_k} \frac{a_{ij}^s(k)}{\lambda_k}\beta_k^s.$$

The expression of θ becomes

$$\begin{array}{rl}\theta = & \sum_k \sum_{s=1}^{n_k} \beta_k^s \omega_0 \wedge \omega_k^s + \sum_{i,j,\lambda_i+\lambda_j=\lambda_k} \frac{a_{ij}^s(k)}{\lambda_k}\beta_k^s \omega_i \wedge \omega_j \\ & + \sum_{r,s,\lambda_r+\lambda_s\neq\lambda_k} c_{rs}(k)\omega_r \wedge \omega_s.\end{array}$$

Thus

$$\begin{array}{rcl}\theta & = & \sum_k (\frac{1}{\lambda_k}\sum_{s=1}^{n_k} \beta_k^s(\lambda_k\omega_0 \wedge \omega_k^s + \sum_{i,j,\lambda_i+\lambda_j=\lambda_k} a_{ij}^s(k)\omega_i \wedge \omega_j) \\ & & + \sum_{r,s,\lambda_r+\lambda_s\neq\lambda_k} c_{rs}(k)\omega_r \wedge \omega_s \\ & = & \sum_s \beta_k^s d\omega_k^s + \sum_{k'\neq k}\sum_{s=1}^{n_{k'}} \beta_{k'}^s \omega_0 \wedge \omega_{k'}^s \\ & & + \sum_{r,s,\lambda_r+\lambda_s\neq\lambda_k} c_{rs}(k)\omega_r \wedge \omega_s.\end{array}$$

If we continue this method for all the non simple roots (i.e which admit a decomposition as sum of two roots, we obtain the heralded result.

For the converse, if 0 is a root, then the cocycle

$$\theta = \omega_0 \wedge \omega_0'$$

where ω_0' is related with the eigenvector associated to the root 0 is not integrable. $\square$

Remark. It is easy to verify that every solvable rigid Lie algebra of rank greater or equal to 2 cannot have 0 as root. Likewise every solvable rigid Lie algebra of rank 1 and of dimension less than 8 has not 0 as root. This confirm in small dimension the following conjecture [Ca]:

If $\mathfrak{g}$ is a complex solvable finite dimensional rigid Lie algebra of rank 1, then 0 is not a root.

Consequences. If $H^2(\mathfrak{g},\mathbb{C}) \neq 0$, there exits $\theta \in \wedge^2\mathfrak{g}^*$ such that $[\theta]_{H^2} \neq 0$. If $rg(\mathfrak{g}) \geq 2$, then we can suppose that $\theta \in \wedge^2\mathfrak{t}^*$ and ω defines a non trivial deformation of $\mathcal{U}(\mathfrak{g})$. If $rg(\mathfrak{g}) = 1$, then 0 is not a root of $\mathfrak{t}$. The Hochschild Serre sequence gives:

$$\begin{array}{rcl}H^2_{CE}(\mathfrak{g},\mathcal{U}(\mathfrak{g})) & = & (\wedge^2\mathfrak{t}^* \otimes Z(\mathcal{U}(\mathfrak{g}))) \oplus (\mathfrak{t}^* \otimes H^1_{CE}(\mathfrak{n},\mathcal{U}(\mathfrak{g}))^t) \oplus H^2_{CE}(\mathfrak{n},\mathcal{U}(\mathfrak{g}))^t \\ & = & \mathfrak{t}^* \otimes H^1_{CE}(\mathfrak{n},\mathcal{U}(\mathfrak{g}))^t \oplus H^2_{CE}(\mathfrak{n},\mathcal{U}(\mathfrak{g}))^t.\end{array}$$

But from the previous proof, if θ is a non trivial 2-cocycle of $Z^2_{CE}(\mathfrak{g},\mathbb{C})$ then $i(X)\theta \neq 0$ for every $X \in \mathfrak{t}$, $X \neq 0$. The 1-form $\omega = i(X)\theta$ is closed. Then θ corresponds to a cocycle belonging to $\mathfrak{t}^* \otimes Z^1_{CE}(\mathfrak{n},\mathcal{U}(\mathfrak{g}))^t$ and defines a deformation of $\mathcal{U}(\mathfrak{g})$.

Theorem 129 *Let $\mathfrak{g}$ be a solvable complex rigid Lie algebra. If its rank is greater than or equal to 2 or if the rank is 1 and 0 is a root, then the enveloping algebra $\mathcal{U}(\mathfrak{g})$ is not rigid.*

Remark. In [103], T.Petit describes some examples of deformations of the enveloping algebra of a rigid Lie algebra $\mathfrak{g}$ in small dimensions and satisfying $H^2_{CE}(\mathfrak{g},\mathbb{C}) = 0$. For them, he shows that every deformation of the linear Poisson structure on the dual $\mathfrak{g}^*$ of $\mathfrak{g}$ induces a non trivial deformation of $\mathcal{U}(\mathfrak{g})$. This reduces the problem to find non trivial deformation of the linear Poisson structure.

12.4.4 Poisson algebras

Recall that a Poisson algebra $\mathcal{P}$ is a (commutative) associative algebra endowed with a second algebra multiplication satisfying the Jacobi's identity and the Leibniz rule

$$[a, bc] = b[a, c] + [a, b]c,$$

for all $a, b, c \in \mathcal{P}$. The tensor product $\mathcal{P}_1 \otimes \mathcal{P}_2$ of two Poisson algebras is again a Poisson algebra with the following associative and Lie products on $\mathcal{P}_1 \otimes \mathcal{P}_2$:

$$\begin{cases} (a_1 \otimes a_2) \cdot (b_1 \otimes b_2) = (a_1 \cdot b_1) \otimes (a_2 \cdot b_2, \\ [(a_1 \otimes a_2), (b_1 \otimes b_2)] = ([a_1, b_1] \otimes a_2 \cdot b_2) + (a_1 \cdot b_1 \otimes [a_2, b_2]), \end{cases}$$

for all $a_1, b_1 \in \mathcal{P}_1, a_2, b_2 \in \mathcal{P}_2$. We can easily verify that these multiplications satisfy the Leibniz rule.

Every commutative associative algebra has a natural Poisson structure, putting $[a, b] = ab - ba = 0$. Then the tensor product of a Poisson algebra by a valued algebra is also a Poisson algebra. In this context we have the notion of valued deformations. For example, if we take as valued algebra the algebra $\mathbb{C}[[t]]$, then the Poisson structure of $\mathcal{P} \otimes \mathbb{C}[[t]]$ is given by

$$\begin{cases} (a_1 \otimes a_2(t)) \cdot (b_1 \otimes b_2(t)) = (a_1 \cdot b_1) \otimes (a_2(t) \cdot b_2(t)), \\ [(a_1 \otimes a_2(t)), (b_1 \otimes b_2(t))] = [a_1, b_1] \otimes a_2(t) \cdot b_2(t), \end{cases}$$

because $\mathbb{C}[[t]]$ is a commutative associative algebra.

Remark. Since we have a tensorial category it is natural to look if we can define a Brauer Group for Poisson algebras. Since the associative product corresponds to the classical tensorial product of associative algebras, we can consider only Poisson algebras which are finite dimensional simple central algebras. The matrix algebras $M_n(\mathbb{C})$ are Poisson algebras. Then, considering the classical equivalence relation for define the Brauer Group, the class of matrix algebra constitutes an unit. Now the opposite algebra A^{op} also is a Poisson algebra. In fact, the associative product is given by $a \cdot_{op} b = ba$ and the Lie bracket by $[a, b]_{op} = ba - ab$. Thus considering

$$\begin{cases} [a, b \cdot_{op} c]_{op} = [a, cb]_{op} = cba - acb, \\ b \cdot_{op} [a, c]_{op} + [a, b]_{op} \cdot_{op} c = cab - acb + cba - cab = cba - acb, \end{cases}$$

we obtain a Poisson structure of A^{op}. The opposite algebra A^{op} is, modulo the equivalence relation, the inverse of A.

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